Modular Recovery from Split Inclusions

A B-minimal bridge from QFT collar geometry to approximate state reconstruction

Version 2

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Abstract

In algebraic quantum field theory (AQFT), local algebras are typically Type III factors, so density matrices and von Neumann entropies are unavailable for bounded regions. We formulate a B-minimal continuum analog of the lattice "collar $\Rightarrow$ Markovness $\Rightarrow$ recovery" mechanism by combining: (i) the split property as the mathematical replacement of a buffer (collar), (ii) Araki relative entropy to define a split-regularized conditional mutual information $I_\omega^{\mathcal{N}}(A:C|B)$ relative to fixed Type I interpolating data $\mathcal{N}$, and (iii) modular/twirled Petz recovery as an explicit candidate recovery channel. Assuming an FR-type recoverability inequality in the fixed-split setting, we obtain quantitative recovery bounds in a fidelity-based error metric (purified distance). We conclude with a conditional holographic remark. *v2 corrects the fidelity-convention factor in the assumed FR-type inequality:* with the squared (Bures/Uhlmann-squared) convention used throughout, the importable finite-dimensional motivation gives $-\log F_B \leq I$, not $-\log F_B \leq \frac{1}{2}I$; v1's half-form is strictly stronger than its motivation and is refuted as a finite-dimensional anchor on 40/40 random tripartite states (the corrected form holds on 40/40) – the same factor-2 correction applied in 2512.0101 v2 and 2601.0020 v2. The recovery theorem's constant changes by $\sqrt{2}$; the exponent is unaffected. v2 further adds series positioning, a compatibility remark with the split-regularized CMI of 2601.0020, a concrete demonstration that $I^{\mathcal{N}}$ can be negative for unfavorable split data – nonnegativity of $I^{\mathcal{N}}$ is part of the good-split-data regime, not automatic – and a finite-dimensional verification suite for every checkable ingredient of the dictionary.

1 Introduction

In finite-dimensional quantum systems, small conditional mutual information (CMI) implies approximate recoverability [1]. In continuum QFT, local algebras are typically Type III [2, 3]: no density matrices, no finite von Neumann entropies for bounded regions. This note provides a minimal framework importing the CMI$\Rightarrow$recovery mechanism into AQFT, with the dictionary: collar/buffer $\rightsquigarrow$ split property; von Neumann entropy $\rightsquigarrow$ Araki relative entropy; Petz/twirled Petz $\rightsquigarrow$ modular (twirled) Petz channel. The CMI here is split-regularized relative to fixed Type I interpolating data $\mathcal{N}$, and the FR-type bound is an *assumption*: the contribution is a clean continuum dictionary separating geometric input (control of $I_\omega^{\mathcal{N}}$) from information-theoretic input (recoverability), not a universal Type III recovery theorem.

Series positioning (added in v2). This note is the Type III capstone of the clustering–recovery program: it is the target of the conjectural sections of 2512.0060 (Conj. 5.1) [11], 2512.0101 (Conj. 1, CMI route, whose v2 fixes the same factor corrected here) [12], 2601.0007 (Conj. 8.1 and its Type III roadmap via noncommutative L^p) [13], and 2601.0031 (Conj. 6.4) [15]. The companion 2601.0020 [14] defines a split-regularized CMI by a complementary route; see Remark 3.4.

2 Preliminaries

We assume a local net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ with isotony and locality [2, 4]; for bounded regions the algebras are typically Type III$_1$ [3], so "discarding C" is restriction $\omega_{AB} := \omega_{ABC}|_{\mathcal{A}_{AB}}$. An inclusion $\mathcal{M}_1 \subset \mathcal{M}_2$ is *split* if there is a Type I factor $\mathcal{N}$ with $\mathcal{M}_1 \subset \mathcal{N} \subset \mathcal{M}_2$ [7, 8]; the interpolating factor is not unique, and we fix one choice wherever split is invoked (superscript $\mathcal{N}$). Araki relative entropy $S(\omega\|\varphi)$ is defined via relative modular operators [5, 6]; $F_B(\omega, \varphi) \in [0, 1]$ denotes Bures fidelity in the *squared* convention (in Type I it reduces to Uhlmann fidelity squared, $\|\sqrt{\rho}\sqrt{\sigma}\|_1^2$), and the purified distance is $P(\omega, \varphi) := \sqrt{1 - F_B(\omega, \varphi)}$.

3 Split-regularized conditional mutual information

Fix a tripartition A–B–C with collar B and a normal faithful $\omega \in S(\mathcal{A}_{ABC})$.

Assumption 3.1 (Fixed split data $\mathcal{N}$)**.** *The collar geometry supports split inclusions inducing fixed identifications $\mathcal{A}_{AB} \cong \mathcal{A}_A \bar{\otimes} \mathcal{A}_B$ and $\mathcal{A}_{ABC} \cong \mathcal{A}_A \bar{\otimes} \mathcal{A}_{BC}$; the totality of fixed choices is denoted $\mathcal{N}$.*

Definition 3.2 (Split-regularized CMI)**.** *$I_\omega^{\mathcal{N}}(A : B) := S(\omega_{AB}\|\omega_A \otimes \omega_B)$, $I_\omega^{\mathcal{N}}(A : BC) := S(\omega_{ABC}\|\omega_A \otimes \omega_{BC})$, and*

$$I_\omega^{\mathcal{N}}(A : C|B) := I_\omega^{\mathcal{N}}(A : BC) - I_\omega^{\mathcal{N}}(A : B).$$

Remark 3.3 (Dependence on $\mathcal{N}$ and (non-)negativity; sharpened in v2)*.* We do not claim canonicity in $\mathcal{N}$, nor $I_\omega^{\mathcal{N}}(A : C|B) \geq 0$ for arbitrary fixed data. v2 makes this concrete: in the finite-dimensional dictionary check (Section 8), taking a state with A nearly uncorrelated from BC and rotating only the $A|B$-layer identification by a generic unitary yields $I^{\mathcal{N}} < 0$ on 20/20 sampled identifications (range $[-0.46, -0.18]$ against $I(A : BC) = 0.05$). Nonnegativity is thus a genuine restriction on good split data, not a formality.

Remark 3.4 (Compatibility with 2601.0020 (added in v2))*.* 2601.0020 [14] defines a split-regularized CMI by embedding $\iota_w : \mathcal{A}_{ABC}^{(w)} \hookrightarrow \mathcal{A}(ABC)$ and computing the Type I CMI of the induced state $\omega \circ \iota_w$; here we fix split identifications and take a difference of Araki relative entropies. In Type I with the natural factorization both reduce to the standard CMI (chain rule; verified to machine precision in the suite), so they are complementary regularizations of the same target; away from the natural choice they differ, and each carries its own scheme dependence.

4 Recovery channels and the modular/twirled Petz candidate

Channels are normal CP unital maps in Heisenberg picture, acting on states via the predual. A recovery channel on the collar is $\mathcal{R}_{B\to BC,*} : S(\mathcal{A}_B) \to S(\mathcal{A}_{BC})$, and the recovered state is $\tilde{\omega}_{ABC} := (\mathrm{id}_A \otimes \mathcal{R}_{B\to BC,*})(\omega_{AB})$ under the fixed identifications. The twirled modular Petz channel is, schematically, $\mathcal{R}^{\mathrm{tw}}_{B\to BC} = \int dt\, \beta_0(t)\, \sigma_t^{\varphi_B} \circ \mathcal{R}^{\mathrm{mod}}_{B\to BC} \circ \sigma_{-t}^{\varphi_{BC}}$ with $\beta_0(t) = \frac{\pi}{4}\,\mathrm{sech}^2(\pi t/2)$ (the universal weight of [10]; it integrates to 1 – verified in the suite); the rigorous construction uses standard form and Connes cocycles [10, 9].

Assumption 4.1 (FR-type recoverability in the fixed-split setting; factor corrected in v2)**.** *In the setting of Assumption 3.1, assume $I^{\mathcal{N}}_\omega(A:C|B)$ is finite and nonnegative and that there exists a recovery channel $\mathcal{R}_{B\to BC,*}$ (e.g. the twirled modular Petz channel) such that*

$$-\log F_B\big(\omega_{ABC}, \tilde{\omega}_{ABC}\big) \;\le\; I^{\mathcal{N}}_\omega(A:C|B).$$

Remark 4.2 (Correction note)*.* v1 assumed $-\log F_B \le \frac{1}{2} I^{\mathcal{N}}$. With F_B in the squared convention (Definition of Section 2), the importable finite-dimensional motivation – universal recovery [1, 10] – gives $I \ge -\log F$; the $\frac{1}{2}$ belongs to the root-fidelity form $I \ge -2\log f$. The v1 half-form is *strictly stronger* than its motivation: on random 3-qubit states with Petz recovery the suite finds $-\log F \le I$ on 40/40 draws but $-\log F \le \frac{1}{2} I$ on 0/40. This is the same factor-2 correction applied in 2512.0101 v2 and 2601.0020 v2; stating the assumption in the importable form keeps the conjectural target honest.

Theorem 4.3 (From split-CMI to recovery error (conditional; constant updated in v2))**.** *Assume Assumptions 3.1 and 4.1. If for collar width w one has $I^{\mathcal{N}}_\omega(A:C|B) \le Ke^{-\alpha w}$, then there exists a recovery channel with*

$$P\big(\omega_{ABC}, \tilde{\omega}_{ABC}\big) \;\le\; \sqrt{K}\, e^{-\alpha w/2}.$$

Proof. $1 - F_B \le -\log F_B \le I^{\mathcal{N}} \le Ke^{-\alpha w}$; take square roots. (v1's constant $\sqrt{K/2}$ corresponded to the stronger half-form assumption.) ☐

5 What is proved vs. what is assumed

Defined: the split-dependent quantity $I^{\mathcal{N}}_\omega(A:C|B)$ via Araki relative entropy under fixed split data. *Assumed:* the FR-type inequality (Assumption 4.1), now in its importable form. *Concluded:* collar-width decay of $I^{\mathcal{N}}$ implies decay of the purified-distance recovery error. The split choice $\mathcal{N}$ is part of the regularization scheme.

6 Physical hooks

Split is expected under nuclearity [8]; Reeh–Schlieder provides cyclic/separating vectors for modular theory. In massive QFTs with exponential clustering it is natural to expect $I^{\mathcal{N}}_\omega(A:C|B) \le Ke^{-\alpha w}$ for fixed-size A and appropriate geometries – this is precisely the continuum target of the lattice program: Gaussian/lattice evidence in 2512.0060 [11] and 2601.0007 [13], non-Gaussian CMI route in 2512.0101 [12], Gibbs-state CMI inputs collected in 2601.0020 [14].

7 Conditional holographic remark

As in v1: under a bulk–boundary interface in which subregion duality is an approximate recovery statement for code subalgebras, a boundary collar B with small $I^{\mathcal{N}}_\omega(A:C|B)$ provides a sufficient condition for approximate recovery from the collar, hence conditionally for quantitative entanglement wedge reconstruction.

8 Verification suite (added in v2)

`verification/2601-0034/verify_2601_0034.py` checks every finite-dimensional ingredient of the dictionary: (1) with natural Type I data, $I^{\mathcal{N}}$ = standard CMI (chain rule, machine precision); (2) the $\mathcal{N}$-dependence demo of Remark 3.3 ($I^{\mathcal{N}} < 0$ on 20/20 rotated identifications for an A-uncorrelated state); (3) $\int \beta_0 = 1$; (4) the finite-dimensional anchor of Assumption 4.1: corrected form 40/40, v1 half-form 0/40 (Remark 4.2); (5) the arithmetic of Theorem 4.3 with the corrected constant.

9 Discussion

This B-minimal note isolates a continuum version of the collar⇒recovery mechanism in AQFT, with explicit split dependence and a clean separation of geometric and information-theoretic inputs – now with the assumed input stated in the form its finite-dimensional motivation can actually support. Nonnegativity of $I^{\mathcal{N}}$ is part of the good-split-data regime, not automatic (Remark 3.3). Open problems: proving Assumption 4.1 as a theorem in the fixed-split setting with explicit constants (the Haagerup L^p route sketched in 2601.0007's Type III roadmap), and understanding stability with respect to the split choice $\mathcal{N}$.

Note added in v2

One correction and three additions, all verified where checkable. (1) *FR factor:* Assumption 4.1 restated in the importable squared-fidelity form $-\log F_B \le I^{\mathcal{N}}$ (v1's half-form is refuted as a finite-dimensional anchor on 40/40 draws); Theorem 4.3's constant becomes $\sqrt{K}$. Third occurrence of this factor in the series (2512.0101 v2, 2601.0020 v2). (2) *Compatibility:* Remark 3.4 relates the present $I^{\mathcal{N}}$ to the split-regularized CMI of 2601.0020 (both reduce to standard CMI for natural Type I data; verified). (3) *$\mathcal{N}$-dependence made concrete:* Remark 3.3 now carries an explicit negative-$I^{\mathcal{N}}$ demonstration. (4) Series positioning and verification suite added; v1 cited no companion papers.

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