

Finite-Dimensional Davies Interface Lemmas and TFIM Witness Tests for Separation-Dependent Decoherence Rate Envelopes

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Abstract

We develop a finite-dimensional technical core for relating separation to effective decoherence-rate envelopes in Davies-type open-system dynamics. We work with energy pinching Δ and quantify coherence by $C(\rho) = S(\rho||\Delta[\rho])$. We import a maintenance inequality $P_{\text{extra}}(\rho) \geq k_B T \dot{C}_{\text{loss}}(\rho)$ as an external input. On the operator side we prove: (i) an exact $\omega = 0$ Dirichlet identity yielding a witness-based lower bound on instantaneous decay envelopes, (ii) a Bohr-block Dirichlet decomposition for a single-channel Davies generator under quantum detailed balance, and (iii) sufficient envelope suppression lemmata under infrared exclusion and quasi-local spectral tails (with explicit constants), including a variant without $\lambda_{\min}(\sigma)^{-1}$. On the state side we give an asymptotic linearization statement on Bohr-block perturbations with fixed diagonal, yielding a direction-dependent effective decay rate. Finite-size TFIM witness diagnostics and reproducibility scripts are provided in an appendix.

Summary of results and assumptions

- **Imported from prior work:** the maintenance inequality in Theorem 2.3 (see [1]).
- **Proved in this paper (finite dimension):** Lemma 4.1, Proposition 4.5, Lemma 4.3 (single-channel Davies, under QDB), Lemma 4.12 and Lemma 4.13, Proposition 6.4 and Corollary 6.5.
- **Finite-size numerics only:** TFIM witness diagnostics in Section 5 and Figures 1–2.
- **No thermodynamic-limit claims:** constants may depend on (N, β) unless stated otherwise; see Remark 4.14.
- **Reproducibility:** Appendix A provides the exact command and a runnable script producing `tfim_ropt_sweep.png` and `tfim_ropt.png`.

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1 Scope, positioning, and non-claims

This manuscript is finite-dimensional and does not make thermodynamic-limit claims unless explicitly stated. It does not propose modified quantum dynamics, collapse mechanisms, derivations of the Born rule, or ontological resolutions of the measurement problem.

Related work (contrast). For contrast, there are works developing theorem-level implication frameworks relating decay of correlations and log-Sobolev-type functional inequalities (and, in some settings, Gibbs-state locality notions); see [4, 5]. Our aim here is different: we isolate a finite-dimensional “operator/state interface” around Davies-type Dirichlet forms and a witness floor, and we provide a reproducible finite-size TFIM diagnostic, without asserting thermodynamic-limit equivalences.

2 Finite-dimensional operational core

2.1 Thermal state, pinching, and coherence

Let \mathcal{H}_S be finite-dimensional and let

$$H_S = \sum_n E_n \Pi_n$$

be the spectral decomposition. Fix bath temperature T and $\beta := 1/(k_B T)$, and define the Gibbs state

$$\sigma := \frac{e^{-\beta H_S}}{\text{Tr}(e^{-\beta H_S})}.$$

Define energy pinching

$$\Delta[\rho] := \sum_n \Pi_n \rho \Pi_n,$$

and relative-entropy coherence

$$C(\rho) := S(\rho \| \Delta[\rho]).$$

We use the Umegaki relative entropy

$$S(\rho \| \tau) := \text{Tr}[\rho(\log \rho - \log \tau)],$$

with the standard support convention. When differentiability is required (e.g. for \dot{C}_{loss}), we assume $\rho \succ 0$.

Remark 2.1 (Norm conventions). We use $\|X\|_2 := \sqrt{\text{Tr}(X^\dagger X)}$ for the Hilbert–Schmidt (Frobenius) norm, and $\|X\|_\infty$ for the operator norm.

Remark 2.2 (Degeneracies and energy-block pinching). If H_S has degeneracies, then the projectors Π_n may have rank > 1 . Accordingly, Δ is an energy-*block* pinching (not necessarily elementwise diagonalization), and the Bohr-zero component $S(0) = \sum_n \Pi_n S \Pi_n$ is block-diagonal on degenerate energy subspaces. The TFIM numerics in Section 5 implement this block pinching (grouping energies within tolerance 10^{-10}), so $S(0)$ is invariant under basis changes inside degenerate eigenspaces.

2.2 Uncontrolled Markovian dynamics and coherence-loss rate

Let the uncontrolled evolution be a Markovian semigroup $\rho_t = e^{t\mathcal{L}}(\rho)$ with GKLS generator \mathcal{L} and stationary Gibbs state σ , i.e. $\mathcal{L}(\sigma) = 0$. Define the instantaneous coherence-loss rate

$$\dot{C}_{\text{loss}}(\rho) := - \left. \frac{d}{dt} C(\rho_t) \right|_{t=0}.$$

2.3 Control model and incremental (extra) power

We assume battery-assisted thermal operations (BATOs) at temperature T . Let $\text{Ctrl}(\rho)$ denote the set of BATO strategies that maintain ρ , and $\text{Ctrl}(\Delta[\rho])$ those that maintain only $\Delta[\rho]$. Define paired-strategy incremental power

$$P_{\text{extra}}(\rho) := \inf_{s \in \text{Ctrl}(\rho), s_{\text{diag}} \in \text{Ctrl}(\Delta[\rho])} \left(P(s) - P(s_{\text{diag}}) \right).$$

2.4 Imported maintenance inequality (input)

Theorem 2.3 (Imported maintenance inequality [1]). *Assume $\text{Ctrl}(\rho) \neq \emptyset$ and $\text{Ctrl}(\Delta[\rho]) \neq \emptyset$. Then*

$$P_{\text{extra}}(\rho) \geq k_B T \dot{C}_{\text{loss}}(\rho).$$

3 Davies generators, KMS Dirichlet form, and a separation envelope

3.1 Davies form and Bohr components

Fix a system coupling operator $S = S^\dagger$ and a Davies-type generator in the weak-coupling–secular limit. Let $S(\omega)$ denote Bohr components:

$$S(\omega) := \sum_{E_m - E_n = \omega} \Pi_m S \Pi_n, \quad S = \sum_{\omega} S(\omega), \quad S(\omega)^\dagger = S(-\omega).$$

Bath rates $\gamma(\omega)$ satisfy KMS:

$$\gamma(-\omega) = e^{-\beta\omega} \gamma(\omega).$$

3.2 KMS inner product and Dirichlet form

Define the KMS (GNS) inner product

$$\langle A, B \rangle_\sigma := \text{Tr} \left(\sigma^{1/2} A^\dagger \sigma^{1/2} B \right), \quad \|A\|_{2,\sigma}^2 := \langle A, A \rangle_\sigma,$$

and the Dirichlet form

$$\mathcal{E}_\sigma(O) := -\Re \left\langle O, \mathcal{L}^\dagger(O) \right\rangle_\sigma.$$

3.3 Separation envelope

Fix a coupling region Ω_0 and graph distance $d(\cdot, \cdot)$. For integer $\epsilon \geq 1$, define the far region

$$\Omega_\epsilon := \{j : d(j, \Omega_0) \geq \epsilon\}.$$

Let \mathcal{A}_ϵ denote the linear subspace of observables supported on Ω_ϵ . Define the effective instantaneous envelope

$$\kappa(\epsilon) := \sup_{O \in \mathcal{A}_\epsilon, O \neq 0} \frac{\mathcal{E}_\sigma(O)}{\|O\|_{2,\sigma}^2}.$$

4 Interface lemmas

4.1 The $\omega = 0$ channel yields a witness floor

Let $S(0) := \sum_n \Pi_n S \Pi_n$ be the $\omega = 0$ Bohr component. Assume $\gamma(0) > 0$. Let $\mathcal{E}_\sigma^{(0)}$ denote the Dirichlet-form contribution associated with the $\omega = 0$ Davies channel.

Lemma 4.1 (Exact $\omega = 0$ Dirichlet identity). *Assume $S(0)$ commutes with H_S (hence with σ). If the $\omega = 0$ Davies channel is*

$$\mathcal{L}_0^\dagger(O) = \gamma(0) \left(S(0)OS(0) - \frac{1}{2}\{S(0)^2, O\} \right),$$

then for all O ,

$$\mathcal{E}_\sigma^{(0)}(O) = \frac{\gamma(0)}{2} \|[S(0), O]\|_{2,\sigma}^2.$$

Proof. Let $L := S(0)$. By assumption, $[L, \sigma] = 0$, hence also $[L, \sigma^{1/2}] = 0$. Write

$$\mathcal{L}_0^\dagger(O) = \gamma(0) \left(LOL - \frac{1}{2}\{L^2, O\} \right).$$

Then

$$\mathcal{E}_\sigma^{(0)}(O) = -\Re \left\langle O, \mathcal{L}_0^\dagger(O) \right\rangle_\sigma = -\gamma(0) \Re \left(\langle O, LOL \rangle_\sigma - \frac{1}{2} \langle O, L^2 O \rangle_\sigma - \frac{1}{2} \langle O, OL^2 \rangle_\sigma \right).$$

Since $[L, \sigma^{1/2}] = 0$, we have $\langle A, LB \rangle_\sigma = \langle LA, B \rangle_\sigma$ for all A, B . Using this to move L between slots and expanding $\langle [L, O], [L, O] \rangle_\sigma$ gives

$$\langle [L, O], [L, O] \rangle_\sigma = 2\Re \left(\langle O, L^2 O \rangle_\sigma - \langle O, LOL \rangle_\sigma \right).$$

Substituting into the expression for $\mathcal{E}_\sigma^{(0)}(O)$ yields

$$\mathcal{E}_\sigma^{(0)}(O) = \frac{\gamma(0)}{2} \langle [L, O], [L, O] \rangle_\sigma = \frac{\gamma(0)}{2} \|[S(0), O]\|_{2,\sigma}^2,$$

as claimed. □

4.2 Bohr-block Dirichlet decomposition

Definition 4.2 (Quantum detailed balance (KMS symmetry)). A quantum Markov semigroup with Heisenberg generator \mathcal{L}^\dagger satisfies quantum detailed balance w.r.t. σ if

$$\left\langle A, \mathcal{L}^\dagger(B) \right\rangle_\sigma = \left\langle \mathcal{L}^\dagger(A), B \right\rangle_\sigma \quad \text{for all } A, B.$$

Lemma 4.3 (Bohr-block Dirichlet decomposition (single-channel Davies)). *Assume quantum detailed balance with respect to σ and assume a single-channel Davies generator in Heisenberg form*

$$\mathcal{L}^\dagger(O) = \sum_\omega \gamma(\omega) \left(S(\omega)^\dagger OS(\omega) - \frac{1}{2}\{S(\omega)^\dagger S(\omega), O\} \right),$$

with $S(\omega)^\dagger = S(-\omega)$ and KMS rates $\gamma(-\omega) = e^{-\beta\omega}\gamma(\omega)$. Then

$$\mathcal{E}_\sigma(O) = \frac{1}{2} \sum_\omega \gamma(\omega) \|[S(\omega), O]\|_{2,\sigma}^2.$$

Proof. Define the derivations $\partial_\omega(O) := [S(\omega), O]$ and let $\partial_{\omega,\sigma}^\dagger$ denote the adjoint with respect to the KMS/GNS inner product $\langle \cdot, \cdot \rangle_\sigma$, i.e.

$$\langle A, \partial_\omega(B) \rangle_\sigma = \left\langle \partial_{\omega,\sigma}^\dagger(A), B \right\rangle_\sigma \quad \text{for all } A, B.$$

Under quantum detailed balance, Davies generators admit a divergence-form representation in the KMS calculus (see, e.g., [5]). In particular, in finite dimension and under GNS symmetry (Definition 4.2), the generator can be written in KMS divergence form, namely

$$\mathcal{L}^\dagger(O) = -\frac{1}{2} \sum_\omega \gamma(\omega) \partial_{\omega,\sigma}^\dagger \partial_\omega(O).$$

Therefore, by the corresponding KMS integration-by-parts (carré-du-champ) identity,

$$\mathcal{E}_\sigma(O) = -\Re \left\langle O, \mathcal{L}^\dagger(O) \right\rangle_\sigma = \frac{1}{2} \sum_\omega \gamma(\omega) \langle \partial_\omega(O), \partial_\omega(O) \rangle_\sigma = \frac{1}{2} \sum_\omega \gamma(\omega) \| [S(\omega), O] \|_{2,\sigma}^2,$$

which is the claimed decomposition (with the prefactor 1/2 consistent with Remark 4.4). \square

Remark 4.4 (Convention for the factor 1/2). With the present choice of Heisenberg generator, KMS inner product $\langle \cdot, \cdot \rangle_\sigma$, and Dirichlet form $\mathcal{E}_\sigma(O) := -\Re \langle O, \mathcal{L}^\dagger(O) \rangle_\sigma$, the decomposition in Lemma 4.3 carries exactly the prefactor 1/2.

4.3 Witness floor

Proposition 4.5 (Witness lower bound at fixed separation). *Assume the Davies/QDB setting of Lemma 4.3. Fix $\epsilon_0 \geq 1$. If there exists $O_{\epsilon_0} \in \mathcal{A}_{\epsilon_0}$ such that*

$$\frac{\| [S(0), O_{\epsilon_0}] \|_{2,\sigma}^2}{\| O_{\epsilon_0} \|_{2,\sigma}^2} \geq c_{\epsilon_0} > 0,$$

then

$$\kappa(\epsilon_0) \geq \frac{\gamma(0)}{2} c_{\epsilon_0}.$$

Proof. By definition, $\kappa(\epsilon_0) \geq \mathcal{E}_\sigma(O_{\epsilon_0}) / \| O_{\epsilon_0} \|_{2,\sigma}^2$. By Lemma 4.3, \mathcal{E}_σ is a sum of nonnegative Bohr-frequency contributions; its $\omega = 0$ summand coincides with $\mathcal{E}_\sigma^{(0)}$, hence $\mathcal{E}_\sigma(O_{\epsilon_0}) \geq \mathcal{E}_\sigma^{(0)}(O_{\epsilon_0})$. Now apply Lemma 4.1. \square

4.4 Envelope suppression lemmata

Definition 4.6 (Infrared exclusion). We say the coupling satisfies IR exclusion at scale $\omega^* > 0$ if, for all $|\omega| < \omega^*$, the corresponding Davies term is absent (either $S(\omega) = 0$ or $\gamma(\omega) = 0$ on that band).

Definition 4.7 (Tensor lattice and partial trace). Let Λ be a finite lattice with local Hilbert space dimension d_{loc} and observable algebra

$$\mathcal{A}(\Lambda) \cong \bigotimes_{j \in \Lambda} M_{d_{\text{loc}}}(\mathbb{C}).$$

For $Y \subseteq \Lambda$, $\text{Tr}_Y : \mathcal{A}(\Lambda) \rightarrow \mathcal{A}(\Lambda \setminus Y)$ denotes the (non-normalized) partial trace over the factor Y .

Definition 4.8 (Support). An observable $O \in \mathcal{A}(\Lambda)$ is supported on $X \subseteq \Lambda$ if $O = O_X \otimes \mathbf{1}_{\Lambda \setminus X}$ for some $O_X \in \mathcal{A}(X)$.

Definition 4.9 (Tail map on tensor lattices). Let Ω_ϵ be the far region and write $\Lambda_{<\epsilon} := \Lambda \setminus \Omega_\epsilon$. Let $d_{\Omega_\epsilon} := d_{\text{loc}}^{|\Omega_\epsilon|}$. Define the trace-preserving conditional expectation onto the near algebra by

$$E_{<\epsilon}(X) := \text{Tr}_{\Omega_\epsilon}(X) \otimes \frac{1}{d_{\Omega_\epsilon}} \mathbf{1}_{\Omega_\epsilon},$$

and define $\text{tail}_\epsilon(X) := X - E_{<\epsilon}(X)$.

Lemma 4.10 (Disjoint support implies commutation). *If A is supported on X and B is supported on Y with $X \cap Y = \emptyset$, then $[A, B] = 0$.*

Proof. Write $A = A_X \otimes \mathbf{1}_{\Lambda \setminus X}$ and $B = B_Y \otimes \mathbf{1}_{\Lambda \setminus Y}$ with $X \cap Y = \emptyset$. Then $AB = BA$ by tensor-factor commutativity, hence $[A, B] = 0$. \square

Lemma 4.11 (Norm comparison). *Write $\lambda_{\min}(\sigma)$ for the smallest eigenvalue of σ . Then for all operators X ,*

$$\|X\|_2^2 \leq \lambda_{\min}(\sigma)^{-1} \|X\|_{2,\sigma}^2.$$

In particular,

$$\|X\|_2 \leq \lambda_{\min}(\sigma)^{-1/2} \|X\|_{2,\sigma}.$$

Proof. Since $\sigma \succeq \lambda_{\min}(\sigma) \mathbf{1}$,

$$\|X\|_{2,\sigma}^2 = \text{Tr}(\sigma^{1/2} X^\dagger \sigma^{1/2} X) \geq \lambda_{\min}(\sigma) \text{Tr}(X^\dagger X) = \lambda_{\min}(\sigma) \|X\|_2^2.$$

Taking square roots gives the second claim. \square

Lemma 4.12 (Sufficient envelope suppression from KMS tails (explicit constant)). *Assume IR exclusion at $\omega^* > 0$ and suppose there exist constants $A > 0$, $\xi > 0$, and $p \geq 0$ such that for all $|\omega| \geq \omega^*$,*

$$\|\text{tail}_\epsilon(S(\omega))\|_{2,\sigma} \leq A(1 + \epsilon)^p e^{-\epsilon/\xi}.$$

Assume also $\sup_{|\omega| \geq \omega^} \gamma(\omega) < \infty$ and let*

$$\Omega_{\geq \omega^*} := \{\omega : |\omega| \geq \omega^*, S(\omega) \neq 0\}.$$

Then for each fixed (N, β) ,

$$\kappa(\epsilon) \leq C(1 + \epsilon)^{2p} e^{-2\epsilon/\xi},$$

with

$$C := 2 |\Omega_{\geq \omega^*}| \left(\sup_{|\omega| \geq \omega^*} \gamma(\omega) \right) \lambda_{\min}(\sigma)^{-1} A^2.$$

Proof sketch. By Lemma 4.3,

$$\mathcal{E}_\sigma(O) = \frac{1}{2} \sum_{\omega \in \Omega_{\geq \omega^*}} \gamma(\omega) \|[S(\omega), O]\|_{2,\sigma}^2.$$

For O supported on the far region Ω_ϵ , write $S(\omega) = E_{<\epsilon}[S(\omega)] + \text{tail}_\epsilon(S(\omega))$. The near term commutes with O by Lemma 4.10, so $[S(\omega), O] = [\text{tail}_\epsilon(S(\omega)), O]$. Then

$$\|[X, O]\|_{2,\sigma} \leq \|XO\|_{2,\sigma} + \|OX\|_{2,\sigma} \leq 2 \|X\|_\infty \|O\|_{2,\sigma}.$$

Finally, use $\|X\|_\infty \leq \|X\|_2$ and Lemma 4.11, insert the tail hypothesis, bound $\gamma(\omega)$ by its supremum, and sum. \square

Variants. The following lemma avoids the factor $\lambda_{\min}(\sigma)^{-1}$ under operator-norm tail control.

Lemma 4.13 (Envelope suppression from operator-norm tails (no $\lambda_{\min}(\sigma)$)). *Assume the setting of Lemma 4.12, but replace the $\|\cdot\|_{2,\sigma}$ tail hypothesis by an operator-norm bound: suppose there exist constants $A > 0$, $\xi > 0$, and $p \geq 0$ such that for all $|\omega| \geq \omega^*$,*

$$\|\text{tail}_\epsilon(S(\omega))\|_\infty \leq A(1 + \epsilon)^p e^{-\epsilon/\xi}.$$

Then for each fixed (N, β) and all $\epsilon \geq 1$,

$$\kappa(\epsilon) \leq C_\infty(1 + \epsilon)^{2p} e^{-2\epsilon/\xi},$$

with

$$C_\infty := 2 |\Omega_{\geq \omega^*}| \left(\sup_{|\omega| \geq \omega^*} \gamma(\omega) \right) A^2.$$

Proof sketch. Proceed as in Lemma 4.12 but estimate directly

$$\|[X, O]\|_{2,\sigma} \leq 2 \|X\|_\infty \|O\|_{2,\sigma},$$

so no conversion through $\lambda_{\min}(\sigma)$ is required. \square

Remark 4.14 (Scaling of prefactors with (N, β)). Lemma 4.12 and Lemma 4.13 offer complementary trade-offs:

- (A) (*KMS-tail version, Lemma 4.12*) The prefactor contains $\lambda_{\min}(\sigma)^{-1}$, which can scale as $e^{\beta O(N)}$ in local spin models.
- (B) (*Operator-tail version, Lemma 4.13*) The prefactor C_∞ is independent of $\lambda_{\min}(\sigma)$, but requires the stronger hypothesis $\|\text{tail}_\epsilon(S(\omega))\|_\infty \leq A(1 + \epsilon)^p e^{-\epsilon/\xi}$.

5 TFIM witness diagnostics (finite size)

We consider the transverse-field Ising chain (open boundary conditions) on N qubits (so $d_{\text{loc}} = 2$):

$$H_S = -J \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x - h \sum_{i=1}^N \sigma_i^z, \quad h > J.$$

We take coupling site $j_0 = \lfloor N/2 \rfloor$ and coupling operator $S = \sigma_{j_0}^z$. We compute σ by exact diagonalization and evaluate the witness ratio

$$R(\epsilon; O) := \frac{\|[S(0), O]\|_{2,\sigma}^2}{\|O\|_{2,\sigma}^2}, \quad R_{\text{opt}}(\epsilon) := \max_{O \in \text{span}\{\sigma_k^x, \sigma_k^y, \sigma_k^z\}} R(\epsilon; O),$$

for $k = j_0 \pm \epsilon$.

By Proposition 4.5,

$$\kappa(\epsilon) \geq \frac{\gamma(0)}{2} R_{\text{opt}}(\epsilon).$$

6 State-side linearization (finite dimension)

Definition 6.1 (BKM inner product). For a faithful state $\tau \succ 0$, define the Bogoliubov–Kubo–Mori (BKM) inner product

$$\langle A, B \rangle_{\text{BKM}, \tau} := \int_0^1 \text{Tr} \left(\tau^s A^\dagger \tau^{1-s} B \right) ds.$$

TFIM witness sweep: $J = 1.0, h = 1.5$

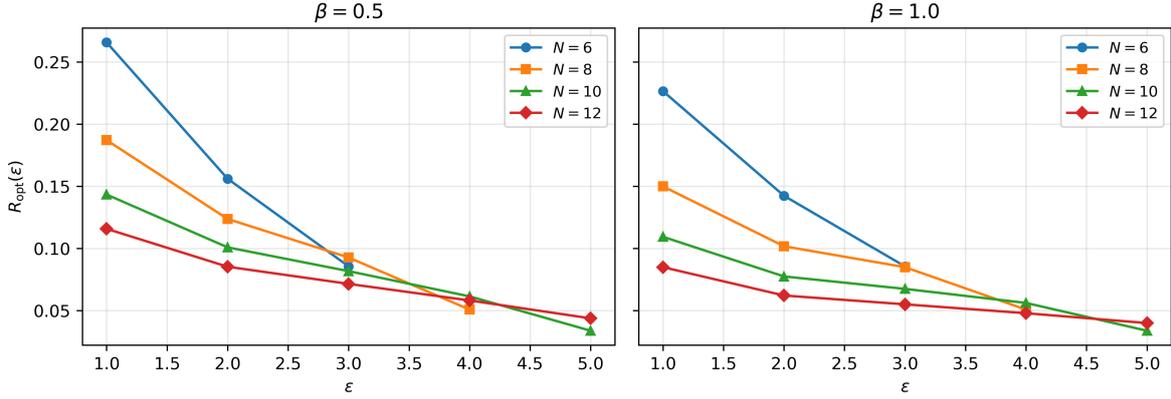


Figure 1: TFIM witness sweep at $J = 1.0, h = 1.5$: $R_{\text{opt}}(\epsilon)$ versus ϵ for $N \in \{6, 8, 10, 12\}$ and $\beta \in \{0.5, 1.0\}$ (panels). Exact diagonalization with energy-block pinching (Remark 2.2).

Lemma 6.2 (Quadratic expansion of relative entropy). *Let $\tau \succ 0$ and let $X = X^\dagger$ satisfy $\text{Tr}(X) = 0$. For sufficiently small real ϵ , define $\rho(\epsilon) := \tau + \epsilon X \succ 0$. Then*

$$S(\rho(\epsilon) \parallel \tau) = \frac{\epsilon^2}{2} \langle X, X \rangle_{\text{BKM}, \tau} + O(\epsilon^3).$$

Definition 6.3 (Bohr-block subspace). Let $H_S = \sum_n E_n \Pi_n$ and let Δ be pinching in the $\{\Pi_n\}$ basis. For a set Ω of Bohr frequencies, define

$$\mathcal{X}_\Omega := \left\{ X = X^\dagger : \Delta[X] = 0, \Pi_m X \Pi_n = 0 \text{ if } E_m - E_n \notin \Omega \right\}.$$

Proposition 6.4 (Asymptotic interface on a Bohr block (general direction)). *Let $\tau \succ 0$ be diagonal in the energy basis of H_S , and let Δ denote pinching in that basis. Assume $\mathcal{L}(\tau) = 0$ and secular block preservation:*

$$\mathcal{L}(\mathcal{X}_\Omega) \subseteq \mathcal{X}_\Omega.$$

Let $X \in \mathcal{X}_\Omega$ satisfy $\text{Tr}(X) = 0$, and define $\rho(\epsilon) := \tau + \epsilon X$ for $|\epsilon|$ small. Then, as $\epsilon \rightarrow 0$,

$$C(\rho(\epsilon)) = S(\rho(\epsilon) \parallel \Delta[\rho(\epsilon)]) = S(\rho(\epsilon) \parallel \tau) = \frac{\epsilon^2}{2} \langle X, X \rangle_{\text{BKM}, \tau} + O(\epsilon^3),$$

and

$$\dot{C}_{\text{loss}}(\rho(\epsilon)) = -\epsilon^2 \Re \langle X, \mathcal{L}(X) \rangle_{\text{BKM}, \tau} + O(\epsilon^3).$$

For $X \neq 0$, define the direction-dependent effective decay rate

$$\Gamma_{\text{eff}}(X) := \frac{-2 \Re \langle X, \mathcal{L}(X) \rangle_{\text{BKM}, \tau}}{\langle X, X \rangle_{\text{BKM}, \tau}}.$$

Then

$$\frac{\dot{C}_{\text{loss}}(\rho(\epsilon))}{C(\rho(\epsilon))} = \Gamma_{\text{eff}}(X) + O(\epsilon).$$

Corollary 6.5 (Eigenmode case as a special instance). *In the setting of Proposition 6.4, if in addition*

$$\mathcal{L}(X) = -\frac{\Gamma}{2} X \quad \text{for some } \Gamma \geq 0,$$

then $\Gamma_{\text{eff}}(X) = \Gamma$ and

$$\dot{C}_{\text{loss}}(\rho(\epsilon)) = \Gamma C(\rho(\epsilon)) + O(\epsilon^3).$$

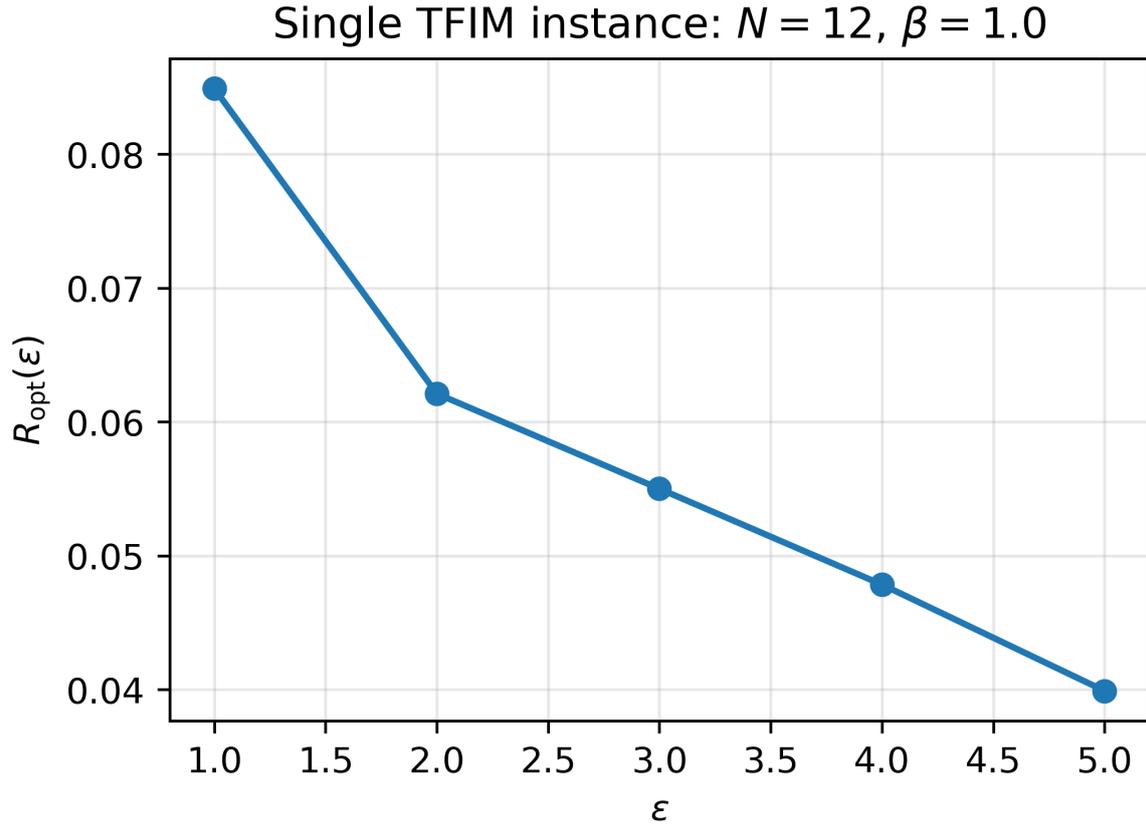


Figure 2: Single TFIM instance at $J = 1.0, h = 1.5$: $R_{\text{opt}}(\epsilon)$ versus ϵ for $N = 12, \beta = 1.0$. Exact diagonalization with energy-block pinching (Remark 2.2).

A Reproducibility

Tested in Google Colab with Python 3, NumPy, SciPy, Matplotlib.

A.1 Commands used to generate the included PNG figures

The following command generates both `tfim_ropt_sweep.png` and `tfim_ropt.png` (and also PDF copies).

Listing 1: Commands used to generate `tfim_ropt_sweep.png` and `tfim_ropt.png`

```
python3 tfim_witness_ropt_all.py --J 1.0 --h 1.5 --eps_max 5 \
  --sweep_N 6 8 10 12 --sweep_beta 0.5 1.0 \
  --single_N 12 --single_beta 1.0
```

A.2 Script used (complete)

Place the file `tfim_witness_ropt_all.py` in the same directory as this `p7.tex` file to reproduce the figures. For convenience, we include the exact script below.

Listing 2: Script `tfim_witness_ropt_all.py` used to generate the TFIM figures (PNG)

```
#!/usr/bin/env python3
import argparse
import time
import numpy as np
```

```

import scipy.linalg as la

def paulis():
    I = np.array([[1, 0], [0, 1]], dtype=complex)
    X = np.array([[0, 1], [1, 0]], dtype=complex)
    Y = np.array([[0, -1j], [1j, 0]], dtype=complex)
    Z = np.array([[1, 0], [0, -1]], dtype=complex)
    return I, X, Y, Z

def kron_all(ops):
    out = ops[0]
    for op in ops[1:]:
        out = np.kron(out, op)
    return out

def local_pauli(N, site, which):
    I, X, Y, Z = paulis()
    P = {"X": X, "Y": Y, "Z": Z}[which.upper()]
    ops = [I] * N
    ops[site] = P
    return kron_all(ops)

def two_site_xx(N, i):
    I, X, Y, Z = paulis()
    ops = [I] * N
    ops[i] = X
    ops[i+1] = X
    return kron_all(ops)

def tfim_hamiltonian(N, J, h):
    dim = 2*N
    H = np.zeros((dim, dim), dtype=complex)
    for i in range(N - 1):
        H += -J * two_site_xx(N, i)
    for i in range(N):
        H += -h * local_pauli(N, i, "Z")
    return (H + H.conj().T) / 2

def energy_blocks(E, tol=1e-10):
    E = np.array(E.real, dtype=float)
    order = np.argsort(E)
    blocks = []
    cur = [order[0]]
    for idx in order[1:]:
        if abs(E[idx] - E[cur[0]]) <= tol:
            cur.append(idx)
        else:
            blocks.append(cur)
            cur = [idx]
    blocks.append(cur)
    return blocks

def SO_component_block(S, E, V, tol=1e-10):
    S_E = V.conj().T @ S @ V
    SO_E = np.zeros_like(S_E)
    for blk in energy_blocks(E, tol=tol):
        ii = np.ix_(blk, blk)
        SO_E[ii] = S_E[ii]

```

```

S0 = V @ S0_E @ V.conj().T
return (S0 + S0.conj().T) / 2

def sigma_sqrt_from_spectrum(E, V, beta):
    Emin = float(np.min(E.real))
    w = np.exp(-beta * (E.real - Emin))
    p = w / np.sum(w)
    sqrtp = np.sqrt(p)
    return (V * sqrtp) @ V.conj().T

def kms_norm_sq_fast(X, sigma_sqrt):
    Y = sigma_sqrt @ X
    return float(np.trace(Y.conj().T @ Y).real)

def R_opt_at_epsilon(N, S0, sigma_sqrt, j0, eps):
    best_R = 0.0
    for side in (+1, -1):
        k = j0 + side * eps
        if k < 0 or k >= N:
            continue
        for pauli in ("X", "Y", "Z"):
            O = local_pauli(N, k, pauli)
            comm = S0 @ O - O @ S0
            den = kms_norm_sq_fast(O, sigma_sqrt)
            if den <= 1e-14:
                continue
            num = kms_norm_sq_fast(comm, sigma_sqrt)
            best_R = max(best_R, num / den)
    return best_R

def compute_ropt_curve(N, J, h, beta, eps_max, degen_tol, cache):
    j0 = N // 2
    if N not in cache:
        print(f"Diagonalizing H for N={N} (dim={2**N}) ...")
        H = tfim_hamiltonian(N, J, h)
        t0 = time.time()
        E, V = la.eigh(H, check_finite=False)
        print(f" eigh done in {time.time()-t0:.2f}s")
        S = local_pauli(N, j0, "Z")
        S0 = S0_component_block(S, E, V, tol=degen_tol)
        cache[N] = (E, V, S0)
    E, V, S0 = cache[N]
    sigma_sqrt = sigma_sqrt_from_spectrum(E, V, beta)
    max_eps = min(eps_max, max(j0, N - 1 - j0))
    eps_list = list(range(1, max_eps + 1))
    ropt_list = [R_opt_at_epsilon(N, S0, sigma_sqrt, j0, eps) for eps in eps_list]
    return eps_list, ropt_list

def plot_two_panel(results, beta_list, N_list, J, h, savepng, savepdf=None):
    import matplotlib.pyplot as plt
    fig, axes = plt.subplots(1, len(beta_list), figsize=(10, 4), sharey=True)
    if len(beta_list) == 1:
        axes = [axes]
    markers = ['o', 's', '^', 'D', 'v', 'P']
    colors = ['#1f77b4', '#ff7f0e', '#2ca02c', '#d62728', '#9467bd', '#8c564b']
    for ax, beta in zip(axes, beta_list):
        for i, N in enumerate(N_list):
            eps_list, ropt_list = results[(beta, N)]

```

```

        ax.plot(eps_list, ropt_list,
                marker=markers[i % len(markers)],
                color=colors[i % len(colors)],
                label=rf"$N={N}$",
                linewidth=1.5, markersize=5)
    ax.set_xlabel(r"$\epsilon$")
    ax.set_title(rf"$\beta={beta}$")
    ax.grid(True, alpha=0.3)
    ax.legend(loc="best", fontsize=9)
    axes[0].set_ylabel(r"$R_{\mathrm{opt}}(\epsilon)$")
    fig.suptitle(rf"TFIM witness sweep: $J={J}$, $h={h}$", fontsize=12)
    plt.tight_layout()
    plt.savefig(savepng, dpi=450, bbox_inches="tight")
    if savepdf:
        plt.savefig(savepdf, bbox_inches="tight")
    print(f"Saved: {savepng}" + (f", {savepdf}" if savepdf else ""))

def plot_single(eps_list, ropt_list, N, beta, savepng, savepdf=None):
    import matplotlib.pyplot as plt
    plt.figure(figsize=(4.8, 3.6))
    plt.plot(eps_list, ropt_list, marker="o", linewidth=1.8)
    plt.grid(True, alpha=0.3)
    plt.xlabel(r"$\epsilon$")
    plt.ylabel(r"$R_{\mathrm{opt}}(\epsilon)$")
    plt.title(rf"Single TFIM instance: $N={N}$, $\beta={beta}$")
    plt.tight_layout()
    plt.savefig(savepng, dpi=450, bbox_inches="tight")
    if savepdf:
        plt.savefig(savepdf, bbox_inches="tight")
    print(f"Saved: {savepng}" + (f", {savepdf}" if savepdf else ""))

def main():
    ap = argparse.ArgumentParser()
    ap.add_argument("--J", type=float, default=1.0)
    ap.add_argument("--h", type=float, default=1.5)
    ap.add_argument("--eps_max", type=int, default=5)
    ap.add_argument("--degen_tol", type=float, default=1e-10)

    ap.add_argument("--sweep_N", type=int, nargs="+", default=[6, 8, 10, 12])
    ap.add_argument("--sweep_beta", type=float, nargs="+", default=[0.5, 1.0])
    ap.add_argument("--sweep_png", type=str, default="tfim_ropt_sweep.png")
    ap.add_argument("--sweep_pdf", type=str, default="tfim_ropt_sweep.pdf")

    ap.add_argument("--single_N", type=int, default=12)
    ap.add_argument("--single_beta", type=float, default=1.0)
    ap.add_argument("--single_png", type=str, default="tfim_ropt.png")
    ap.add_argument("--single_pdf", type=str, default="tfim_ropt.pdf")

    args = ap.parse_args()
    cache = {}
    results = {}

    for N in args.sweep_N:
        for beta in args.sweep_beta:
            eps_list, ropt_list = compute_ropt_curve(
                N, args.J, args.h, beta, args.eps_max, args.degen_tol, cache
            )
            results[(beta, N)] = (eps_list, ropt_list)

```

```

plot_two_panel(results, args.sweep_beta, args.sweep_N, args.J, args.h, args.
sweep_png, args.sweep_pdf)

eps_list, ropt_list = compute_ropt_curve(
    args.single_N, args.J, args.h, args.single_beta, args.eps_max, args.degen_tol,
    cache
)
plot_single(eps_list, ropt_list, args.single_N, args.single_beta, args.single_png,
args.single_pdf)

if __name__ == "__main__":
    main()

```

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