From Static Recoverability to Maintenance Power:
A Typed Pipeline with $\omega = 0$ Obstructions

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Abstract

We study when geometric separation in gapped quantum systems yields a genuine reduction in thermodynamic resources required to maintain coherence against uncontrolled open-system dynamics. Our analysis separates three layers. **First**, in a regularized Gaussian split regime motivated by algebraic QFT, we state an explicit static reconstruction bound: collar suppression of vacuum cross-correlations enables approximate state recovery via the Petz map with fidelity error controlled by a cross-block recovery norm. **Second**, we show why static recoverability does not automatically imply suppression of dynamical decay rates: fixed-point structure and Bohr-zero ($\omega = 0$) channels can generate obstructions invisible to static clustering alone. We formalize this obstruction using an exact $\omega = 0$ Dirichlet identity and implement finite-size commutator-witness diagnostics in the transverse-field Ising chain (one-site and two-site Pauli test families), finding no evidence of a size-independent $\omega = 0$ floor in that benchmark regime for tested sizes (with an extended one-site check at $N = 14$ consistent with the observed trend). **Third**, we give an autocontained finite-dimensional core linking coherence loss to incremental maintenance power under an explicit battery-assisted thermal-operations model (with paired strategies), and we state a typed rate-inheritance hypothesis identifying precisely what additional dynamical input is required to propagate collar suppression into power suppression on an operational target family. We conclude with a Type III blueprint for extending the pipeline to AQFT.

1 Introduction

Geometric separation in gapped many-body systems suppresses static correlations, motivating the expectation that thick buffers (collars) reduce cross-region influence. A distinct operational question is dynamical: even if a global state is recoverable from local data, what resources are required to keep a target state stable under uncontrolled open-system dynamics? In particular, does increasing a collar width r reduce the minimal power required to maintain coherence against noise and thermalization?

This paper isolates a frequent logical pitfall: static locality controls an *amplitude* (e.g. reconstruction error), whereas maintenance cost is governed by *rates* (how fast coherence is lost and must be replenished). Conflating these types leads to invalid inferences of the form "small static residue implies small maintenance cost." Our contribution is a typed pipeline separating static recoverability from dynamic rate control, identifying the unique missing hinge (rate inheritance on a family), and clarifying the role of $\omega = 0$/fixed-point obstructions.

2 Setup: conditional expectations, coherence functionals, and operational families

Let $\mathcal{H}$ be finite-dimensional and $\mathcal{A} = \mathcal{B}(\mathcal{H})$. Let H_S be the system Hamiltonian with spectral projectors $\{\Pi_n\}$. Fix $\beta := 1/(k_B T)$ and the thermal reference state

$$\sigma := \frac{e^{-\beta H_S}}{\mathrm{Tr}(e^{-\beta H_S})}. \tag{1}$$

Let $\mathcal{T}_t = e^{t\mathcal{L}_*}$ be the uncontrolled Markov semigroup (Schrödinger picture) with GKLS generator $\mathcal{L}_*$ satisfying $\mathcal{L}_*[\sigma] = 0$.

2.1 A single coherence functional with two instances

Definition 2.1 (Coherence relative to a conditional expectation). Let $E : \mathcal{A} \to \mathcal{B} \subset \mathcal{A}$ be a σ-preserving conditional expectation and E_* its predual acting on states. Define

$$C_E(\rho) := S(\rho \,\|\, E_*[\rho]). \tag{2}$$

We use two canonical instances.

Instance 1 (law-grade): energy pinching. Define

$$\Delta[A] := \sum_n \Pi_n A \Pi_n. \tag{3}$$

Then Δ is σ-preserving and $C_\Delta(\rho)$ quantifies coherence in the energy basis.

Instance 2 (static, geometric): split expectation (regularized posture). In a split configuration $O_1 \Subset O_2$ with collar width r, let E_{split} denote a vacuum-preserving conditional expectation onto a Type I split factor. In the Gaussian realization (Section 3), this yields explicit Petz recovery at covariance level.

2.2 Loss rate and typed envelopes (sign convention)

Definition 2.2 (Loss rate and local rate). For $\rho_t := \mathcal{T}_t(\rho)$ define

$$\dot{C}_{E,\mathrm{loss}}(\rho) := -\frac{d}{dt}\Big|_{t=0} C_E(\rho_t). \tag{4}$$

For $C_E(\rho) > 0$ define

$$r_E(\rho) := \frac{\dot{C}_{E,\mathrm{loss}}(\rho)}{C_E(\rho)}. \tag{5}$$

Definition 2.3 (Typed envelopes on a family). Given a family $\mathcal{F}_r$, define

$$r_E^\downarrow(r) := \inf_{\rho \in \mathcal{F}_r : C_E(\rho) > 0} r_E(\rho), \qquad r_E^\uparrow(r) := \sup_{\rho \in \mathcal{F}_r : C_E(\rho) > 0} r_E(\rho). \tag{6}$$

The lower envelope $r_E^\downarrow$ governs necessary lower bounds (e.g. power floors), while $r_E^\uparrow$ governs sufficient upper envelopes.

2.3 Collar geometry and local preparation

We consider a tripartition A–B–C with collar width $r = \text{dist}(A, C)$ (buffer region B separating A from C). Operationally local preparation is specified via admissible CPTP maps supported in A.

Choice of $\mathsf{Prep}_A(r)$. Unless stated otherwise, $\mathsf{Prep}_A(r)$ denotes the set of CPTP maps on $\mathcal{A}$ admitting a Stinespring dilation localized in A, i.e. maps of the form

$$\Lambda_A(\cdot) = \text{Tr}_{\text{anc}}\Big(U_{A\text{anc}}\left((\cdot) \otimes \tau_{\text{anc}}\right)U_{A\text{anc}}^\dagger\Big), \tag{7}$$

with an ancilla state τ_{anc} and a unitary $U_{A\text{anc}}$ supported in A (equivalently $U_{A\text{anc}} = U_{A\text{anc}} \otimes \mathbb{1}_{BC}$ in the ABC decomposition). In the quasi-free split instantiation (Section 3), one may restrict to localized unitaries or bounded-depth circuits without changing the logical role of $\mathcal{F}_r$.

2.4 Operational target family $\mathcal{F}_r$

Definition 2.4 (Operational target family $\mathcal{F}_r$). Fix collar width r and budgets $(E_{\max}, \epsilon_\Delta, \delta_{\text{fix}})$. Define $\mathcal{F}_r$ as the set of states

$$\rho = \Lambda_A(\sigma), \qquad \Lambda_A \in \mathsf{Prep}_A(r), \tag{8}$$

such that:

(F1) $\text{Tr}(H_S\rho) \leq E_{\max}$ (energy budget);

(F2) $C_\Delta(\rho) \leq \epsilon_\Delta$ (coherence budget; optional in some applications);

(F3) $\|\rho - E_{\text{fix},*}[\rho]\|_1 \geq \delta_{\text{fix}}$ (fixed-point exclusion at resolution δ_{fix}).

Fixed-point projection (Schrödinger picture). In finite dimension we define the fixed-point projection on states by the ergodic mean

$$E_{\text{fix},*}[\rho] := \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathcal{T}_t(\rho)\, dt, \tag{9}$$

which exists and satisfies $\mathcal{T}_t(E_{\text{fix},*}[\rho]) = E_{\text{fix},*}[\rho]$ for all $t \geq 0$. When the QMS satisfies quantum detailed balance, this corresponds to the σ-preserving conditional expectation onto the fixed-point algebra in the Heisenberg picture.

Remark 2.1 (Resolution parameter). The threshold δ_{fix} encodes the operational resolution at which we exclude approximately invariant targets; taking δ_{fix} very small corresponds to allowing slow/metastable components, which can drive $r_\Delta^\downarrow(r)$ near zero on relevant time windows (cf. Section 4).

3 Static layer: quantitative recoverability from collar suppression (Gaussian split regime)

We state the static reconstruction input used by the pipeline. In a regularized Gaussian split regime, collar suppression controls an amplitude of cross-region dependence and yields explicit Petz recoverability.

3.1 Gaussian CCR posture and Petz recovery in block form

Let Γ and Γ_0 be covariances of centered quasi-free states with block decomposition

$$\Gamma = \begin{pmatrix} A & X \\ X^T & B \end{pmatrix}, \qquad \Gamma_0 = \begin{pmatrix} A_0 & X_0 \\ X_0^T & B_0 \end{pmatrix}. \tag{10}$$

Define the vacuum correlation factor

$$\eta_{\mathrm{vac}} := \left\| A_0^{-1/2} X_0 B_0^{-1/2} \right\|_{\mathrm{op}}, \tag{11}$$

and cross-correlation perturbation

$$\delta_X := \left\| A_0^{-1/2}(X - X_0) B_0^{-1/2} \right\|_{\mathrm{op}}. \tag{12}$$

In the Gaussian sector, the Petz-recovered covariance $\tilde{\Gamma}$ has blocks

$$\tilde{A} = A, \qquad \tilde{X} = A A_0^{-1} X_0, \qquad \tilde{B} = B_0 + X_0^T A_0^{-1}(A - A_0) A_0^{-1} X_0, \tag{13}$$

and we define the cross-block error

$$\Delta^{(12)} := X - \tilde{X} = X - A A_0^{-1} X_0. \tag{14}$$

Theorem 3.1 (Gaussian clustering–recovery bridge (static)). *Under standard quasi-free regularity and a perturbative regime (coercivity of A relative to A_0, cross-correlation control, and $\|K\|_{\mathrm{op}} \leq 1/2$ for the normalized covariance perturbation), the Petz-recovered state $\tilde{\omega}$ satisfies a fidelity bound of the form*

$$1 - F(\omega, \tilde{\omega}) \ \leq \ \mathsf{C}(\varepsilon, \eta_{\mathrm{vac}}, \delta) \ \left\| \Delta^{(12)} \right\|_{\mathrm{HS}}^2, \tag{15}$$

where the prefactor is controlled by ε and the correlation denominator $1 - \varepsilon^{-1}(\eta_{\mathrm{vac}} + \delta)^2$.

Remark 3.1 (Static only). Theorem 3.1 is a static reconstruction guarantee. It does not imply any separation-dependent statement about dynamical rates.

4 The $\omega = 0$ obstruction and witness diagnostics

We formalize an obstruction to inferring dynamical rate suppression from static collar suppression alone.

4.1 An exact $\omega = 0$ identity

Let $S = S^\dagger$ be a coupling operator and define its Bohr-zero component

$$S(0) := \sum_n \Pi_n S \Pi_n. \tag{16}$$

Assume an $\omega = 0$ Davies contribution

$$\mathcal{L}_0(O) = \gamma(0)\Big(S(0) O S(0) - \tfrac{1}{2}\{S(0)^2, O\} \Big), \qquad \gamma(0) > 0. \tag{17}$$

σ-weighted inner product (matches the numerical diagnostics). We use the σ-weighted inner product

$$\langle A, B \rangle_\sigma := \mathrm{Tr}\big(A^\dagger \sigma B\big), \qquad \|A\|_{2,\sigma}^2 := \mathrm{Tr}(\sigma A^\dagger A). \tag{18}$$

This convention matches the witness computation in Appendix A; for Pauli strings O one has $O^\dagger O = \mathbb{1}$ and hence $\|O\|_{2,\sigma}^2 = \mathrm{Tr}(\sigma) = 1$.

Lemma 4.1 (Exact $\omega = 0$ Dirichlet identity). *If $[S(0), \sigma] = 0$, then for all observables O,*

$$\mathcal{E}_\sigma^{(0)}(O) := -\Re \langle O, \mathcal{L}_0(O)\rangle_\sigma = \frac{\gamma(0)}{2} \, \|[S(0), O]\|_{2,\sigma}^2 \,. \tag{19}$$

4.2 Witness families and finite-size diagnostics

Fix a coupling location j_0 and a distance parameter $\epsilon \geq 1$. For test families $\mathcal{A}_\epsilon^{(k)}$ (one-site, two-site), define the optimized witness

$$R_{\mathrm{opt}}^{(k)}(\epsilon; N) := \max_{O \in \mathcal{A}_\epsilon^{(k)} \setminus \{0\}} \frac{\|[S(0), O]\|_{2,\sigma}^2}{\|O\|_{2,\sigma}^2}. \tag{20}$$

By Lemma 4.1, there exists at least one $O \in \mathcal{A}_\epsilon^{(k)}$ such that

$$\frac{\mathcal{E}_\sigma^{(0)}(O)}{\|O\|_{2,\sigma}^2} \geq \frac{\gamma(0)}{2} R_{\mathrm{opt}}^{(k)}(\epsilon; N). \tag{21}$$

Accordingly, we define the family-certified witness scale

$$\kappa_{\mathrm{wit}}^{(k)}(\epsilon; N) := \frac{\gamma(0)}{2} R_{\mathrm{opt}}^{(k)}(\epsilon; N), \tag{22}$$

which quantifies a *potential* obstruction within the chosen test family (but does not by itself imply a worst-case floor across an arbitrary family of targets).

Remark 4.1 (Extended check). A one-site extension at $N = 14$ for $\epsilon \in \{1, 2\}$ yields values consistent with the decreasing-with-N trend observed in Figure 1.

5 Thermodynamic control model and an autocontained proof of the power bound

This section defines the control task and proves the finite-dimensional extra-power lower bound. Two technical points are handled explicitly: (i) battery–system correlations in the one-step work bound, and (ii) the definition of P_{extra} via *paired strategies* (so we never subtract unrelated infima).

5.1 Battery-assisted thermal operations (finite dimension)

Let B be a bath with Hamiltonian H_B prepared in the Gibbs state $\gamma_B := e^{-\beta H_B}/\mathrm{Tr}(e^{-\beta H_B})$. Let W be a battery with Hamiltonian H_W. We consider global energy-conserving unitaries U on *SBW* satisfying $[U, H_S + H_B + H_W] = 0$.

Definition 5.1 (Thermal operation with battery). A battery-assisted thermal operation is any CPTP map on *SW* of the form

$$\Phi_{SW}(\cdot) = \mathrm{Tr}_B\Big(U\big((\cdot) \otimes \gamma_B\big) U^\dagger\Big), \qquad [U, H_S + H_B + H_W] = 0. \tag{23}$$
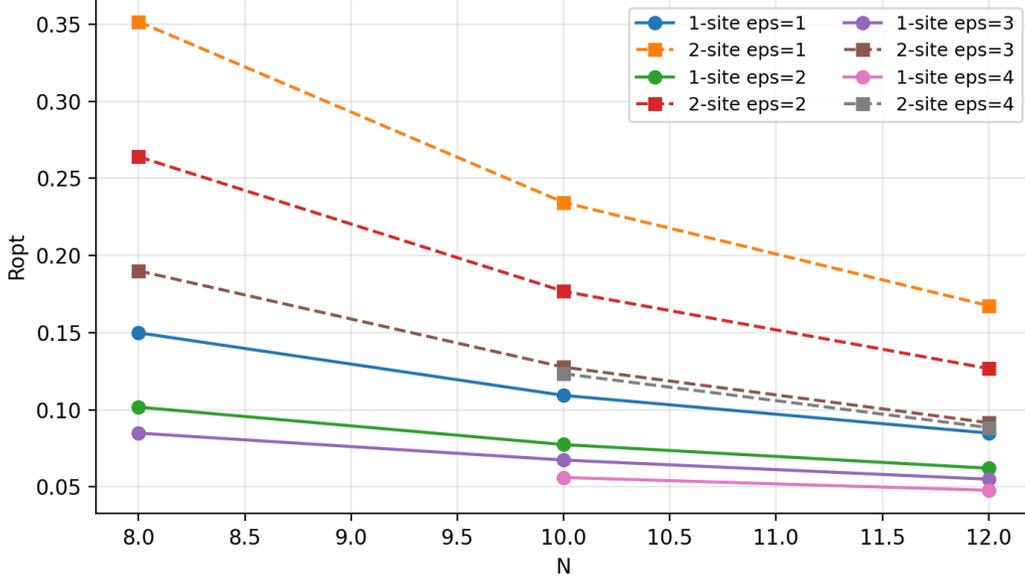
Figure 1: Optimized $\omega = 0$ commutator witnesses in the TFIM ($J = 1$, $h = 1.5$, $\beta = 1$). Solid: one-site family; dashed: adjacent two-site family. Both decrease with N at fixed ϵ for tested sizes $N \leq 12$, indicating no evidence of a size-independent $\omega = 0$ floor in this benchmark regime.

Work bookkeeping. Let $\gamma_W := e^{-\beta H_W}/\mathrm{Tr}(e^{-\beta H_W})$ and define the battery non-equilibrium free energy

$$F_\beta(\omega_W) := k_B T \, S(\omega_W \,\|\, \gamma_W). \tag{24}$$

For a process $\omega_W \mapsto \omega_W'$ we define the (battery) work cost as

$$W := F_\beta(\omega_W') - F_\beta(\omega_W). \tag{25}$$

5.2 Maintenance in the instantaneous (differential) sense

Fix an uncontrolled Markov drift $\rho_t = \mathcal{T}_t(\rho)$ on S. For a small step size $\delta t > 0$ we consider protocols of the form

$$\rho \xrightarrow{\mathcal{T}_{\delta t}} \rho_{\delta t} := \mathcal{T}_{\delta t}(\rho) \xrightarrow{\Phi_{\delta t}} \rho', \tag{26}$$

where $\Phi_{\delta t}$ is a battery-assisted thermal operation acting on SW (Definition 5.1) and the bath is freshly prepared in γ_B in each step (reset model).

Definition 5.2 (Instantaneous maintenance). A family of controls $\{\Phi_{\delta t}\}_{\delta t > 0}$ *maintains* ρ instantaneously if the post-control system marginal satisfies

$$\mathrm{Tr}_W\big[\Phi_{\delta t}(\rho_{\delta t} \otimes \omega_W)\big] = \rho + o(\delta t) \qquad \text{as } \delta t \to 0, \tag{27}$$

for the chosen initial battery state ω_W.

Definition 5.3 (Maintenance power and paired incremental power). Let $W_{\delta t}(\rho; \Phi_{\delta t})$ denote the battery work (free-energy increase) incurred by applying a control step $\Phi_{\delta t}$ to restore $\rho_{\delta t}$ to ρ up to $o(\delta t)$. Define the minimal maintenance power as

$$P(\rho) := \inf_{\{\Phi_{\delta t}\} \text{ maintains } \rho} \limsup_{\delta t \to 0} \frac{W_{\delta t}(\rho; \Phi_{\delta t})}{\delta t}. \tag{28}$$

To isolate the incremental cost of maintaining *coherence*, we define extra power via *paired strategies*. A paired strategy is a pair of control families $\left(\{\Phi_{\delta t}^{\rho}\}, \{\Phi_{\delta t}^{\Delta}\}\right)$ that: (i) maintain ρ and maintain $\Delta[\rho]$ (Definition 5.2), (ii) use the *same* bath model and the *same* initial battery state ω_W at each step. Define the paired-step incremental work by

$$W_{\delta t}^{\text{extra}}(\rho) := W_{\delta t}(\rho; \Phi_{\delta t}^{\rho}) - W_{\delta t}(\Delta[\rho]; \Phi_{\delta t}^{\Delta}), \tag{29}$$

and the incremental extra power as

$$P_{\text{extra}}(\rho) := \inf_{\text{paired strategies}} \limsup_{\delta t \to 0} \frac{W_{\delta t}^{\text{extra}}(\rho)}{\delta t}. \tag{30}$$

Remark 5.1. This paired definition avoids the invalid inference that can arise from subtracting two unrelated lower bounds on two unrelated infima. One always has $P_{\text{extra}}(\rho) \leq P(\rho) - P(\Delta[\rho])$, but we do not need this inequality.

5.3 A free-energy lemma for one-step restoration (with correlations)

Define the non-equilibrium free energy on S relative to σ by

$$F_{\beta}^{S}(\rho) := k_B T\, S(\rho \,\|\, \sigma) + F_{\beta}^{S}(\sigma), \tag{31}$$

where $F_{\beta}^{S}(\sigma)$ is a constant offset (irrelevant for differences).

Lemma 5.1 (One-step work lower bound). *Consider a battery-assisted thermal operation Φ_{SW} (Definition 5.1) that maps $\rho \otimes \omega_W$ to a final SW state with system marginal ρ' and battery marginal ω_W'. Then*

$$F_{\beta}(\omega_W') - F_{\beta}(\omega_W) \;\geq\; F_{\beta}^{S}(\rho') - F_{\beta}^{S}(\rho). \tag{32}$$

Proof. Let $\Gamma_{SBW} := \sigma \otimes \gamma_B \otimes \gamma_W$ be the global Gibbs state at inverse temperature β. Energy conservation implies $U\Gamma_{SBW}U^{\dagger} = \Gamma_{SBW}$. By monotonicity of relative entropy under the CPTP map $\Psi(\cdot) := \text{Tr}_B\big(U(\cdot)U^{\dagger}\big)$,

$$S(\rho \otimes \omega_W \otimes \gamma_B \,\|\, \sigma \otimes \gamma_W \otimes \gamma_B) \;\geq\; S\big(\rho_{SW}' \,\|\, \sigma \otimes \gamma_W\big), \tag{33}$$

where $\rho_{SW}' := \Phi_{SW}(\rho \otimes \omega_W)$.

The left-hand side is additive:

$$S(\rho \otimes \omega_W \otimes \gamma_B \,\|\, \sigma \otimes \gamma_W \otimes \gamma_B) = S(\rho \,\|\, \sigma) + S(\omega_W \,\|\, \gamma_W). \tag{34}$$

The right-hand side need not be additive because ρ_{SW}' can be correlated. Using the chain rule for relative entropy against a product reference,

$$S\big(\rho_{SW}' \,\|\, \sigma \otimes \gamma_W\big) = S\big(\rho' \,\|\, \sigma\big) + S\big(\omega_W' \,\|\, \gamma_W\big) + I(S{:}W)_{\rho_{SW}'}, \tag{35}$$

where $I(S{:}W)_{\rho_{SW}'} \geq 0$ is the mutual information in the final SW state. Hence

$$S(\rho \,\|\, \sigma) + S(\omega_W \,\|\, \gamma_W) \;\geq\; S\big(\rho' \,\|\, \sigma\big) + S\big(\omega_W' \,\|\, \gamma_W\big). \tag{36}$$

Multiplying by $k_B T$ and rewriting in terms of $F_{\beta}(\cdot)$ and $F_{\beta}^{S}(\cdot)$ yields (32). $\qquad\square$

5.4 Autocontained proof of the paired extra-power bound

Assumption 5.1 (Energy covariance of the drift). The uncontrolled semigroup $\mathcal{T}_t$ commutes with energy pinching:

$$\Delta[\mathcal{T}_t(\rho)] = \mathcal{T}_t(\Delta[\rho]) \qquad \text{for all } t \geq 0 \text{ and all } \rho. \tag{37}$$

Remark 5.2. Assumption 5.1 holds for Davies/secular generators that are time-translation covariant with respect to H_S (the standard weak-coupling, rotating-wave setting).

Lemma 5.2 (Relative-entropy Pythagoras for pinching). *For the Gibbs state σ and the pinching map Δ of* (3)*, one has for all states ρ,*

$$S(\rho \,\|\, \sigma) = S(\rho \,\|\, \Delta[\rho]) + S(\Delta[\rho] \,\|\, \sigma). \tag{38}$$

Proof. Since σ is diagonal in the energy basis, Δ is a σ-preserving conditional expectation. The identity (38) is the standard Pythagorean relation for relative entropy with respect to conditional expectations. $\qquad\square$

Theorem 5.3 (Incremental maintenance power lower bound (finite dimension, paired strategies)). *Assume the instantaneous maintenance model (Definitions 5.2 and 5.3) with battery-assisted thermal operations (Definition 5.1), and assume Assumption 5.1. If paired strategies exist that maintain ρ and maintain $\Delta[\rho]$, then*

$$P_{\text{extra}}(\rho) \;\geq\; k_B T\, \dot{C}_{\Delta,\text{loss}}(\rho), \tag{39}$$

where $C_\Delta(\rho) = S(\rho \,\|\, \Delta[\rho])$ and $\dot{C}_{\Delta,\text{loss}}(\rho) = -\frac{d}{dt}\big|_{t=0} C_\Delta(\mathcal{T}_t(\rho))$.

Proof. Fix a target ρ and consider a paired strategy for a one-step duration δt. Let $\rho_{\delta t} := \mathcal{T}_{\delta t}(\rho)$ and $\Delta[\rho]_{\delta t} := \mathcal{T}_{\delta t}(\Delta[\rho]) = \Delta[\rho_{\delta t}]$ by Assumption 5.1.

Apply Lemma 5.1 to the ρ-maintenance step:

$$W_{\delta t}(\rho; \Phi^\rho_{\delta t}) \;\geq\; F^S_\beta(\rho) - F^S_\beta(\rho_{\delta t}) + o(\delta t) = k_B T\big(S(\rho \,\|\, \sigma) - S(\rho_{\delta t} \,\|\, \sigma)\big) + o(\delta t). \tag{40}$$

Apply Lemma 5.1 to the $\Delta[\rho]$-maintenance step:

$$W_{\delta t}(\Delta[\rho]; \Phi^\Delta_{\delta t}) \;\geq\; k_B T\big(S(\Delta[\rho] \,\|\, \sigma) - S(\Delta[\rho_{\delta t}] \,\|\, \sigma)\big) + o(\delta t). \tag{41}$$

Subtracting these two valid inequalities (for the same fixed δt and paired strategy) yields

$$W^{\text{extra}}_{\delta t}(\rho) \;\geq\; k_B T\Big[\big(S(\rho \,\|\, \sigma) - S(\Delta[\rho] \,\|\, \sigma)\big) - \big(S(\rho_{\delta t} \,\|\, \sigma) - S(\Delta[\rho_{\delta t}] \,\|\, \sigma)\big)\Big] + o(\delta t). \tag{42}$$

By Lemma 5.2,

$$S(\rho \,\|\, \sigma) - S(\Delta[\rho] \,\|\, \sigma) = S(\rho \,\|\, \Delta[\rho]) = C_\Delta(\rho), \qquad S(\rho_{\delta t} \,\|\, \sigma) - S(\Delta[\rho_{\delta t}] \,\|\, \sigma) = S(\rho_{\delta t} \,\|\, \Delta[\rho_{\delta t}]) = C_\Delta(\rho_{\delta t}). \tag{43}$$

Therefore

$$W^{\text{extra}}_{\delta t}(\rho) \;\geq\; k_B T\big(C_\Delta(\rho) - C_\Delta(\rho_{\delta t})\big) + o(\delta t). \tag{44}$$

Divide by δt, take $\limsup_{\delta t \to 0}$, and then take the infimum over paired strategies as in (30) to obtain

$$P_{\text{extra}}(\rho) \;\geq\; k_B T\left(-\frac{d}{dt}\Big|_{t=0} C_\Delta(\mathcal{T}_t(\rho))\right) = k_B T\, \dot{C}_{\Delta,\text{loss}}(\rho). \tag{45}$$

$$\square$$

Remark 5.3 (Integrated version). Integrating the differential inequality over a time interval yields an extra-work bound $W_{\text{extra}}([0,T]) \geq k_B T\big(C_\Delta(\rho_0) - C_\Delta(\rho_T)\big)$ for protocols that track a target over time, under the same bookkeeping conventions.

6 Main results: finite-dimensional core and a Type III blueprint

6.1 Finite-dimensional core: rates imply power (proved)

Corollary 6.1 (Typed family lower bound)**.** *Fix r and a family $\mathcal{F}_r$. For any $\rho \in \mathcal{F}_r$ with $C_\Delta(\rho) > 0$,*

$$P_{\text{extra}}(\rho) \ \geq \ k_B T \, \dot{C}_{\Delta,\text{loss}}(\rho) = k_B T \, r_\Delta(\rho) \, C_\Delta(\rho) \ \geq \ k_B T \, r_\Delta^\downarrow(r) \, C_\Delta(\rho). \tag{46}$$

6.2 The missing hinge: typed rate inheritance (explicit hypothesis)

Hypothesis 6.2 (Typed rate inheritance on $\mathcal{F}_r$)**.** *On the operational family $\mathcal{F}_r$, the lower envelope $r_\Delta^\downarrow(r)$ is strictly positive in the regime of interest and admits a collar-dependent envelope (for example, $r_\Delta^\downarrow(r) \gtrsim e^{-\alpha r}$ up to polynomial prefactors). Moreover, fixed-point/$\omega = 0$ mechanisms do not induce a collar-independent floor on $\mathcal{F}_r$ in the relevant regime.*

Corollary 6.3 (Collar-dependent power suppression (conditional))**.** *Assuming Hypothesis 6.2, if $C_\Delta(\rho) \geq C_0 > 0$ on $\mathcal{F}_r$, then*

$$\inf_{\rho \in \mathcal{F}_r} P_{\text{extra}}(\rho) \ \gtrsim \ k_B T \, r_\Delta^\downarrow(r) \, C_0. \tag{47}$$

6.3 Type III blueprint (conjectural)

Conjecture 6.4 (Typed pipeline in AQFT (blueprint))**.** *Let $\mathcal{M}$ be a Type III local algebra in AQFT with vacuum ω_0 and a split inclusion $O_1 \Subset O_2$ of collar width r. Let E_{split} be the vacuum-preserving conditional expectation onto the split factor. Define the static residue by Araki relative entropy*

$$C_{E_{\text{split}}}(\omega) := S_{\text{Araki}}(\omega \| \omega \circ E_{\text{split}}). \tag{48}$$

Assume a physically specified dissipative dynamics and a well-posed maintenance-power notion under a regularized thermodynamic bookkeeping model. Then, on a typed fast-sector family and under a Type III analogue of Hypothesis 6.2 (with explicit control of fixed-point/$\omega = 0$ obstructions), one obtains a collar-dependent bound of the schematic form

$$P_{\text{extra}}(\omega) \ \gtrsim \ k_B T \, r_\Delta^\downarrow(r) \, C_{E_{\text{split}}}(\omega), \qquad \omega \in \mathcal{F}_r. \tag{49}$$

7 Conclusion

We have presented a typed pipeline connecting geometric isolation to thermodynamic maintenance cost. The central lesson is that while geometry buys recoverability (Theorem 3.1), it does not buy efficiency freely: maintenance is governed by rate envelopes such as $r_\Delta^\downarrow(r)$, which are vulnerable to fixed-point and $\omega = 0$ obstructions that static clustering alone cannot rule out. By making the missing dynamical hinge explicit (Hypothesis 6.2) and providing commutator-witness diagnostics for the $\omega = 0$ mechanism, we delineate the conditions under which the intuition "isolation implies stability" becomes thermodynamically valid.

Reproducibility

Figure 1 is reproduced by running the Python script embedded in Appendix A. Copy the listing into a file named `reproduce_witness_figure.py` and execute it from the project root. The script performs exact diagonalization for $N \in \{8, 10, 12\}$ and generates the file `witness_sweep_Nle12_plot.png` in the working directory.

A Reproducibility script (Python)

Usage (local):

- Install: `pip install numpy scipy matplotlib`

- Run: `python reproduce_witness_figure.py`

- Output: `witness_sweep_Nle12_plot.png`

Listing 1: reproduce_witness_figure.py (generates Figure 1).

```python
#!/usr/bin/env python3
import time, gc
import numpy as np
import scipy.linalg as la
import matplotlib.pyplot as plt

np.set_printoptions(precision=6, suppress=True)

# -------- Parameters for Figure 1 --------
J = 1.0
h = 1.5
beta = 1.0
degen_tol = 1e-10

Ns = [8, 10, 12]
eps_targets = [1, 2, 3, 4]
OUT_PNG = "witness_sweep_Nle12_plot.png"

# -------- Helpers --------
def paulis():
    I = np.array([[1,0],[0,1]], dtype=complex)
    X = np.array([[0,1],[1,0]], dtype=complex)
    Y = np.array([[0,-1j],[1j,0]], dtype=complex)
    Z = np.array([[1,0],[0,-1]], dtype=complex)
    return I, X, Y, Z

def kron_all(ops):
    out = ops[0]
    for op in ops[1:]:
        out = np.kron(out, op)
    return out

def local_pauli(N, site, which):
    I, X, Y, Z = paulis()
    P = {"X":X,"Y":Y,"Z":Z}[which.upper()]
    ops = [I]*N
    ops[site] = P
    return kron_all(ops)

def two_site_xx(N, i):
    I, X, Y, Z = paulis()
    ops = [I]*N
    ops[i] = X
```

```python
44          ops[i+1] = X
45      return kron_all(ops)
46
47  def tfim_hamiltonian(N, J, h):
48      dim = 2**N
49      H = np.zeros((dim, dim), dtype=complex)
50      for i in range(N-1):
51          H += -J * two_site_xx(N, i)
52      for i in range(N):
53          H += -h * local_pauli(N, i, "Z")
54      return (H + H.conj().T)/2
55
56  def thermal_probs(E, beta):
57      Emin = float(np.min(E.real))
58      w = np.exp(-beta*(E.real - Emin))
59      return w/np.sum(w)
60
61  def thermal_state_from_eigh(E, V, beta):
62      p = thermal_probs(E, beta)
63      rho = (V * p) @ V.conj().T
64      return (rho + rho.conj().T)/2
65
66  def energy_blocks(E, tol=1e-10):
67      E = np.array(E.real, dtype=float)
68      order = np.argsort(E)
69      blocks, cur = [], [order[0]]
70      for idx in order[1:]:
71          if abs(E[idx] - E[cur[0]]) <= tol:
72              cur.append(idx)
73          else:
74              blocks.append(cur); cur=[idx]
75      blocks.append(cur)
76      return blocks
77
78  def S0_component(S, E, V, tol=1e-10):
79      S_E = V.conj().T @ S @ V
80      S0_E = np.zeros_like(S_E)
81      for blk in energy_blocks(E, tol=tol):
82          ii = np.ix_(blk, blk)
83          S0_E[ii] = S_E[ii]
84      S0 = V @ S0_E @ V.conj().T
85      return (S0 + S0.conj().T)/2
86
87  def R_ratio(O, S0, sigma):
88      # sigma-weighted 2-norm: ||O||_{2,sigma}^2 = Tr(sigma O^\dagger O)
89      # for Pauli strings O: O^\dagger O = I, hence denominator = Tr(
            sigma)=1.
90      comm = S0@O - O@S0
91      return float(np.trace(comm.conj().T @ (sigma @ comm)).real)
92
93  def Ropt_1site(N, S0, sigma, j0, eps):
94      best = 0.0
95      for side in (+1, -1):
96          k = j0 + side*eps
```

11

```
 97            if k < 0 or k >= N:
 98                continue
 99            for p in ("X","Y","Z"):
100                O = local_pauli(N, k, p)
101                best = max(best, R_ratio(O, S0, sigma))
102        return best
103
104    def Ropt_2site_adj(N, S0, sigma, j0, eps):
105        I, X, Y, Z = paulis()
106        P = {"X":X, "Y":Y, "Z":Z}
107        best = 0.0
108        for side in (+1, -1):
109            k0 = j0 + side*eps
110            if k0 < 0 or k0+1 >= N:
111                continue
112            for a in ("X","Y","Z"):
113                for b in ("X","Y","Z"):
114                    ops = [I]*N
115                    ops[k0]   = P[a]
116                    ops[k0+1] = P[b]
117                    O = kron_all(ops)
118                    best = max(best, R_ratio(O, S0, sigma))
119        return best
120
121    def main():
122        R1 = {}
123        R2 = {}
124
125        for N in Ns:
126            tN = time.time()
127            print(f"--- N={N} dim={2**N} ---")
128            H = tfim_hamiltonian(N, J, h)
129
130            t_eigh = time.time()
131            E, V = la.eigh(H, check_finite=False)
132            print(f"  eigh: {time.time()-t_eigh:.2f}s")
133
134            j0 = N//2
135            S = local_pauli(N, j0, "Z")
136            S0 = S0_component(S, E, V, tol=degen_tol)
137            sigma = thermal_state_from_eigh(E, V, beta)
138
139            for eps in eps_targets:
140                if eps <= min(j0, N-1-j0):
141                    R1[(N,eps)] = Ropt_1site(N, S0, sigma, j0, eps)
142                    R2[(N,eps)] = Ropt_2site_adj(N, S0, sigma, j0, eps)
143                else:
144                    R1[(N,eps)] = np.nan
145                    R2[(N,eps)] = np.nan
146
147            del H,E,V,S,S0,sigma
148            gc.collect()
149            print(f"  total for N: {time.time()-tN:.2f}s")
150
```

```python
        plt.figure(figsize=(7.2,4.2))
        for eps in eps_targets:
            y1 = [R1.get((N,eps), np.nan) for N in Ns]
            y2 = [R2.get((N,eps), np.nan) for N in Ns]
            plt.plot(Ns, y1, marker="o", label=f"1-site eps={eps}")
            plt.plot(Ns, y2, marker="s", linestyle="--", label=f"2-site eps
                ={eps}")
        plt.grid(True, alpha=0.3)
        plt.xlabel("N")
        plt.ylabel("Ropt")
        plt.legend(ncol=2, fontsize=9)
        plt.tight_layout()
        plt.savefig(OUT_PNG, dpi=200)
        plt.close()
        print("Saved:", OUT_PNG)

if __name__ == "__main__":
        main()
```

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