Petz recoverability in AQFT via conditional expectations

A framework and a conditional exponential recovery bound

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Abstract

We formulate an operational notion of recoverability in algebraic quantum field theory (AQFT) for type III local von Neumann algebras. Fixing a faithful normal KMS reference state and assuming the existence of a state-preserving conditional expectation, we define a Petz-type recovery channel as the Petz dual (Accardi–Cecchini adjoint). To make tripartite tensor-product notation meaningful, we work in a fixed split implementation for each separation parameter r. For a tripartition A–B–C with B a collar separating A from C, we assume (i) exponential decay of split-implemented conditional mutual information and (ii) a CMI-to-recovery inequality for the chosen Petz-type map. Under these explicit bridge assumptions we obtain a conditional exponential recoverability bound $E_{\mathrm{rec}}(r) \leq g(C_1 e^{-m_{\mathrm{CMI}} r})$, and in particular $E_{\mathrm{rec}}(r) \leq (cC_1)e^{-m_{\mathrm{CMI}} r}$ when g is locally linear. We include a finite-dimensional warm-up, prove finite-dimensional consistency of the Petz-dual definition with the standard Petz map, and discuss how nuclearity/modular nuclearity may support a future derivation of the bridge assumptions. The main theorem is therefore a conditional framework statement: it isolates AQFT-correct algebraic ingredients for Petz-type recovery in type III settings and makes the missing bridge assumptions explicit.

Contents

1 Introduction

Local algebras in AQFT are typically type III, so reduced density matrices and partial traces are not available in the naive finite-dimensional sense. Nevertheless, one can pose operational questions about *recoverability*: given access to a region AB, how well can one reconstruct information about a distant region C?

This note is intentionally conservative. We do not claim new field-theoretic results on the Yang–Mills mass gap. Instead, we isolate a clean conditional statement: assuming quantitative conditional-independence and a CMI-to-recovery bridge (both made explicit), one obtains an exponential bound on a fidelity-based recovery error for a Petz-type recovery channel defined using conditional expectations.

We emphasize that our main statement is a conditional theorem: its purpose is to isolate AQFT-correct algebraic infrastructure for Petz-type recovery in type III settings and to make all bridge assumptions explicit.

Contribution. Our contribution is to (i) formulate a Petz-type recovery channel in the type III AQFT setting using the Accardi–Cecchini (Petz-dual) construction relative to a faithful KMS state and a state-preserving conditional expectation, and (ii) package the "clustering-to-recoverability" route into two explicit bridge assumptions: exponential decay of split-implemented conditional mutual information and a CMI-to-recovery inequality for the chosen Petz-type map. Under these assumptions we obtain a conditional exponential recoverability bound. We emphasize that the bridge assumptions are not proved here and constitute the main open technical problem.

2 Tripartition geometry and split implementations

We work in a Haag–Kastler framework. Let $O \mapsto \mathcal{A}(O)$ be a local net of von Neumann algebras on Minkowski space, satisfying isotony and locality. We fix a faithful normal KMS state ω_β at inverse temperature $\beta < \infty$ with respect to time translations.

Definition 2.1 (Tripartition and separation)**.** Let A, B, C denote spacetime regions such that B is an intermediate collar separating A from C. We write r for a chosen notion of separation between A and C compatible with the collar geometry.

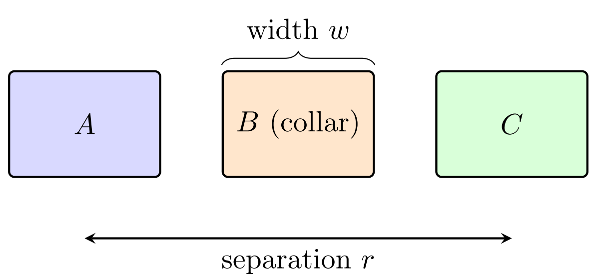

Figure 1: Schematic collar geometry.

Assumption 2.2 (Fixed split implementation)**.** For each separation parameter r under consideration, we fix a normal $*$-isomorphism

$$\alpha_r : \mathcal{A}(A \cup B \cup C) \;\to\; \mathcal{M}^{ABC^{(r)}},$$

where $\mathcal{M}_{ABC}^{(r)}$ is a type I von Neumann algebra of the form

$$\mathcal{M}_{ABC}^{(r)} \cong \mathcal{B}(\mathcal{H}_A^{(r)}) \,\bar{\otimes}\, \mathcal{B}(\mathcal{H}_B^{(r)}) \,\bar{\otimes}\, \mathcal{B}(\mathcal{H}_C^{(r)}),$$

such that $\alpha_r(\mathcal{A}(A))$, $\alpha_r(\mathcal{A}(B))$, $\alpha_r(\mathcal{A}(C))$ act nontrivially only on the corresponding tensor factors. We write $\omega_\beta^{r:=\omega_\beta\circ\alpha_r^{-1}}$ for the pushed-forward normal state on $\mathcal{M}_{ABC}^{(r)}$.

Remark 2.3. Assumption 2.2 is a bookkeeping device: it makes tensor-factor notation meaningful by working in a fixed split implementation. We do not claim uniqueness of α_r.

Assumption 2.4 (Generation in the split implementation)**.** For each r, in the fixed split implementation α_r we assume that the von Neumann algebra associated with the union is generated by the corresponding subalgebras, i.e.

$$\alpha_r(\mathcal{A}(A\cup B)) = \big(\alpha_r(\mathcal{A}(A))\vee\alpha_r(\mathcal{A}(B))\big)'' \quad \text{and} \quad \alpha_r(\mathcal{A}(A\cup B\cup C)) = \big(\alpha_r(\mathcal{A}(A))\vee\alpha_r(\mathcal{A}(B))\vee\alpha_r(\mathcal{A}(C))\big)''.$$

3 Finite-dimensional warm-up: the CMI-to-recovery mechanism

We briefly recall the finite-dimensional picture motivating Assumption 7.4 below. Let ρ_{ABC} be a density operator on $\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_C$ with $\dim\mathcal{H}_A, \dim\mathcal{H}_B, \dim\mathcal{H}_C < \infty$. The conditional mutual information is

$$I_\rho(A:C|B) = S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_B) - S(\rho_{ABC}),$$

with $S(\rho) = -\mathrm{Tr}(\rho\log\rho)$.

Proposition 3.1 (Finite-dimensional CMI-to-recovery prototype)**.** *Let ρ_{ABC} be a density operator on $\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_C$ with finite dimensions. Then there exists a recovery channel $\mathcal{R}_{B\to BC}$ and a universal constant $c_{\mathrm{FR}} > 0$ such that*

$$-\log F\big(\rho_{ABC}, (\mathrm{id}_A\otimes\mathcal{R}_{B\to BC})(\rho_{AB})\big) \le c_{\mathrm{FR}}\, I_\rho(A:C|B).$$

Proof. This follows from the Fawzi–Renner recovery theorem; see [6]. The constant depends on the convention for fidelity (squared vs. unsquared); we absorb this into c_{FR}. □

Remark 3.2. Proposition 3.1 is stated with a universal constant c_{FR} to avoid convention issues (squared vs. unsquared fidelity). We also stress that the recovery map whose existence is guaranteed by Fawzi–Renner need not coincide with the Petz dual; in this note we fix a Petz-type candidate and encode the required estimate as an explicit assumption (Assumption 7.4).

3.1 Finite-dimensional consistency of the Petz dual

Proposition 3.3 (Finite-dimensional consistency of the Petz dual)**.** *Let $\mathcal{H}_B$ and $\mathcal{H}_C$ be finite-dimensional Hilbert spaces and let*

$$\mathcal{M} := \mathcal{B}(\mathcal{H}_B\otimes\mathcal{H}_C), \qquad \mathcal{N} := \mathcal{B}(\mathcal{H}_B)\otimes\mathbf{1}_C \subset \mathcal{M}.$$

Let $\rho_{BC} > 0$ be a faithful density operator on $\mathcal{H}_B\otimes\mathcal{H}_C$ and let $\omega(X) := \mathrm{Tr}(\rho_{BC}X)$. Define $\rho_B := \mathrm{Tr}_C(\rho_{BC})$.

Define the ω-preserving conditional expectation $E_{BC\to B} : \mathcal{M}\to\mathcal{N}$ by

$$E_{BC\to B}(Z) := \rho_B^{-1/2}\,\mathrm{Tr}_C\big(\rho_{BC}^{1/2}\,Z\,\rho_{BC}^{1/2}\big)\,\rho_B^{-1/2} \;\otimes\; \mathbf{1}_C.$$

Let $\mathcal{R}^{\mathrm{Petz}}_{B\to BC} : \mathcal{N} \to \mathcal{M}$ be the Petz dual (Accardi–Cecchini adjoint) of $E_{BC\to B}$ with respect to ω, i.e. the unique map such that for all $Z \in \mathcal{M}$ and $X \in \mathcal{N}$,

$$\omega\Big(Z^{\,\mathcal{R}^{\mathrm{Petz}}_{B\to BC}(X)}\Big) = \omega\big(E_{BC\to B}(Z)^{\,X}\big)\,.$$

Then $\mathcal{R}^{\mathrm{Petz}}_{B\to BC}$ coincides with the standard Petz recovery map:

$$\mathcal{R}^{\mathrm{Petz}}_{B\to BC}(X) = \rho_{BC}^{1/2}\left(\rho_B^{-1/2}\,X_B\,\rho_B^{-1/2}\otimes \mathbf{1}_C\right)\rho_{BC}^{1/2}, \qquad X = X_B\otimes \mathbf{1}_C.$$

Proof. First note that $E_{BC\to B}$ is ω-preserving:

$$\begin{aligned}
\omega(E_{BC\to B}(Z)) &= \mathrm{Tr}\Big(\rho_{BC}\,\rho_B^{-1/2}\,\mathrm{Tr}_C(\rho_{BC}^{1/2} Z \rho_{BC}^{1/2})\,\rho_B^{-1/2}\otimes \mathbf{1}_C\Big)\\
&= \mathrm{Tr}\Big(\rho_B\,\rho_B^{-1/2}\,\mathrm{Tr}_C(\rho_{BC}^{1/2} Z \rho_{BC}^{1/2})\,\rho_B^{-1/2}\Big)\\
&= \mathrm{Tr}\Big(\mathrm{Tr}_C(\rho_{BC}^{1/2} Z \rho_{BC}^{1/2})\Big) = \mathrm{Tr}\Big(\rho_{BC}^{1/2} Z \rho_{BC}^{1/2}\Big) = \mathrm{Tr}(\rho_{BC} Z) = \omega(Z),
\end{aligned}$$

using $\mathrm{Tr}(\mathrm{Tr}_C(\cdot)) = \mathrm{Tr}(\cdot)$.

Now define $\mathcal{R} : \mathcal{N} \to \mathcal{M}$ by

$$\mathcal{R}(X_B\otimes \mathbf{1}_C) := \rho_{BC}^{1/2}\left(\rho_B^{-1/2}\,X_B\,\rho_B^{-1/2}\otimes \mathbf{1}_C\right)\rho_{BC}^{1/2}.$$

We check the adjointness relation. For $Z \in \mathcal{M}$ and $X = X_B \otimes \mathbf{1}_C \in \mathcal{N}$,

$$\begin{aligned}
\omega\Big(Z^{\,\mathcal{R}(X)}\Big) &= \mathrm{Tr}\Big(\rho_{BC}\,Z^{\,\rho_{BC}^{1/2}\,(\rho_B^{-1/2} X_B \rho_B^{-1/2}\otimes \mathbf{1}_C)\,\rho_{BC}^{1/2}}\Big)\\
&= \mathrm{Tr}\Big(\rho_{BC}^{1/2}\,Z^{\,\rho_{BC}^{1/2}\,(\rho_B^{-1/2} X_B \rho_B^{-1/2}\otimes \mathbf{1}_C)}\Big)\\
&= \mathrm{Tr}\Big(\mathrm{Tr}_C(\rho_{BC}^{1/2}\,Z^{\,\rho_{BC}^{1/2})\,\rho_B^{-1/2} X_B \rho_B^{-1/2}}\Big)\\
&= \omega\big(E_{BC\to B}(Z)^{\,X}\big)\,,
\end{aligned}$$

which is the defining adjointness condition. By uniqueness of the Accardi–Cecchini adjoint, $\mathcal{R} = \mathcal{R}^{\mathrm{Petz}}_{B\to BC}$. □

4 Fidelity and recovery error for normal states

Definition 4.1 (Fidelity / transition probability)**.** Let φ, ψ be normal states on a von Neumann algebra $\mathcal{M}$. We denote by $F(\varphi,\psi)$ the squared transition probability (Uhlmann–Araki fidelity) for normal states on $\mathcal{M}$, as defined in the standard form representation (see [3, 4]). We use only that $F(\varphi,\psi) \in [0,1]$ and that F is monotone under normal completely positive unital maps.

Definition 4.2 (Recovery error)**.** Given a target state ω and a reconstructed state $\widetilde{\omega}$ on the same algebra, define

$$E_{\mathrm{rec}} := -\log F(\omega, \widetilde{\omega}).$$

5 Conditional expectations and the Petz dual (Accardi–Cecchini)

Assumption 5.1 (ω_β-preserving conditional expectation)**.** There exists a normal conditional expectation

$${}_{BC\to B} : \mathcal{A}(B\cup C) \to \mathcal{A}(B)$$

such that $\omega_\beta \circ_{BC\to B} = \omega_\beta|_{\mathcal{A}(B\cup C)}$.

Remark 5.2. The existence of a ω_β-preserving conditional expectation is a nontrivial condition and is related to modular invariance (Takesaki-type criteria). We take it as an explicit assumption; see [2].

Definition 5.3 (Petz dual map)**.** Assume Theorem 5.1. The Petz dual (Accardi–Cecchini adjoint) of ${}_{BC\to B}$ with respect to ω_β is the unique normal completely positive unital map

$$\mathcal{R}^{\mathrm{Petz}}_{B\to BC} : \mathcal{A}(B) \to \mathcal{A}(B\cup C)$$

satisfying, for all $Z \in \mathcal{A}(B\cup C)$ and all $X_B \in \mathcal{A}(B)$,

$$\omega_\beta\Big(Z^{\,\mathcal{R}^{\mathrm{Petz}}_{B\to BC}(X_B)}\Big) = \omega_\beta\big({}_{BC\to B}(Z)^{\,X_B}\big)\,.$$

Remark 5.4. Definition 5.3 avoids any appeal to traces on $\mathcal{B}(\mathcal{H})$. In finite dimensions it reproduces the usual Petz transpose channel; see Proposition 3.3 and [1].

Lemma 5.5 (Existence and basic properties of the Petz dual)**.** *Assume Assumption 5.1. Then there exists a unique normal completely positive unital map*

$$\mathcal{R}^{\mathrm{Petz}}_{B\to BC} : \mathcal{A}(B) \to \mathcal{A}(B\cup C)$$

satisfying the adjointness relation in Definition 5.3. Moreover,

$$\omega_\beta\big(\mathcal{R}^{\mathrm{Petz}}_{B\to BC}(X_B)\big) = \omega_\beta(X_B) \qquad \textit{for all } X_B \in \mathcal{A}(B).$$

Proof. Existence and uniqueness of the ω_β-adjoint (Accardi–Cecchini adjoint / Petz dual) associated with a ω_β-preserving normal conditional expectation are part of the general theory developed in [2]. In particular, the adjoint map is normal, completely positive, and unital. The final identity follows by taking $Z = \mathbf{1}$ in Definition 5.3, giving

$$\omega_\beta\big(\mathcal{R}^{\mathrm{Petz}}_{B\to BC}(X_B)\big) = \omega_\beta\Big(\mathbf{1}^{\,\mathcal{R}^{\mathrm{Petz}}_{B\to BC}(X_B)}\Big) = \omega_\beta\big({}_{BC\to B}(\mathbf{1})^{\,X_B}\big) = \omega_\beta(X_B).$$

□

6 Recovered state and recovery error in the split implementation

Definition 6.1 (Recovered state)**.** Assume Theorems 2.2 and 5.3. Fix r. Define the split-implemented Petz map as the conjugated map on the relevant subalgebra:

$$\mathcal{R}^{\mathrm{Petz}_{r,B\to BC}:=\alpha_r\circ\mathcal{R}^{\mathrm{Petz}}_{B\to BC}\circ\alpha_r^{-1}\big|_{\alpha_r(\mathcal{A}(B))}:\alpha_r(\mathcal{A}(B))\to\alpha_r(\mathcal{A}(B\cup C)).}$$

Since $\mathcal{R}^{\mathrm{Petz}}_{B\to BC}$ is normal, completely positive, and unital, the conjugated map $\mathcal{R}^{\mathrm{Petz}_{r,B\to BC}}$ inherits the same properties. Let $\omega_\beta^{r,AB}$ denote the restriction of ω_β^r to $\alpha_r(\mathcal{A}(A\cup B))$. We define the recovered split state on $\mathcal{M}^{ABC(r)}$ by

$$\widetilde{\omega_\beta}^{r,ABC:=\omega_\beta^{r,AB\circ\Phi_r},}$$

where $\Phi_r : \alpha_r(\mathcal{A}(A\cup B)) \to \alpha_r(\mathcal{A}(A\cup B\cup C))$ is the normal map determined on products XY with $X \in \alpha_r(\mathcal{A}(A))$ and $Y \in \alpha_r(\mathcal{A}(B))$ by

$$\Phi_r(XY) = X\,\mathcal{R}^{\mathrm{Petz}_{r,B\to BC}(Y),}$$

and extends uniquely to a normal map on $\alpha_r(\mathcal{A}(A\cup B))$ by Assumption 2.4. Finally, pull back to $\mathcal{A}(A\cup B\cup C)$ by

Remark 6.2. In the fixed tensor-product realization provided by Assumption 2.2, the map Φ_r coincides with the natural extension that acts as the identity on the A-factor and applies $\mathcal{R}^{\mathrm{Petz}_{r,B\to BC}}$ on the B-subalgebra.

$$\widetilde{\omega_{\beta}}_{r,ABC} := \widetilde{\omega_{\beta}}^{r,ABC} \circ \alpha_r.$$

Definition 6.3 (Recovery error as a function of separation)**.** Assume Theorems 4.1 and 6.1. Define

$$E_{\mathrm{rec}}(r) := -\log F\big(\omega_\beta|_{\mathcal{A}(A\cup B\cup C)}, \widetilde{\omega_{\beta}}_{r,ABC}\big).$$

Remark 6.4. This statement is made precise by Lemma 6.5 and Corollary 6.6 below.

Lemma 6.5 ($*$-isomorphism invariance of fidelity)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras and let $\alpha : \mathcal{M} \to \mathcal{N}$ be a normal $*$-isomorphism. For normal states φ, ψ on $\mathcal{M}$ one has*

$$F_{\mathcal{M}}(\varphi, \psi) = F_{\mathcal{N}}(\varphi \circ \alpha^{-1}, \psi \circ \alpha^{-1}).$$

Consequently,

$$-\log F_{\mathcal{M}}(\varphi, \psi) = -\log F_{\mathcal{N}}(\varphi \circ \alpha^{-1}, \psi \circ \alpha^{-1}).$$

Proof. By monotonicity of F under normal completely positive unital maps, applying the $*$-isomorphism α gives

$$F_{\mathcal{M}}(\varphi, \psi) \le F_{\mathcal{N}}(\varphi \circ \alpha^{-1}, \psi \circ \alpha^{-1}).$$

Applying the same monotonicity to α^{-1} yields the reverse inequality. Therefore equality holds. ☐

Corollary 6.6 (Computing $E_{\mathrm{rec}}(r)$ in the split algebra)**.** *With α_r as in Assumption 2.2, one has*

$$E_{\mathrm{rec}}(r) = -\log F\big(\omega_\beta^{r,\widetilde{\omega_\beta}^{r,ABC}}\big),$$

so the recovery error can be computed entirely in the type I split algebra $\mathcal{M}_{ABC}{}^{(r)}$.

Proof. Apply Lemma 6.5 with $\alpha = \alpha_r$ and the pair of normal states $\varphi = \omega_\beta|_{\mathcal{A}(A\cup B\cup C)}$ and $\psi = \widetilde{\omega_{\beta}}_{r,ABC}$, using the definitions of the pushed-forward states in the split implementation. ☐

7 Bridge assumptions: CMI decay and CMI-to-recovery

Definition 7.1 (Split-implemented CMI)**.** Fix r and the split implementation α_r from Assumption 2.2. In the split algebra $\mathcal{M}_{ABC}{}^{(r)}$, let $\rho_{ABC}{}^{(r)}$ be a density operator representing the normal state $\omega_\beta^{r}{}^{=\omega_\beta\circ\alpha_r^{-1}}$. Define the reduced density operators by partial traces in the fixed tensor-product realization: here Tr_A, Tr_B, Tr_C denote the usual partial traces in a chosen identification $\mathcal{M}_{ABC}{}^{(r)} \cong \mathcal{B}(\mathcal{H}_A^{(r)} \otimes \mathcal{H}_B^{(r)} \otimes \mathcal{H}_C^{(r)})$. We set

$$\rho^{AB}{}^{(r)} := \mathrm{Tr}_C(\rho^{ABC}{}^{(r)}), \qquad \rho^{BC}{}^{(r)} := \mathrm{Tr}_A(\rho^{ABC}{}^{(r)}), \qquad \rho^{B}{}^{(r)} := \mathrm{Tr}_{AC}(\rho^{ABC}{}^{(r)}).$$

Since $\mathcal{M}_{ABC}{}^{(r)}$ is type I, every normal state admits such a trace-class density-operator representative in a concrete $\mathcal{B}(\mathcal{H})$ realization; we fix such a realization for the purpose of defining entropies. Assume the following entropies are finite and define

$$I^{\omega_\beta,r}(A{:}C|B) := S(\rho^{AB}{}^{(r)}) + S(\rho^{BC}{}^{(r)}) - S(\rho^{B}{}^{(r)}) - S(\rho^{ABC}{}^{(r)}),$$

where $S(\rho) := -\mathrm{Tr}(\rho \log \rho)$ is the von Neumann entropy in the chosen split implementation.

Remark 7.2. The quantity $I^{\omega_\beta,r}(A:C|B)$ depends on the choice of split implementation α_r; we fix a choice and treat the result as an operational notion of CMI. When the above entropies are infinite, one should instead use an Araki relative-entropy formulation; we do not pursue that refinement here.

Assumption 7.3 (Exponential decay of split-implemented CMI)**.** There exist constants $C_1 > 0$ and $m_{\mathrm{CMI}} > 0$ such that

$$I^{\omega_\beta,r}(A:C|B) \le C_1 e^{-m_{\mathrm{CMI}} r}$$

for all separations r in the collar geometry under consideration.

Assumption 7.4 (CMI-to-recovery inequality)**.** There exists a monotone function $g : [0,\infty) \to [0,\infty)$ with $g(t) \to 0$ as $t \to 0$ such that

$$E_{\mathrm{rec}}(r) \le g\big(I^{\omega_\beta,r}(A:C|B)\big),$$

where $E_{\mathrm{rec}}(r)$ is the Petz-type recovery error from Theorem 6.3.

Remark 7.5. In finite dimensions, quantitative CMI-to-recovery bounds are known for suitable recovery maps (e.g. Fawzi–Renner [6]). In the present type III setting we encode the needed estimate for our chosen Petz-type recovery in Assumption 7.4.

8 Conditional theorem (framework statement)

Theorem 8.1 (Conditional exponential recoverability bound)**.** *Assume Theorems 2.2, 2.4, 5.1, 7.3 and 7.4 and the constructions of Theorems 5.3, 6.3 and 7.1. Then*

$$E_{\mathrm{rec}}(r) \le g(C_1 e^{-m_{\mathrm{CMI}} r}).$$

In particular, if $g(t) \le c\,t$ for t in the relevant range, then

$$E_{\mathrm{rec}}(r) \le (cC_1)\, e^{-m_{\mathrm{CMI}} r}.$$

Proof. Combine Theorem 7.4 with Theorem 7.3. □

Remark 8.2 (Role of assumptions). Assumption 2.2 provides a fixed type I split implementation in which tensor-product notation and density operators are meaningful. Assumption 5.1 guarantees that the Petz dual recovery channel is well-defined (Lemma 5.5). Assumption 7.3 encodes exponential conditional independence in the chosen split implementation, and Assumption 7.4 bridges this information-theoretic decay to the operational recovery error $E_{\mathrm{rec}}(r)$.

9 Discussion and outlook

Physical motivation: exponential clustering. In massive QFTs one expects exponential decay of connected correlation functions,

$$\big|\omega_\beta(XY) - \omega_\beta(X)\omega_\beta(Y)\big| \le C_{\mathrm{cl}} \|X\| \|Y\| e^{-m_{\mathrm{cl}} r},$$

for observables $X \in \mathcal{A}(A)$ and $Y \in \mathcal{A}(C)$ separated by distance r. This clustering behavior is a standard hallmark of gapped phases and motivates Assumption 7.3. Establishing a rigorous implication from such clustering to CMI decay in the continuum type III setting remains open in general.

Toward deriving the bridge from nuclearity (program). Assumptions 7.3 and 7.4 are the technical frontier. One possible route to deriving them in AQFT (rather than assuming them) is:

1. Use nuclearity/modular nuclearity to obtain quantitative split inclusions and control localized degrees of freedom at inverse temperature β [5].
2. Construct finite-rank approximations of restricted states/channels with explicit error bounds in the predual norm.
3. Apply finite-dimensional recovery inequalities to the approximants and pass to the limit using stability of fidelity/relative entropy under the chosen approximation.

We leave this program for future work.

Numerical outlook. The framework suggests numerical tests in lattice models by computing $I(A : C|B)$ (or suitable proxies) and Petz-based recovery errors as functions of collar width and separation. Systematic simulations, including lattice gauge theories, are deferred.

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