Finite-size scaling of Petz recovery length in the TFIM

Threshold-dependent operational exponents from exact diagonalization

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January 2026

Abstract

We study finite-size scaling of an operational recovery length extracted from Petz-map recovery in the transverse-field Ising chain (TFIM). For a tripartition A–B–C of a 1D chain with a collar B of width w separating A from C, we define a recovery error $E_{\mathrm{Petz}}(w) = -\log F(\rho_{ABC}, \widetilde{\rho}^{\mathrm{Petz}}_{ABC}(w))$ (squared Uhlmann fidelity) and an effective recovery distance $d_{\mathrm{eff}}(\varepsilon)$ as the minimal collar width achieving $E_{\mathrm{best}}(w) \le \varepsilon$, where $E_{\mathrm{best}}(w) = \min_{w' \le w} E_{\mathrm{Petz}}(w')$. Using exact diagonalization at $h_z = 0$ and $\beta = 12$ for $N \in \{9, 10, 11, 12\}$, we analyze the peak height $d^{\max}_{\mathrm{eff}}(\varepsilon; N) = \max_{h_x} d_{\mathrm{eff}}(\varepsilon; N, h_x)$ in a censoring-free threshold regime. We find power-law growth $d^{\max}_{\mathrm{eff}}(\varepsilon; N) \sim N^{\kappa(\varepsilon)}$ with a threshold-dependent exponent, e.g. $\kappa(3 \times 10^{-3}) \approx 0.44$ and $\kappa(5 \times 10^{-3}) \approx 0.26$ over this size range. The peak location $h_x^{(N)}$ drifts toward the critical region as N increases; however, the drift exponent extracted from $|h_x^{(N)-1|\sim N^{-1/\nu_{\mathrm{eff}}}}$ is threshold-dependent in this finite-size, finite-temperature window, and we report it as an effective operational exponent rather than a universal estimate of the TFIM correlation-length exponent.

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1 Setup

We consider a 1D chain and a tripartition A–B–C, where B is a collar of width w separating A from C. For each w we form a Petz-based reconstructed state $\tilde{\rho}^{\mathrm{Petz}}_{ABC}(w)$ and define the recovery error

$$E_{\mathrm{Petz}}(w) := -\log F\big(\rho_{ABC}, \tilde{\rho}^{\mathrm{Petz}}_{ABC}(w)\big)\,, \qquad F(\rho,\sigma) := \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}\right)^2.$$

To stabilize non-monotonic behavior in finite-size numerics we use

$$E_{\mathrm{best}}(w) := \min_{w' \le w} E_{\mathrm{Petz}}(w').$$

Given a target threshold $\varepsilon > 0$, define the effective recovery distance

$$d_{\mathrm{eff}}(\varepsilon) := \min\{w : \; E_{\mathrm{best}}(w) \le \varepsilon\},$$

with log-linear interpolation between adjacent w values when needed.

Petz recovery map and numerical regularization. We construct $\tilde{\rho}^{\mathrm{Petz}}_{ABC}(w)$ using the Petz recovery map for the channel Tr_C [1]. Concretely, we apply the recovery map $\mathcal{R}^{\mathrm{Petz}}_{B\to BC}$ to the input marginal ρ_{AB}:

$$\tilde{\rho}^{\mathrm{Petz}}_{ABC}(w) = (\mathrm{id}_A \otimes \mathcal{R}^{\mathrm{Petz}}_{B\to BC})(\rho_{AB}), \qquad \mathcal{R}^{\mathrm{Petz}}_{B\to BC}(X) = \rho_{BC}^{1/2}\,\rho_B^{-1/2}\,X\,\rho_B^{-1/2}\,\rho_{BC}^{1/2},$$

where ρ_{BC} and ρ_B are marginals of the target state ρ_{ABC}. Inverse powers of ρ_B are regularized by an eigenvalue floor in the spectral decomposition, and the reconstructed operator is projected to the positive semidefinite cone and trace-renormalized prior to evaluating the fidelity.

Censoring and censoring-free thresholds. For a run with accessible window $w \le w_{\max}$, if $E_{\mathrm{best}}(w_{\max}) > \varepsilon$ then $d_{\mathrm{eff}}(\varepsilon)$ is censored and only a lower bound $d_{\mathrm{eff}}(\varepsilon) > w_{\max}$ is available. In this paper we focus on thresholds ε for which censoring is absent across the scanned h_x window for the studied N values, enabling direct finite-size fits.

2 Model and data set

We analyze the transverse-field Ising chain with open boundary conditions [2]:

$$H = -\sum_{i=1}^{N-1} Z_i Z_{i+1} - h_x \sum_{i=1}^{N} X_i - h_z \sum_{i=1}^{N} Z_i,$$

with $h_z = 0$. We consider Gibbs states $\rho \propto e^{-\beta H}$ at $\beta = 12$. We fix $|A| = 2$ at the left edge, vary the collar width w, and sweep $h_x \in [0.90, 1.10]$ with step 0.01. We analyze $N \in \{9, 10, 11, 12\}$.

3 Results

3.1 Peak extraction

For each N and threshold ε, we define the peak height and location

$$d^{\max}_{\mathrm{eff}}(\varepsilon; N) := \max_{h_x} d_{\mathrm{eff}}(\varepsilon; N, h_x), \qquad h_x^{(N):= \arg\max_{h_x} d_{\mathrm{eff}}(\varepsilon; N, h_x).}$$

To reduce discretization bias from the finite h_x grid, we estimate $h_x^{(N)}$ by a local quadratic (parabolic) refinement around the discrete maximizer.

ε	N	h_x	$d_{\text{eff}}^{\max}(\varepsilon)$	censored
3e-3	9	0.938730	5.727977	0
3e-3	10	0.942005	6.123411	0
3e-3	11	0.949397	6.362774	0
3e-3	12	0.957045	6.506568	0
5e-3	9	0.941619	5.199258	0
5e-3	10	0.948430	5.427081	0
5e-3	11	0.956581	5.547560	0
5e-3	12	0.963713	5.594329	0

Table 1: Peak locations $h_x^{(N)}$ and peak heights $d_{\text{eff}}^{\max}(\varepsilon; N)$ for censoring-free thresholds.

3.2 Finite-size scaling of the peak height

In the censoring-free regime, we fit the peak height to a power law

$$d_{\text{eff}}^{\max}(\varepsilon; N) \sim N^{\kappa(\varepsilon)},$$

and report the threshold-dependent operational exponent $\kappa(\varepsilon)$.

3.3 Pseudocritical drift as an effective operational exponent

We also quantify the drift of the peak location toward the critical region by fitting

$$|h_x^{(N)-1|\sim N^{-1/\nu_{\text{eff}}}}.$$

Because ν_{eff} varies significantly with the threshold ε within the accessible (N, β) window, we interpret it as a threshold-dependent operational drift exponent rather than as a universal estimate of the TFIM thermodynamic exponent.

ε	κ	R^2_κ	b (shift)	ν_{eff}	R^2_ν
3e-03	0.442622	0.9640	-1.240740	0.805973	0.9402
5e-03	0.255161	0.9301	-1.659450	0.602610	0.9849

Table 2: Finite-size scaling fits in the censoring-free regime. We report $\kappa(\varepsilon)$ from $d_{\text{eff}}^{\max}(\varepsilon; N) \sim N^{\kappa(\varepsilon)}$ and an effective drift exponent ν_{eff} from $|h_x^{(N)-1|\sim N^{-1/\nu_{\text{eff}}}}$.

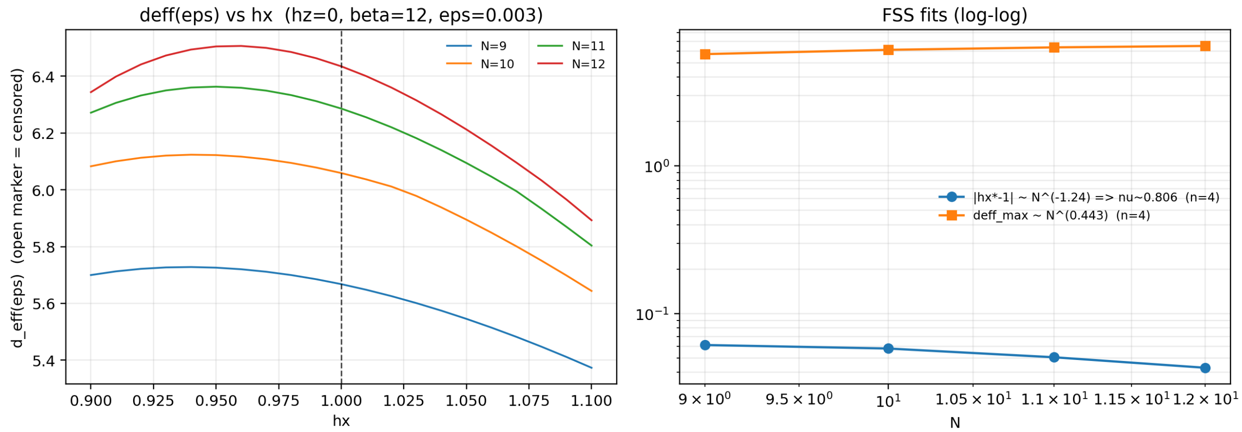

Figure 1: Censoring-free threshold $\varepsilon = 3 \times 10^{-3}$ at $\beta = 12$ and $h_z = 0$. Left: $d_{\text{eff}}(\varepsilon)$ versus h_x for $N \in \{9, 10, 11, 12\}$. Right: log-log fits for peak height and peak-location drift.

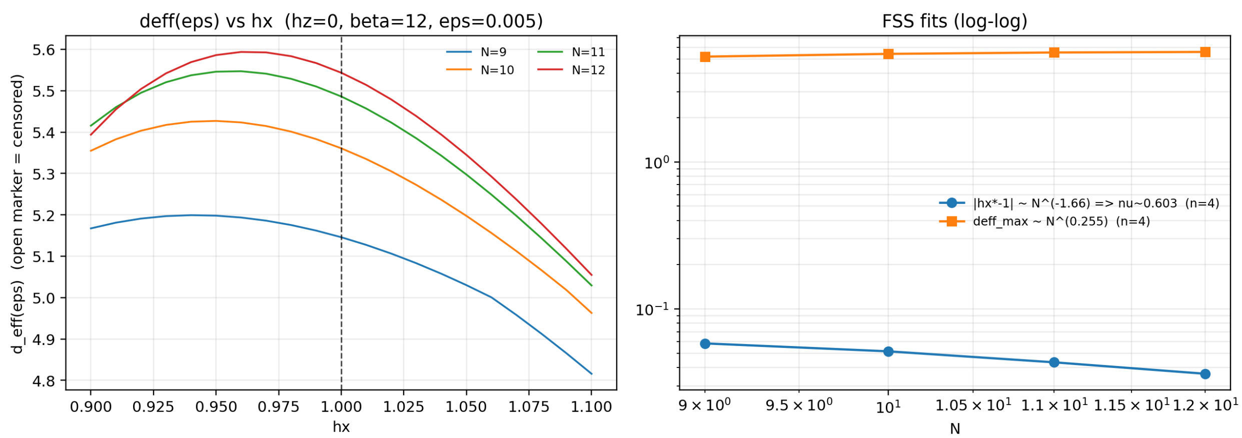

Figure 2: Censoring-free threshold $\varepsilon = 5 \times 10^{-3}$ at $\beta = 12$ and $h_z = 0$. Left: $d_{\text{eff}}(\varepsilon)$ versus h_x for $N \in \{9, 10, 11, 12\}$. Right: log-log fits for peak height and peak-location drift.

4 Discussion

Across the studied finite-size, finite-temperature window, the peak height $d_{\text{eff}}^{\max}(\varepsilon; N)$ exhibits clear power-law growth with N, defining a threshold-dependent operational exponent $\kappa(\varepsilon)$. Tightening the recovery threshold increases $\kappa(\varepsilon)$, consistent with the intuition that stricter recovery requirements probe longer operational length scales.

The peak location $h_x^{(N)}$ drifts toward the TFIM critical region as N increases, but the corresponding drift exponent ν_{eff} depends on the threshold ε even in the censoring-free regime. This suggests that $h_x^{(N)}$ defined via $\arg\max_{h_x} d_{\text{eff}}(\varepsilon)$ tracks an operational crossover whose scaling need not coincide with the thermodynamic correlation-length exponent within the accessible system sizes and collar geometry.

Future work includes extending the size range, refining h_x resolution, and comparing operational scaling to diagnostics based on conditional mutual information and correlation functions.

References

[1] Dénes Petz. Sufficient subalgebras and the relative entropy of states of a von neumann algebra. *Communications in Mathematical Physics*, 105(1):123–131, 1986.

[2] Subir Sachdev. *Quantum Phase Transitions.* Cambridge University Press, 2 edition, 2011.