Recoverability Geometry: Distances and Embeddings from Quantum Markov Data

Definitions, diagnostics, and a reconstruction protocol

Version 2

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Abstract

We propose an operational route from recoverability data to effective geometry. Given a tripartition A–$B(w)$–C and a collar width w, we consider a Petz-type recoverability error $E_{\rm rec}^{\rm Petz}(w)$ defined via fidelity and extracted from a fixed collaring rule $(A, C, w) \mapsto B_{A,C}(w)$. We define distance-like functionals from the minimal buffer needed to suppress $E_{\rm rec}^{\rm Petz}(w)$ below a threshold, and from exponential fit scales when such a regime exists; these are organized into a (generally non-metric) dissimilarity matrix on coarse regions, symmetrized when needed, and embedded via multidimensional scaling or diffusion maps. The paper emphasizes precise definitions (collaring rule, symmetrization, censoring below numerical floors) and falsifiable diagnostics (approximate triangle inequalities, robustness to thresholds and regularization). A minimal control experiment in the 1D transverse-field Ising model illustrates the pipeline and the growth of a recoverability length near criticality. *v2 (definitions unchanged) adds:* a regenerable suite replacing the "representative run" of v1 – the control table is regenerated from scratch, its $g = 0.5$ row is flagged as unstable by the paper's own fit-window policy, and a three-point-fit caveat is stated; the first in-model test of the tracking conjecture – from the same ground states, $\xi_{\rm rec}/\xi_{\rm corr} \in \{0.98, 0.53, 0.52\}$ across regimes, same order of magnitude throughout; a first numerical illustration of the embedding machinery (four coarse regions, MDS recovering the chain order exactly, triangle violations bounded by discretization), which also surfaces an operational lesson: tracing out the region between $B(w)$ and C fakes separation and inverts monotonicity, so the separation condition of the collaring rule is essential, and profiles below the separating width are not admissible; the observation that the fixed-$|A|, |C|$/traced-environment design of the control is precisely the fixed-target protocol that resolves the $|C|$-shrinkage confound identified in the companion $d_{\rm eff}$ notes; and series positioning.

1 Introduction

Geometry from quantum-information patterns appears as entanglement entropy and minimal surfaces [4], tensor networks [5], and QEC toy models [6]. This note's input is *recoverability*: the buffer width needed to reconstruct BC from B acting only on B is treated as a proxy for effective distance between A and C, and such distances are embedded and compared to lattice geometry and physical scales. Scope: no unique emergent manifold and no theorem-level continuum limit are claimed; the output is a family of operational distance-like objects that can succeed (approximate metric behavior) or fail informatively.

Series positioning (added in v2). This note formalizes the d_{eff} mini-block of the recovery program: 2601.0038 [10] (criticality signature in h_x), 2601.0040 [11] (finite-size scaling and its caveats), 2601.0042 [12] (temperature/perturbation dependence), on the pipeline of 2601.0035 [13] with the non-Gaussian bridge upstream [14]. Notably, the control experiment of Section 3 uses fixed $|A|, |C|$ with the complement traced as environment – precisely the fixed-target protocol that the v2 caveats of [10, 11, 12] call for to resolve the $|C|$-shrinkage confound; the present framework thus supplies the clean geometry for future scaling studies of the mini-block.

2 Preliminaries, collaring rule, distances, embeddings

Definitions as in v1 (unchanged): squared Uhlmann fidelity and $E_{\text{rec}} = -\log F$; regularized Petz-type map with $\rho_{B,\delta} = \rho_B + \delta \mathbf{1}$ (CP, trace-normalized output) – because the δ-regularized construction is trace-normalized after application, it is used throughout as an *operational diagnostic* rather than as a literal CPTP recovery channel, unless the unregularized Petz map is well-conditioned; collaring rule as part of the input (disjointness, nestedness, separation); directed threshold distance $d^{(\varepsilon)}_{\text{rec}}(A \to C)$, max-symmetrization, persistent variant; exponential-fit scale ξ_{rec} with floor censoring; affinity kernel $K_{ij} = e^{-\alpha D_{ij}}$ with missing-data conventions; MDS/diffusion-map embeddings as diagnostic summaries; effective monotonicity, robustness, and triangle-violation score Δ_{ijk} as falsifiable criteria.

Remark 2.1 (The separation condition is essential (added in v2)). The suite's embedding demo surfaced an operational subtlety: if $B(w)$ does not yet separate A from C (i.e. w is below the minimal separating width) and the region between $B(w)$ and C is traced out, the traced environment itself decorrelates A from C and the recoverability error becomes *small for distant pairs at tiny* w, inverting monotonicity and corrupting the distance matrix. Profiles must therefore be restricted to w at or above the separating width (or the collaring rule chosen so that $B(w) \cup C$ exhausts the region between, as in the Section 3 design); condition (3) of the collaring rule is not decorative.

Remark 2.2 (Three-point fits (added in v2)). The control fits below use $w \in \{2, 3, 4\}$: three points. As quantified for the companion scaling note (2601.0040 v2), such windows cannot discriminate functional forms; ξ_{rec} values are descriptive scales, and rows failing the stability policy (low R^2) must be flagged rather than quoted.

3 Minimal numerical illustration: TFIM control (regenerated in v2)

Model and geometry as in v1: $H = -J \sum Z_i Z_{i+1} - g \sum X_i - h \sum Z_i$, open chain, $L = 14$, contiguous A–$B(w)$–C with $|A| = 3$, $|C| = 4$ fixed and the complement traced as environment; $h = 10^{-3}$ at $g = 0.5$ to select a symmetry-broken vacuum; global $\delta = 10^{-12}$; semilog fits on $w \in \{2, 3, 4\}$ with floor 10^{-14}. The suite regenerates the control from scratch (sparse Lanczos ground states):

g	h	ξ_{rec} (v1)	ξ_{rec} (suite)	R^2 (suite)	
0.5	10^{-3}	1.433	0.986	0.8325	[flagged; see below]
1.0	0	4.58	3.794	0.9800	
2.0	0	0.6515	0.640	1.0000	

The $g = 2.0$ row reproduces v1 closely; the $g = 1.0$ row agrees in scale (the residual difference is consistent with undeclared v1 details such as exact block placement); the $g = 0.5$ row is *unstable under regeneration* (0.986 vs 1.433) with low R^2 in both runs – by the paper's own policy (Remark on floors and fit windows) this row should not be quoted as a scale, and v2 flags it accordingly. The qualitative pattern – short ξ_{rec} in gapped regimes, growth near criticality – stands.

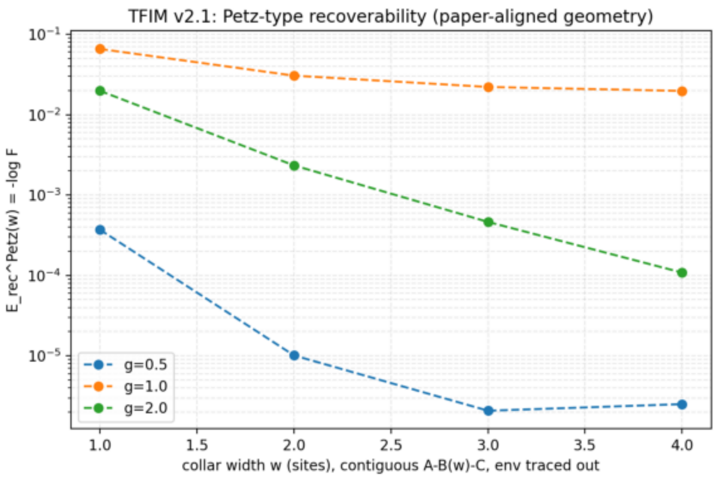

Figure 1: TFIM control (v1 figure, unchanged): $E_{\mathrm{rec}}^{\mathrm{Petz}}(w)$ vs w.

3.1 First in-model test of the tracking conjecture (added in v2)

The conjecture that ξ_{rec} tracks a conventional correlation length is now tested inside the control: from the *same* ground states, the suite extracts ξ_{corr} from connected $\langle Z_1 Z_r \rangle$ decay and compares:

$$g = 0.5: \ \xi_{\mathrm{rec}}/\xi_{\mathrm{corr}} = 0.98; \qquad g = 1.0: \ 0.53; \qquad g = 2.0: \ 0.52.$$

Same order of magnitude throughout, with both scales growing together at criticality – a first, in-model confirmation of Conjecture 4.1 (which remains open in general, and whose interesting content is its failure modes in constrained/sectored systems).

3.2 Embedding demo (added in v2)

The Section 5 machinery of v1 was never illustrated. The suite adds a minimal demo: $L = 12$, $g = 2.0$, four single-site coarse regions at positions $\{1, 4, 7, 10\}$; distances from the separating-width profile (Remark 2.1) with $\varepsilon = 10^{-3}$ and censoring cap; max-symmetrization; classical MDS in one dimension. Result: the embedding recovers the chain order exactly, and all triangle-violation scores are bounded by the discretization/cap unit ($\max \Delta_{ijk} = 1.0$). Modest as it is, this closes the loop: the full pipeline – profiles → distances → symmetrization → embedding → diagnostics – runs end to end on a case with known geometry.

4 Conjectures and model comparisons

Conjecture 4.1 (Recoverability tracks a physical length scale; unchanged, now with in-model support)**.** *In gapped phases of local Hamiltonians, for suitable collaring rules and region families, $E_{\mathrm{rec}}^{\mathrm{Petz}}(w)$ admits an exponential window and the extracted ξ_{rec} is finite and of the same order as a conventional correlation length, up to model- and geometry-dependent constants.*

5 Protocol summary and discussion

As in v1 (protocol steps 1–7 unchanged), with two v2 refinements: profiles are admissible only at or above the separating width (Remark 2.1), and fit-window stability is enforced before quoting ξ_{rec} (Remark 2.2). Immediate extensions: higher-dimensional systems, free-fermion benchmarks with known correlation lengths, and constrained/gauge systems where subsystem prescriptions add structure [9].

6 Verification suite (added in v2)

`verification/2601-0043/verify_2601_0043.py` (sparse-Lanczos ground states, chunked JSON cache): regenerates the control table (`-chunk g`), runs the in-model conjecture test from the same states, and the embedding demo (`-demo`). Reference output ships alongside.

Note added in v2

Definitions and protocol are unchanged; the control's qualitative pattern stands. v2 adds: (1) regeneration of the control table with the $g = 0.5$ row flagged as unstable under the paper's own policy; (2) the three-point-fit caveat; (3) the first in-model test of Conjecture 4.1 (ratios $0.98, 0.53, 0.52$); (4) the first end-to-end illustration of the embedding machinery, which surfaced the operational necessity of the separation condition (Remark 2.1); (5) the observation that the control's fixed-$|A|, |C|$/traced-environment design is the fixed-target protocol resolving the $|C|$ confound of the companion notes; (6) series positioning (v1 cited two forthcoming preprints but none of the published d_{eff} companions).

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