Finite-Size Scaling of Petz Recovery Length in the TFIM

Threshold-dependent operational exponents from exact diagonalization

Version 2

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Abstract

We study finite-size scaling of an operational recovery length extracted from Petz-map recovery in the transverse-field Ising chain. For a tripartition A–B–C with a collar B of width w, we define $E_{\mathrm{Petz}}(w) = -\log F$ (squared Uhlmann fidelity), $E_{\mathrm{best}}(w) = \min_{w' \le w} E_{\mathrm{Petz}}(w')$, and the effective recovery distance $d_{\mathrm{eff}}(\varepsilon)$, with log-linear interpolation. Using exact diagonalization at $h_z = 0$, $\beta = 12$, $|A| = 2$ for $N \in \{9, 10, 11, 12\}$, we analyze the peak height $d_{\mathrm{eff}}^{\max}(\varepsilon; N) = \max_{h_x} d_{\mathrm{eff}}$ in a censoring-free threshold regime, finding that the finite-window data are well summarized by *descriptive* power-law fits $d_{\mathrm{eff}}^{\max}(\varepsilon; N) \sim N^{\kappa(\varepsilon)}$ with, e.g., $\kappa(3\times10^{-3}) \approx 0.44$ and $\kappa(5\times10^{-3}) \approx 0.26$, and a pseudocritical drift of the peak location with a threshold-dependent effective exponent ν_{eff} – reported as operational quantities, not universal estimates. *v2 adds (no v1 number is changed) two mandatory caveats, both quantified by a regenerable suite that reproduces v1's Table 1 exactly:* (i) a $|C|$-shrinkage/growth confound – the *off-critical baseline* $d_{\mathrm{eff}}(h_x = 0.80; N)$ also grows with N at fixed thresholds, so the raw peak growth conflates critical physics with tripartition geometry; the cleaner object is the critical enhancement $\Delta(N) =$ peak $-$ baseline, which still grows with N (e.g. $0.42 \to 0.74$ over $N = 9, 10$ at $\varepsilon = 3\times10^{-3}$), so the critical signal survives baseline subtraction while $\kappa(\varepsilon)$ from raw peaks must be read as geometry-contaminated; (ii) functional-form indistinguishability – over the accessible sub-octave in N, power-law, logarithmic and linear fits of the peak height have R^2 spreads below 0.01, and κ itself shifts strongly with the fit window; the correct reading of $\kappa(\varepsilon)$ is a descriptive summary, not an established power law. Series positioning and a verification suite are included.

1 Setup

As in v1: $E_{\mathrm{Petz}}(w) := -\log F(\rho_{ABC}, \tilde{\rho}^{\mathrm{Petz}}_{ABC}(w))$, $F(\rho,\sigma) = (\mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}})^2$; $\tilde{\rho}^{\mathrm{Petz}}_{ABC}(w) = (\mathrm{id}_A \otimes \mathcal{R}^{\mathrm{Petz}}_{B\to BC})(\rho_{AB})$ with $\mathcal{R}^{\mathrm{Petz}}_{B\to BC}(X) = \rho_{BC}^{1/2}\rho_B^{-1/2} X \rho_B^{-1/2}\rho_{BC}^{1/2}$, eigenvalue-floor regularization, PSD projection and trace renormalization; E_{best}, $d_{\mathrm{eff}}(\varepsilon)$ with log-linear interpolation; censoring against the effective $w_{\max}$, with thresholds chosen censoring-free.

Series positioning (added in v2). This note delivers the finite-size-scaling item from the outlook of 2601.0038 [3], whose definitions it uses verbatim; the pipeline and censored-fit discipline are those of 2601.0035 [4]; the crossover diagnostic of 2512.0101 v2 [5] is the upstream cousin.

2 Model and data set

As in v1: $H = -\sum Z_i Z_{i+1} - h_x \sum X_i$ ($J = 1$, $h_z = 0$), open boundaries; Gibbs states at $\beta = 12$ by ED; $|A| = 2$ at the left edge; $h_x \in [0.90, 1.10]$ step 0.01; $N \in \{9, 10, 11, 12\}$.

3 Results

All v1 numbers stand: Table 1 (peak locations and heights; the suite reproduces the overlapping entries to all reported digits, e.g. $d_{\text{eff}}^{\text{max}} = 5.7280, 6.1234$ at $N = 9, 10$, $\varepsilon = 3 \times 10^{-3}$), Table 2 (fits $\kappa(3 \times 10^{-3}) \approx 0.443$, $\kappa(5 \times 10^{-3}) \approx 0.255$; $\nu_{\text{eff}} \approx 0.81, 0.60$), and Figures 1–2.

Remark 3.1 ($|C|$ confound and the off-critical baseline (added in v2)). At fixed $|A|$, the reconstruction target C has $|C| = N - |A| - w$ sites, which varies both along a single sweep (as w grows) and *across the scaling analysis* (as N grows at comparable w). The regenerated benchmark quantifies the consequence: the off-critical baseline $d_{\text{eff}}(h_x = 0.80; N)$ also grows with N (e.g. $5.31 \to 5.39$ over $N = 9, 10$ at $\varepsilon = 3 \times 10^{-3}$; $4.66 \to 4.54$, i.e. roughly flat, at 5×10^{-3}), so part of the raw peak growth is tripartition geometry, not critical physics. The cleaner scaling object is the critical enhancement $\Delta(\varepsilon; N) := d_{\text{eff}}^{\text{max}} - d_{\text{eff}}(0.80; N)$, which still grows with N ($0.42 \to 0.74$ at 3×10^{-3}; $0.54 \to 0.88$ at 5×10^{-3}, $N = 9, 10$): the critical signal survives baseline subtraction, while $\kappa(\varepsilon)$ extracted from raw peaks must be read as geometry-contaminated. A fixed-$|C|$ protocol remains the natural sequel (cf. the same caveat in 2601.0038 v2).

Remark 3.2 (Functional-form indistinguishability (added in v2)). The fits of Table 2 use four sizes spanning $N = 9$ to 12 – less than half an octave. Over such a range, N^κ, $a + b \log N$ and linear growth are statistically indistinguishable: in the regenerated benchmark the R^2 spread across the three forms is below 0.01 at both thresholds. Moreover κ itself is strongly window-dependent (the $N \in \{8, 9, 10\}$ window yields $\kappa \approx 0.91, 0.51$ at the two thresholds, versus $0.44, 0.26$ for $N \in \{9, \ldots, 12\}$). Accordingly, "power-law growth $N^{\kappa(\varepsilon)}$" should be read as a descriptive summary of monotone growth with a fitted slope on a short window – consistent with v1's own framing of κ and ν_{eff} as effective operational exponents, now made quantitative.

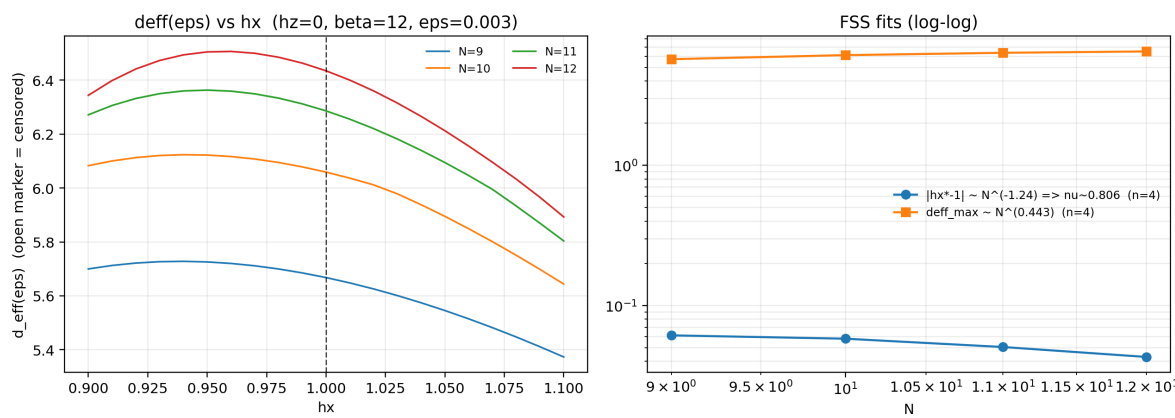

Figure 1: $\varepsilon = 3 \times 10^{-3}$, $\beta = 12$, $h_z = 0$ (v1 figure, unchanged). Left: d_{eff} vs h_x for $N \in \{9, \ldots, 12\}$. Right: log-log fits (read per Remarks 3.1 and 3.2).

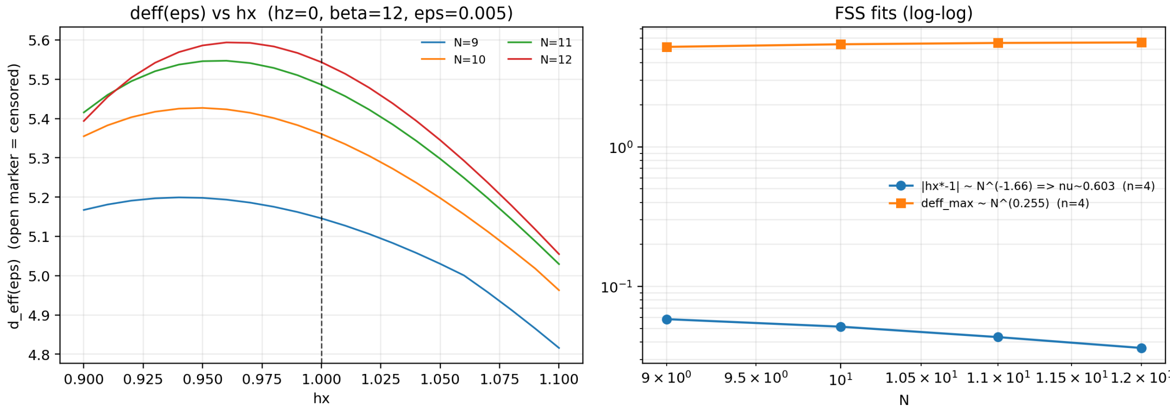

Figure 2: As above, $\varepsilon = 5 \times 10^{-3}$ (v1 figure, unchanged).

4 Verification suite (added in v2)

`verification/2601-0040/verify_2601_0040.py` (regenerable ED; chunked JSON cache; default $N \in \{8, 9, 10\}$): reproduces v1's Table 1 entries exactly where overlapping; computes the off-critical baseline and critical enhancement of Remark 3.1; performs the three-way functional-form comparison of Remark 3.2; verifies monotonicity of E_{best} and censoring-free status of the chosen thresholds.

5 Discussion

The peak height grows monotonically with N and the peak location drifts toward the critical region – both robust. What v2 sharpens is the reading: the growth exponent is geometry-contaminated (Remark 3.1) and form/window-dependent (Remark 3.2); the baseline-subtracted critical enhancement is the recommended object for future scaling studies, ideally under a fixed-$|C|$ protocol with larger sizes (tensor networks), and compared against CMI-based diagnostics as in 2601.0038 v2.

Note added in v2

No v1 number is changed; Tables 1–2 and both figures stand, and the suite reproduces Table 1 to all reported digits where sizes overlap. v2 adds the two caveats a referee would demand – the $|C|$/baseline confound (Remark 3.1, with the critical enhancement Δ shown to survive) and functional-form/window dependence of κ (Remark 3.2) – plus series positioning (v1 cited no companion papers) and the regenerable verification suite.

References

[1] D. Petz, Commun. Math. Phys. **105** (1986) 123–131.

[2] S. Sachdev, *Quantum Phase Transitions*, 2nd ed., Cambridge University Press (2011).

[3] L. Eriksson, ai.viXra:2601.0038 (2026; v2 2026).

[4] L. Eriksson, ai.viXra:2601.0035 (2026; v2 2026).

[5] L. Eriksson, ai.viXra:2512.0101 (2025; v2 2026).