Finite-Dimensional Davies Interface Lemmas and TFIM Witness Tests for Separation-Dependent Decoherence Rate Envelopes

Version 2

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January 2026 (v1); July 2026 (v2)

Abstract

We develop a finite-dimensional technical core for relating separation to effective decoherence-rate envelopes in Davies-type open-system dynamics. We work with energy pinching Δ and quantify coherence by $C(\rho) = S(\rho\|\Delta[\rho])$. We import a maintenance inequality $P_{\text{extra}}(\rho) \geq k_B T\,\dot{C}_{\text{loss}}(\rho)$ – in the corrected operational sense of its source (paired strategies against efficient/free baselines, 2512.0061 v2) – as an external input. On the operator side we prove: (i) an exact $\omega = 0$ Dirichlet identity yielding a witness-based lower bound on instantaneous decay envelopes, (ii) a Bohr-channel Dirichlet decomposition for a single-channel Davies generator under quantum detailed balance – *corrected in v2*: the v1 form $\frac{1}{2}\sum_\omega \gamma(\omega)\|[S(\omega), O]\|^2_{2,\sigma}$ is false as an identity (numerical deviations of order 20%); the exact form is $\sum_\omega \gamma(\omega)e^{\beta\omega/2}\,\Re\langle S(\omega)O, [S(\omega), O]\rangle_\sigma$, which reduces to (i) at $\omega = 0$ and is nonnegative pairwise in $\pm\omega$ – and (iii) envelope suppression lemmata under infrared exclusion and quasi-local spectral tails, *reproved in v2*: the v1 proofs used a KMS submultiplicativity step that is false in general (a one-qubit counterexample saturates the corrected constant); the corrected constants carry $c_\sigma^2 = (\lambda_{\max}/\lambda_{\min})^{1/2}$ and $\sup_\omega \gamma(\omega)e^{\beta\omega/2}$, and the $e^{-2\epsilon/\xi}$ exponent survives via a new *positivity-pinning* argument: for $S = S_{\text{near}} + \delta\, S_{\text{tail}}$ and far-supported O, detailed balance at every δ forces $\mathcal{E}_\delta(O) = \delta^2\mathcal{E}_{\text{tail}}(O)$ exactly. Consequently Lemma 4.13's v1 claim of a $\lambda_{\min}(\sigma)$-free prefactor is downgraded to a quadratically improved dependence. On the state side we give an asymptotic linearization on Bohr-block perturbations with fixed diagonal, yielding a direction-dependent effective decay rate. Finite-size TFIM witness diagnostics are provided, and every corrected statement is verified to machine precision by the shipped suite.

Summary of results and assumptions (updated in v2). Imported: the maintenance inequality (Theorem 2.1) from [1]. Proved here (finite dimension): Lemma 4.1, Lemma 4.5 (new), Lemma 4.3 (corrected), Lemma 4.6 (new), Proposition 4.4, Lemmas 4.7–4.8 (corrected constants), Proposition 6.1 and Corollary 6.2. Finite-size numerics: Section 5. No thermodynamic-limit claims. All identities and counterexamples are verified in `verification/2601-0023/` (Section 7).

1 Scope, positioning, and non-claims

This manuscript is finite-dimensional and makes no thermodynamic-limit claims unless stated. It does not propose modified dynamics, collapse mechanisms, Born-rule derivations, or ontological resolutions of the measurement problem. For theorem-level frameworks relating correlation decay and log-Sobolev inequalities see [4, 5]; our aim is an operator/state interface around Davies Dirichlet forms and a witness floor, with reproducible finite-size diagnostics. *Series context (added in v2):* this note is the theory core behind the TFIM witness diagnostics of 2601.0022 [6] (whose ED parameters are the ones declared in Appendix A here), and feeds the $\omega = 0$-floor block 2512.0064/2512.0070 [7, 8] and the entropic interface 2601.0020 [9].

2 Finite-dimensional operational core

Let $H_S = \sum_n E_n \Pi_n$, $\sigma := e^{-\beta H_S} / \operatorname{Tr}(e^{-\beta H_S})$, $\Delta[\rho] := \sum_n \Pi_n \rho \Pi_n$, and $C(\rho) := S(\rho \| \Delta[\rho])$ with Umegaki relative entropy. If H_S is degenerate, Δ is energy-*block* pinching and $S(0)$ below is block-diagonal on degenerate subspaces (the numerics group energies within 10^{-10}). The uncontrolled evolution is a GKLS semigroup with $\mathcal{L}(\sigma) = 0$; $\dot{C}_{\text{loss}}(\rho) := -\frac{d}{dt} C(\rho_t)|_{t=0}$. Battery-assisted thermal operations define $P_{\text{extra}}(\rho) := \inf [P(s) - P(s_{\text{diag}})]$ over paired strategies maintaining ρ resp. $\Delta[\rho]$.

Theorem 2.1 (Imported maintenance inequality [1])**.** *If* $\operatorname{Ctrl}(\rho) \neq \emptyset$ *and* $\operatorname{Ctrl}(\Delta[\rho]) \neq \emptyset$*, then* $P_{\text{extra}}(\rho) \geq k_B T \, \dot{C}_{\text{loss}}(\rho)$*, with* P_{extra} *understood in the corrected operational sense of 2512.0061 v2 (paired strategies against efficient/free baselines; the v1 "assumption-free" variant of that source was vacuous and is not used here).*

3 Davies generators, KMS Dirichlet form, and a separation envelope

Fix $S = S^\dagger$ and a Davies generator in the weak-coupling–secular limit, with Bohr components $S(\omega) := \sum_{E_m - E_n = \omega} \Pi_m S \Pi_n$, $S(\omega)^\dagger = S(-\omega)$, and KMS rates $\gamma(-\omega) = e^{-\beta\omega}\gamma(\omega)$. Define $\langle A, B\rangle_\sigma := \operatorname{Tr}(\sigma^{1/2} A^\dagger \sigma^{1/2} B)$, $\mathcal{E}_\sigma(O) := -\Re\langle O, \mathcal{L}^\dagger(O)\rangle_\sigma$, far regions $\Omega_\epsilon := \{j : d(j, \Omega_0) \geq \epsilon\}$ with observable subspace $\mathcal{A}_\epsilon$, and $\kappa(\epsilon) := \sup_{O \in \mathcal{A}_\epsilon} \mathcal{E}_\sigma(O)/\|O\|^2_{2,\sigma}$.

Remark 3.1 (Modular relations)*.* Since $(S(\omega))_{mn} \neq 0$ only if $E_m - E_n = \omega$: $\sigma^s S(\omega) \sigma^{-s} = e^{-s\beta\omega} S(\omega)$ for all s, and $S(\omega)^\dagger S(\omega)$ commutes with σ. These two facts drive all corrected identities below.

4 Interface lemmas

4.1 The $\omega = 0$ channel yields a witness floor

Lemma 4.1 (Exact $\omega = 0$ Dirichlet identity)**.** *Assume* $[S(0), H_S] = 0$ *and* $\gamma(0) > 0$*. For the* $\omega = 0$ *Davies channel* $\mathcal{L}_0^\dagger(O) = \gamma(0)(S(0) O S(0) - \frac{1}{2}\{S(0)^2, O\})$*, one has* $\mathcal{E}_\sigma^{(0)}(O) = \frac{\gamma(0)}{2} \|[S(0), O]\|^2_{2,\sigma}$ *for all* O*.*

Proof. As in v1: with $L := S(0)$, $[L, \sigma^{1/2}] = 0$ makes left/right multiplication by L KMS-self-adjoint; expanding $\langle [L, O], [L, O]\rangle_\sigma$ gives the identity. (Verified to 4×10^{-16} in the suite.) □

4.2 Bohr-channel Dirichlet decomposition (corrected in v2)

Remark 4.2 (The v1 decomposition is false as an identity)*.* v1 asserted $\mathcal{E}_\sigma(O) = \frac{1}{2}\sum_\omega \gamma(\omega) \|[S(\omega), O]\|^2_{2,\sigma}$, citing the divergence-form representation of [5]. That representation involves *modular-deformed* derivations, not plain commutators; the plain-commutator form is exact only at $\omega = 0$. On a TFIM $N = 4$ single-channel Davies model the suite finds relative deviations up to 0.22 between $\mathcal{E}_\sigma$ and the v1 right-hand side.

Lemma 4.3 (Bohr-channel Dirichlet decomposition, corrected)**.** *Under quantum detailed balance and the single-channel Davies form, for Hermitian O,*

$$\mathcal{E}_\sigma(O) \;=\; \sum_\omega \gamma(\omega)\, e^{\beta\omega/2}\, \Re\,\big\langle S(\omega)O,\ [S(\omega),O]\big\rangle_\sigma .$$

Each pair $\{+\omega,-\omega\}$ of summands is jointly nonnegative (each such pair, with its KMS rates, is itself a detailed-balanced generator), and the $\omega=0$ summand equals $\mathcal{E}^{(0)}_\sigma(O)$ of Lemma 4.1.

Proof. Per channel, using Remark 3.1: $\langle O, S(\omega)^\dagger O S(\omega)\rangle_\sigma = e^{\beta\omega/2}\langle S(\omega)O, OS(\omega)\rangle_\sigma$ and $\langle O, S(\omega)^\dagger S(\omega)O\rangle_\sigma = \Re\,\langle O, OS(\omega)^\dagger S(\omega)\rangle_\sigma = e^{\beta\omega/2}\|S(\omega)O\|^2_{2,\sigma}$ (move $\sigma^{1/2}$ through $S(\omega)^\dagger$, use $[S^\dagger S,\sigma]=0$ and cyclicity). Substituting into $-\Re\langle O, \mathcal{L}^\dagger_\omega(O)\rangle_\sigma$ gives the ω summand $\gamma(\omega)e^{\beta\omega/2}[\|S(\omega)O\|^2_{2,\sigma} - \Re\langle S(\omega)O, OS(\omega)\rangle_\sigma]$. Pairwise nonnegativity follows since the $\{\pm\omega\}$ sub-generator is KMS-symmetric with stationary σ, hence has nonnegative Dirichlet form. At $\omega=0$, $\|OS(0)\|_{2,\sigma} = \|S(0)O\|_{2,\sigma}$ (adjoint invariance of the KMS norm), and the summand equals $\frac{\gamma(0)}{2}\|[S(0),O]\|^2_{2,\sigma}$. All statements are verified to machine precision in the suite (2×10^{-16}). □

4.3 Witness floor

Proposition 4.4 (Witness lower bound at fixed separation)**.** *In the setting of Lemma 4.3, fix $\epsilon_0 \geq 1$. If there is $O_{\epsilon_0} \in \mathcal{A}_{\epsilon_0}$ with $\|[S(0),O_{\epsilon_0}]\|^2_{2,\sigma}/\|O_{\epsilon_0}\|^2_{2,\sigma} \geq c_{\epsilon_0} > 0$, then $\kappa(\epsilon_0) \geq \frac{\gamma(0)}{2} c_{\epsilon_0}$.*

Proof. By Lemma 4.3, $\mathcal{E}_\sigma \geq$ its $\omega=0$ summand (the remaining $\pm\omega$ pairs are jointly nonnegative), which equals $\mathcal{E}^{(0)}_\sigma$; apply Lemma 4.1. (Note the v1 proof used termwise nonnegativity of the – incorrect – commutator decomposition; pairwise nonnegativity suffices.) □

4.4 KMS multiplication bound (new in v2)

Lemma 4.5 (KMS multiplication bound and sharpness)**.** *For all X, O: $\|XO\|_{2,\sigma} \leq \|\sigma^{1/4}X\sigma^{-1/4}\|_\infty \|O\|_{2,\sigma} \leq c_\sigma \|X\|_\infty \|O\|_{2,\sigma}$, and likewise for OX, with $c_\sigma := (\lambda_{\max}(\sigma)/\lambda_{\min}(\sigma))^{1/4}$. The naive bound $\|XO\|_{2,\sigma} \leq \|X\|_\infty\|O\|_{2,\sigma}$, used implicitly in v1's Lemmas 4.12–4.13, is false: for one qubit with gap Δ, $X=\sigma^x$ and $O = |1\rangle\langle 0|$ give $\|XO\|_{2,\sigma}/\|O\|_{2,\sigma} = e^{\beta\Delta/4}$, which saturates c_σ.*

Proof. $\|A\|_{2,\sigma} = \|\sigma^{1/4}A\sigma^{1/4}\|_2$, so $\|XO\|_{2,\sigma} = \|(\sigma^{1/4}X\sigma^{-1/4})(\sigma^{1/4}O\sigma^{1/4})\|_2 \leq \|\sigma^{1/4}X\sigma^{-1/4}\|_\infty \|O\|_{2,\sigma}$, and $\|\sigma^{1/4}X\sigma^{-1/4}\|_\infty \leq \lambda_{\max}^{1/4}\lambda_{\min}^{-1/4}\|X\|_\infty$. The one-qubit computation is direct. (300 random draws in the suite: no violations of the corrected bound.) □

4.5 Envelope suppression (reproved in v2)

Definitions of IR exclusion at ω^*, tensor lattice, support, tail map $\mathrm{tail}_\epsilon(X) := X - E_{<\epsilon}(X)$, and the norm comparison $\|X\|_2^2 \leq \lambda_{\min}(\sigma)^{-1}\|X\|^2_{2,\sigma}$ are as in v1 (that lemma is correct).

Lemma 4.6 (Positivity pinning (new in v2))**.** *Let O be supported on Ω_ϵ and decompose each coupling as $S = S_{\mathrm{near}} + S_{\mathrm{tail}}$ with $S_{\mathrm{near}} := E_{<\epsilon}(S)$. For $\delta \in \mathbb{R}$ let $\mathcal{E}_\delta$ denote the Dirichlet form of the Davies generator built from $S_{\mathrm{near}} + \delta S_{\mathrm{tail}}$ (same H_S, same rates). Then*

$$\mathcal{E}_\delta(O) \;=\; \delta^2\, \mathcal{E}_{\mathrm{tail}}(O) \qquad \textit{exactly},$$

where $\mathcal{E}_{\mathrm{tail}}$ is the Dirichlet form built from S_{tail} alone.

Proof. Each δ yields a detailed-balanced Davies generator with stationary σ, so $\delta \mapsto \mathcal{E}_\delta(O) \geq 0$ for all real δ. The dissipator is exactly quadratic in the coupling, so $\mathcal{E}_\delta(O) = a + b\delta + c\delta^2$ with $a = \mathcal{E}_{\mathrm{near}}(O) = 0$ (near jumps commute with far O) and $c = \mathcal{E}_{\mathrm{tail}}(O)$. Nonnegativity of a quadratic vanishing at $\delta = 0$ forces $b = 0$. (Verified exactly over four decades of δ in the suite.) $\square$

Lemma 4.7 (Envelope suppression from KMS tails; corrected constant)**.** *Assume IR exclusion at $\omega^* > 0$ and $\|\mathrm{tail}_\epsilon(S(\omega))\|_{2,\sigma} \leq A(1+\epsilon)^p e^{-\epsilon/\xi}$ for $|\omega| \geq \omega^*$, with $\tilde{\gamma}_{\mathrm{sup}} := \sup_{|\omega|\geq\omega^*} \gamma(\omega) e^{\beta\omega/2} < \infty$. Then for fixed (N, β),*

$$\kappa(\epsilon) \leq C(1+\epsilon)^{2p} e^{-2\epsilon/\xi}, \qquad C := 2\,|\Omega_{\geq\omega^*}|\,\tilde{\gamma}_{\mathrm{sup}}\, c_\sigma^2\, \lambda_{\min}(\sigma)^{-1} A^2.$$

Lemma 4.8 (Envelope suppression from operator-norm tails; corrected)**.** *With the tail hypothesis in operator norm, $\|\mathrm{tail}_\epsilon(S(\omega))\|_\infty \leq A(1+\epsilon)^p e^{-\epsilon/\xi}$, the same bound holds with $C_\infty := 2\,|\Omega_{\geq\omega^*}|\,\tilde{\gamma}_{\mathrm{sup}}\, c_\sigma^2\, A^2$.*

Proof of Lemmas 4.7–4.8. By Lemma 4.6 (applied with $\delta = 1$), $\mathcal{E}_\sigma(O) = \mathcal{E}_{\mathrm{tail}}(O)$ for far-supported O. By Lemma 4.3 applied to the tail generator and Cauchy–Schwarz, each channel contributes at most $\gamma(\omega)e^{\beta\omega/2}\,\|T(\omega)O\|_{2,\sigma}\,\|[T(\omega),O]\|_{2,\sigma} \leq 2c_\sigma^2\|T(\omega)\|_\infty^2\|O\|_{2,\sigma}^2\,\gamma(\omega)e^{\beta\omega/2}$ with $T(\omega) := \mathrm{tail}_\epsilon(S(\omega))$ (Lemma 4.5 twice). Sum over $\Omega_{\geq\omega^*}$ and insert the tail hypothesis; for Lemma 4.7 convert $\|T\|_\infty \leq \|T\|_2 \leq \lambda_{\min}^{-1/2}\|T\|_{2,\sigma}$. $\square$

Remark 4.9 (Corrected trade-off; v1's Remark 4.14 revised). v1 claimed Lemma 4.8's prefactor is independent of $\lambda_{\min}(\sigma)$. With the corrected multiplication bound this is no longer true: $c_\sigma^2 = (\lambda_{\max}/\lambda_{\min})^{1/2} \leq \lambda_{\min}^{-1/2}$. The honest comparison is: KMS-tail version $\sim \lambda_{\min}^{-3/2}$ (worst case), operator-tail version $\sim \lambda_{\min}^{-1/2}$ – a quadratic improvement, not $\lambda_{\min}$-freeness. Both versions also replace $\sup\gamma$ by $\sup\gamma(\omega)e^{\beta\omega/2}$, reflecting the corrected decomposition. The $e^{-2\epsilon/\xi}$ *exponent* of v1 is confirmed: by Lemma 4.6 the cross (near×tail) terms vanish identically, which v1 had implicitly and incorrectly justified via the plain-commutator decomposition.

5 TFIM witness diagnostics (finite size)

TFIM with open boundaries, $H_S = -J\sum\sigma_i^x\sigma_{i+1}^x - h\sum\sigma_i^z$, $h > J$; coupling site $j_0 = \lfloor N/2 \rfloor$, $S = \sigma_{j_0}^z$; witness ratio $R(\epsilon; O) := \|[S(0), O]\|_{2,\sigma}^2 / \|O\|_{2,\sigma}^2$ optimized over single-site Paulis at $k = j_0 \pm \epsilon$. By Proposition 4.4, $\kappa(\epsilon) \geq \frac{\gamma(0)}{2} R_{\mathrm{opt}}(\epsilon)$ – unaffected by the v2 corrections, since only the $\omega = 0$ channel enters.

6 State-side linearization (finite dimension)

Definitions (BKM inner product, Bohr-block subspace X_Ω) and the quadratic expansion $S(\tau + \epsilon X\|\tau) = \frac{\epsilon^2}{2}\langle X, X\rangle_{\mathrm{BKM},\tau} + O(\epsilon^3)$ are as in v1.

Proposition 6.1 (Asymptotic interface on a Bohr block)**.** *Let $\tau \succ 0$ be diagonal in the energy basis, $\mathcal{L}(\tau) = 0$, $\mathcal{L}(X_\Omega) \subseteq X_\Omega$, $X \in X_\Omega$ traceless, $\rho(\epsilon) := \tau + \epsilon X$. Then $C(\rho(\epsilon)) = \frac{\epsilon^2}{2}\langle X, X\rangle_{\mathrm{BKM},\tau} + O(\epsilon^3)$, $\dot{C}_{\mathrm{loss}}(\rho(\epsilon)) = -\epsilon^2\Re\langle X, \mathcal{L}(X)\rangle_{\mathrm{BKM},\tau} + O(\epsilon^3)$, and $\dot{C}_{\mathrm{loss}}/C = \Gamma_{\mathrm{eff}}(X) + O(\epsilon)$ with $\Gamma_{\mathrm{eff}}(X) := -2\Re\langle X, \mathcal{L}(X)\rangle_{\mathrm{BKM}}/\langle X, X\rangle_{\mathrm{BKM}}$.*

Corollary 6.2 (Eigenmode case)**.** *If additionally $\mathcal{L}(X) = -\frac{\Gamma}{2}X$, then $\Gamma_{\mathrm{eff}}(X) = \Gamma$ and $\dot{C}_{\mathrm{loss}} = \Gamma\, C(\rho(\epsilon)) + O(\epsilon^3)$.*

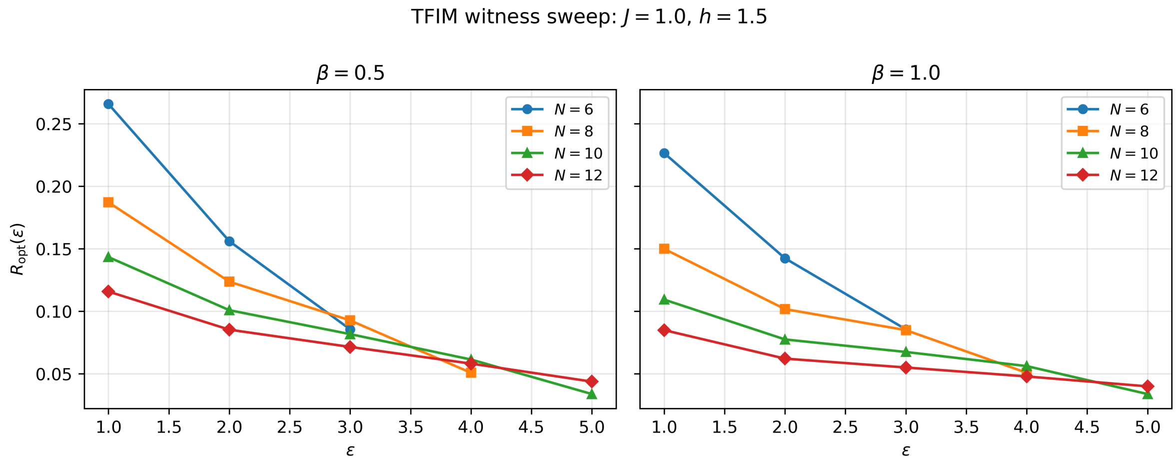

Figure 1: TFIM witness sweep at $J = 1.0$, $h = 1.5$: $R_{\mathrm{opt}}(\epsilon)$ for $N \in \{6, 8, 10, 12\}$, $\beta \in \{0.5, 1.0\}$. Exact diagonalization with energy-block pinching.

7 Verification suite (added in v2)

`verification/2601-0023/verify_2601_0023.py` verifies: Lemma 4.1 and the corrected Lemma 4.3 to machine precision on a TFIM $N = 4$ Davies model; the falsity of the v1 decomposition (deviations up to 0.22); pairwise $\pm\omega$ positivity (0 violations); the one-qubit counterexample of Lemma 4.5 (ratio $= e^{\beta\Delta/4}$ exactly) and the corrected bound (0/300 violations); and the pinning identity of Lemma 4.6 exactly over four decades of δ. An optional flag `--with-tfim` runs the $N = 10$ witness at the declared parameters ($J = 1$, $h = 1.5$, $S = \sigma^z_{j_0}$), for comparison with the companion 2601.0022 [6], whose v1 ED used these parameters.

Note added in v2

Three corrections, all verified in the suite; the witness mechanism and all $\omega = 0$ statements of v1 stand unchanged. (1) *Lemma 4.3 (Bohr decomposition):* the v1 plain-commutator identity is false in general (deviations $\sim 20\%$); the exact form is Lemma 4.3, which reduces to the $\omega = 0$ identity and is pairwise nonnegative – the witness bound (Proposition 4.4) survives with a repaired proof. (2) *Lemmas 4.12–4.13 (suppression):* the v1 proofs used the false KMS submultiplicativity $\|XO\|_{2,\sigma} \leq \|X\|_\infty \|O\|_{2,\sigma}$ (Lemma 4.5 gives the counterexample and the corrected c_σ bound); the corrected constants carry c_σ^2 and $\sup \gamma(\omega) e^{\beta\omega/2}$, and v1's "no $\lambda_{\min}^{-1}$" claim is downgraded per Remark 4.9. The $e^{-2\epsilon/\xi}$ exponent is confirmed by the new positivity-pinning Lemma 4.6, which replaces the incorrect route through the v1 decomposition. (3) Series cross-references added; the appendix script is relocated to `verification/2601-0023/` in the series repository (the inline listing of v1 is superseded).

A Reproducibility

The TFIM figures were generated with `tfim_witness_ropt_all.py --J 1.0 --h 1.5 --eps_max 5 --sweep_N 6 8 10 12 --sweep_beta 0.5 1.0 --single_N 12 --single_beta 1.0`. In v2 the script and the full verification suite live in `verification/2601-0023/` of the series repository; the inline listing of v1 is superseded.

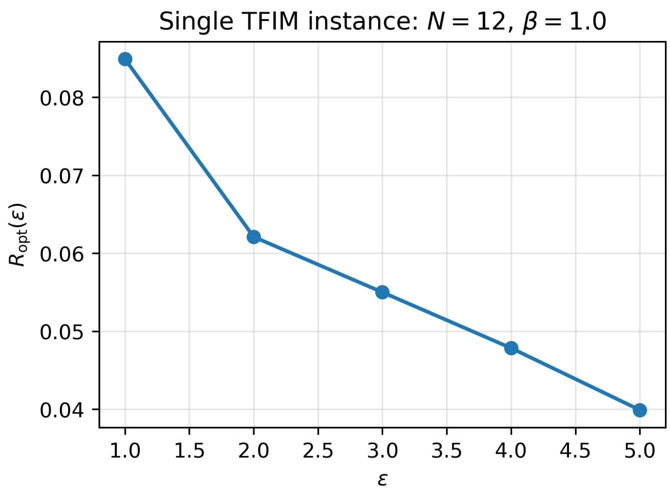

Figure 2: Single instance: $N = 12$, $\beta = 1.0$.

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