Modular Recovery from Split Inclusions

A B-minimal bridge from QFT collar geometry to approximate state reconstruction

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Abstract

In algebraic quantum field theory (AQFT), local algebras are typically Type III factors, so density matrices and von Neumann entropies are unavailable for bounded regions. We formulate a B-minimal continuum analog of the lattice "collar ⇒ Markovness ⇒ recovery" mechanism by combining: (i) the split property as the mathematical replacement of a buffer (collar), (ii) Araki relative entropy to define a split-regularized conditional mutual information $I_\omega^{\mathcal{N}}(A:C|B)$ relative to fixed Type I interpolating data $\mathcal{N}$, and (iii) modular/twirled Petz recovery as an explicit candidate recovery channel. Assuming an FR-type recoverability inequality in the fixed-split setting, we obtain quantitative recovery bounds in a fidelity-based error metric (purified distance). We conclude with a conditional holographic remark in the spirit of the quantum error-correction interpretation of subregion duality.

Contents

1 Introduction

In finite-dimensional quantum systems, small conditional mutual information (CMI) implies approximate recoverability: there exists a channel acting only on the conditioning system that reconstructs the global state from an appropriate marginal [1]. In continuum QFT, however, local algebras are typically Type III, so there are no density matrices and no finite von Neumann entropies for bounded regions [2, 3]. This paper provides a minimal (but technically careful) framework for importing the CMI⇒recovery mechanism into AQFT. We define channels in Heisenberg picture and apply them to states via the predual (Schrödinger picture).

Our guiding dictionary is:

- Collar/buffer geometry $\rightsquigarrow$ split property (Type I interpolation).
- von Neumann entropy $\rightsquigarrow$ Araki relative entropy.
- Petz / twirled Petz $\rightsquigarrow$ modular (twirled) Petz channel defined via modular flow.

We emphasize from the outset that our conditional mutual information is a *split-regularized* quantity relative to a fixed choice of Type I interpolating data $\mathcal{N}$, and that the FR-type recoverability bound is stated as an assumption in this B-minimal note. The main contribution is a clean continuum dictionary and a logically separated "geometry/input" (control of $I_\omega^{\mathcal{N}}(A:C|B)$) versus "information-theory/input" (recoverability) framework, rather than a new proof of a universal Type III recovery theorem.

2 Preliminaries: nets, split property, and Type III information measures

2.1 AQFT nets and locality

We assume a local net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ satisfying isotony and locality (see, e.g., [2, 4]).

For a region label X (e.g. A, B, C, AB, BC, ABC), write $\mathcal{A}_X := \mathcal{A}(X)$.

Remark 2.1 (Type III obstruction). For many QFTs, $\mathcal{A}(\mathcal{O})$ is a Type III$_1$ factor for bounded $\mathcal{O}$. In particular, there is no normal trace and no density matrix representation for $\omega|_{\mathcal{A}(\mathcal{O})}$ in the algebra itself [2, 3].

2.2 States and restriction

Let $\mathcal{S}(\mathcal{M})$ denote the set of normal states on a von Neumann algebra $\mathcal{M}$.

Remark 2.2 (Discarding a region). By isotony, $\mathcal{A}_{AB} \subset \mathcal{A}_{ABC}$. Thus the continuum analog of "discarding C" is restriction: if $\omega_{ABC} \in \mathcal{S}(\mathcal{A}_{ABC})$, then $\omega_{AB} := \omega_{ABC}|_{\mathcal{A}_{AB}} \in \mathcal{S}(\mathcal{A}_{AB})$.

2.3 Split property (continuum collar)

Definition 2.3 (Split inclusion)**.** An inclusion $\mathcal{M}_1 \subset \mathcal{M}_2$ of von Neumann algebras is *split* if there exists a Type I factor $\mathcal{N}$ such that

$$\mathcal{M}_1 \subset \mathcal{N} \subset \mathcal{M}_2.$$

In AQFT, split is expected (and often proved) for nested regions with a nonzero "gap"; it is closely tied to nuclearity [7, 8].

Remark 2.4 (Fixed split data). The interpolating Type I factor is not unique. In this paper we *fix* one choice wherever split is invoked, and we record this dependence in the notation (superscript $\mathcal{N}$).

2.4 Araki relative entropy, Bures fidelity, and purified distance

Definition 2.5 (Araki relative entropy)**.** Let $\omega, \varphi \in \mathcal{S}(\mathcal{M})$ be normal faithful states on a von Neumann algebra $\mathcal{M}$. The Araki relative entropy $S(\omega\|\varphi) \in [0, \infty]$ is the standard extension of Umegaki relative entropy to von Neumann algebras, defined via relative modular operators in standard form [5, 6]. We use natural logarithms throughout.

Definition 2.6 (Bures fidelity)**.** Let $\omega, \varphi \in \mathcal{S}(\mathcal{M})$ be normal states. We write $F_{\mathrm{B}}(\omega, \varphi) \in [0, 1]$ for the Bures fidelity (squared transition probability) between ω and φ. In Type I this reduces to Uhlmann fidelity squared.

Definition 2.7 (Purified distance)**.** We define the purified distance

$$P(\omega, \varphi) := \sqrt{1 - F_{\mathrm{B}}(\omega, \varphi)}.$$

This is a standard fidelity-based error measure; different Bures/fidelity conventions are related by fixed monotone transforms.

3 Split-regularized conditional mutual information

3.1 Collar geometry A–B–C

Fix a tripartition A–B–C with B a collar separating A from C. Let $\omega \in \mathcal{S}(\mathcal{A}_{ABC})$ be normal and faithful.

3.2 Fixed split identifications

Assumption 3.1 (Fixed split data $\mathcal{N}$)**.** *Assume that the collar geometry supports split inclusions sufficient to induce identifications*

$$\mathcal{A}_{AB} \cong \mathcal{A}_A \bar{\otimes} \mathcal{A}_B, \qquad \mathcal{A}_{ABC} \cong \mathcal{A}_A \bar{\otimes} \mathcal{A}_{BC},$$

with $\bar{\otimes}$ the von Neumann tensor product, and that these identifications are fixed once and for all. We denote the totality of fixed split choices by $\mathcal{N}$.

3.3 Definition of $I_\omega^{\mathcal{N}}(A : C|B)$

Definition 3.2 (Split mutual informations)**.** Under Theorem 3.1, use the fixed identifications to form the product states $\omega_A \otimes \omega_B$ on $\mathcal{A}_{AB}$ and $\omega_A \otimes \omega_{BC}$ on $\mathcal{A}_{ABC}$. Define

$$I_\omega^{\mathcal{N}}(A : B) := S(\omega_{AB} \,\|\, \omega_A \otimes \omega_B)\,, \qquad I_\omega^{\mathcal{N}}(A : BC) := S(\omega_{ABC} \,\|\, \omega_A \otimes \omega_{BC})\,.$$

Definition 3.3 (Split-regularized conditional mutual information)**.** Define

$$I^{\mathcal{N}}_{\omega}(A:C|B) := I^{\mathcal{N}}_{\omega}(A:BC) - I^{\mathcal{N}}_{\omega}(A:B).$$

Remark 3.4 (Dependence on $\mathcal{N}$ and (non-)negativity). We do not claim canonicity of $I^{\mathcal{N}}_{\omega}(A:C|B)$ with respect to the split choice. The purpose of the explicit superscript $\mathcal{N}$ is to make this dependence transparent. In contrast to the finite-dimensional CMI, we do not claim $I^{\mathcal{N}}_{\omega}(A:C|B) \geq 0$ for arbitrary fixed split data $\mathcal{N}$; in applications one may restrict to regimes/geometries where it is small and nonnegative.

4 Recovery channels and the modular/twirled Petz candidate

4.1 Channels: Heisenberg vs Schrödinger

Definition 4.1 (Quantum channel)**.** A (Heisenberg-picture) channel from $\mathcal{M}$ to $\mathcal{N}$ is a normal completely positive unital map $\mathcal{T}:\mathcal{M}\to\mathcal{N}$. Its Schrödinger-picture action on normal states is the predual map

$$\mathcal{T}_*:\mathcal{S}(\mathcal{N})\to\mathcal{S}(\mathcal{M}), \qquad (\mathcal{T}_*\omega)(x) := \omega(\mathcal{T}(x)).$$

4.2 Recovery by acting only on B

Definition 4.2 (Recovery channel on the collar)**.** We label the recovery channel by its Schrödinger-picture action. Let

$$\mathcal{R}_{B\to BC,*}:\ \mathcal{S}(\mathcal{A}_B)\to\mathcal{S}(\mathcal{A}_{BC})$$

be a normal completely positive map mapping normal states to normal states, and let its Heisenberg adjoint be

$$\mathcal{R}_{B\to BC}:\ \mathcal{A}_{BC}\to\mathcal{A}_B,$$

which is normal completely positive and unital. Given $\omega_{AB}\in\mathcal{S}(\mathcal{A}_{AB})$, define the recovered state

$$\widetilde{\omega}_{ABC} := (\mathrm{id}_A\otimes\mathcal{R}_{B\to BC,*})(\omega_{AB}),$$

where $\mathrm{id}_A\otimes\mathcal{R}_{B\to BC,*}$ is defined as the tensor-product extension under the fixed split identifications in Theorem 3.1.

4.3 Twirled modular Petz (conceptual)

In finite dimensions, Petz recovery and its twirled/averaged variants can be written using $\rho^{\pm 1/2}$ and modular rotations $\rho^{it}(\cdot)\rho^{-it}$. In Type III algebras, the correct replacement uses Tomita–Takesaki modular theory: for a faithful normal state φ on $\mathcal{M}$, one has a modular automorphism group σ^{φ}_t on $\mathcal{M}$.

Remark 4.3 (Schematic twirled modular Petz formula). Fix a reference state φ on $\mathcal{A}_{BC}$ (and its induced state on $\mathcal{A}_B$). The twirled modular Petz channel can be written schematically in Heisenberg picture as

$$\mathcal{R}^{\mathrm{tw}}_{B\to BC} \;=\; \int_{-\infty}^{\infty} dt\ \beta(t)\ \big(\sigma^{\varphi_B}_t\big)\circ\mathcal{R}^{\mathrm{mod}}_{B\to BC}\circ\big(\sigma^{\varphi_{BC}}_{-t}\big),$$

where $\mathcal{R}^{\mathrm{mod}}_{B\to BC}$ denotes the modular Petz map, $\sigma^{\varphi_{BC}}_t$ acts on the input algebra $\mathcal{A}_{BC}$, $\sigma^{\varphi_B}_t$ acts on the output algebra $\mathcal{A}_B$, and $\beta(t)$ is a fixed even weight (often chosen proportional to $\mathrm{sech}^2\big(\frac{\pi t}{2}\big)$ in strengthened data-processing results). We use this only as intuition; the rigorous construction uses standard form / Connes cocycles [10, 9].

4.4 Main recoverability input (assumed for B-minimal)

Assumption 4.4 (FR-type recoverability in the fixed-split setting). *In the setting of Theorem 3.1, assume that* $I_\omega^{\mathcal{N}}(A:C|B)$ *is finite and nonnegative and that there exists a recovery channel* $\mathcal{R}_{B\to BC,*}$ *(e.g. the twirled modular Petz channel built from the fixed split data and a suitable reference state) such that*

$$-\log F_{\mathrm{B}}(\omega_{ABC},\ \widetilde{\omega}_{ABC}) \le \tfrac{1}{2}\, I_\omega^{\mathcal{N}}(A:C|B).$$

Theorem 4.5 (From split-CMI to recovery error (conditional on an FR-type inequality)). *Assume Theorems 3.1 and 4.4. If for collar width* w *one has*

$$I_\omega^{\mathcal{N}}(A:C|B) \le K e^{-\alpha w},$$

then there exists a recovery channel $\mathcal{R}_{B\to BC,*}$ *such that*

$$P(\omega_{ABC},\ \widetilde{\omega}_{ABC}) \le \sqrt{\tfrac{1}{2}\,K}\, e^{-\alpha w/2}.$$

Proof sketch. From Theorem 4.4 and $1-x \le -\log x$ for $x \in (0,1]$,

$$1 - F_{\mathrm{B}}(\omega_{ABC},\widetilde{\omega}_{ABC}) \le -\log F_{\mathrm{B}}(\omega_{ABC},\widetilde{\omega}_{ABC}) \le \tfrac{1}{2} K e^{-\alpha w}.$$

Taking square roots and using $P(\omega,\varphi) = \sqrt{1 - F_{\mathrm{B}}(\omega,\varphi)}$ yields the claim. □

5 What is proved vs. what is assumed

We stress the logical structure of the note:

- **Defined:** a split-dependent, collar-adapted quantity $I_\omega^{\mathcal{N}}(A:C|B)$ using Araki relative entropy under fixed split data.
- **Assumed (imported):** an FR-type recoverability inequality in the fixed-split setting (Theorem 4.4), motivated by universal recovery map technology.
- **Concluded:** decay of $I_\omega^{\mathcal{N}}(A:C|B)$ in the collar width implies decay of a fidelity-based recovery error (purified distance).

We view the split choice $\mathcal{N}$ as part of the regularization scheme and do not claim canonicity.

6 Physical hooks (brief)

The split property is expected (and often proved) under nuclearity-type conditions [8]. For vacuum states, Reeh–Schlieder provides the cyclic/separating vectors needed for modular theory on local algebras. In massive QFTs with exponential clustering, it is natural to expect collar-width decay $I_\omega^{\mathcal{N}}(A:C|B) \le K e^{-\alpha w}$ for fixed-size A and appropriate geometries, which then implies exponential recovery by Theorem 4.5.

7 Conditional holographic remark

Assumption 7.1 (Bulk–boundary QEC/modular interface)**.** *Assume a holographic bulk–boundary interface in which subregion duality can be phrased as an approximate recovery statement for appropriate (code) subalgebras, with boundary recovery error controlling bulk reconstruction error.*

Under such an interface, a boundary collar B yielding small $I_\omega^{\mathcal{N}}(A:C|B)$ provides a sufficient condition for approximate recovery from the boundary collar, hence (conditionally) for quantitative entanglement wedge reconstruction in the corresponding regime.

8 Discussion

This B-minimal note isolates a continuum version of the collar⇒recovery mechanism in AQFT. The key technical points are: (i) making explicit the split dependence ($\mathcal{N}$) of the regularized information measure, and (ii) separating geometric/QFT input (establishing decay of $I_\omega^{\mathcal{N}}(A:C|B)$) from information-theoretic input (Theorem 4.4).

Open problems include proving Theorem 4.4 as a theorem in the fixed-split setting with explicit constants, and understanding stability with respect to different split choices $\mathcal{N}$.

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