Geometric Markov Bounds and Rate Inheritance Modulo Fixed Points:

A Scalar Entropic Interface from Static Locality to Davies Dynamics

Version 2

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Abstract

We propose an entropic interface between locality, recoverability, and dynamical decay rates across a geometric collar. The central scalar invariant is the conditional mutual information (CMI) $I_\rho(A:C|B)$, where B is a buffer separating A and C. In finite dimension (Type I algebras), the Fawzi–Renner theorem implies that small CMI yields a quantitative recovery channel acting on B. We formulate a volume-uniform geometric Markov bound with a boundary prefactor, $I_{\rho_\Lambda}(A:C|B) \leq \sigma(\partial B)\, g(w)$, and summarize recent literature inputs establishing exponential CMI decay in shielded/high-temperature regimes. On the dynamical side, we formulate a Rate Inheritance Principle (RIP) for Davies/KMS-symmetric generators: static Markovness across the collar constrains decay rates on the fast sector $\mathcal{F}^\perp$ modulo the fixed-point algebra $\mathcal{F} = \ker \mathcal{L}$ (the $\omega = 0$ floor), with a dynamical input stated as a Poincaré inequality for a local collar Dirichlet form. The only remaining nontrivial link is isolated as an explicit Dirichlet comparison assumption. We verify a diagonal (classical) heat-bath comparison and derive a diagonal subsector corollary with an explicit transfer coefficient. Finally, we define a split reduction datum and a split-regularized CMI target quantity for an AQFT lift and include finite-size illustrations/diagnostics. *v2 corrects two points and adds verification:* (i) the Fawzi–Renner factor under the squared-fidelity convention is $-\log F$, not $-2\log F$ (as in the companion 2512.0101 v2), so the geometric recovery bound reads $1-F \leq \sigma(\partial B) g(w)$; (ii) the v1 bulk–collar comparison for diagonal heat-bath observables is *false as stated* – exact-enumeration counterexamples are exhibited – and is replaced by a proved version with the collar extension taken on the enlarged neighborhood B^{+2r} (commuting-projection argument). A full verification suite (exact enumeration, $N = 9$ Ising-Z) ships with the paper, checking the corrected comparison (0 violations), the transfer constants, the exact vanishing of CMI for the 1D Markov field, and the corrected FR factor on random tripartite states. Throughout, the target RIP theorem remains *conditional* on the bulk–collar comparison assumption; the diagonal heat-bath result verifies only a corrected commuting-subsector comparison, and also exhibits a 1D degeneracy of the transfer constant.

Contents

1 Introduction

Many hard problems in quantum field theory and mathematical physics are not blocked by "more calculation" but by the lack of a representation in which the controlling structure is manifest. For locality-to-dynamics questions, we propose a convenient such representation in terms of a single scalar entropic quantity: the conditional mutual information across a geometric collar, for a tripartition A–B–C where B separates A and C and has collar width w: $I_\rho(A : C|B)$. In our setting, the relevant global constraint is approximate quantum Markovness across a collar, quantified by small CMI.

In finite dimension (Type I algebras), small CMI implies quantitative recoverability via the Fawzi–Renner theorem. We then ask a dynamical question: does static recoverability constrain decay rates under Davies/KMS-symmetric dynamics? The intrinsic obstruction is the fixed-point algebra $\mathcal{F} = \ker \mathcal{L}$ (the $\omega = 0$ floor). Accordingly, any rate-inheritance statement must be formulated on the fast sector $\mathcal{F}^\perp$ (or assume primitivity).

In AQFT (Type III algebras), von Neumann entropies are not directly available. We therefore introduce a split reduction datum at collar width w and define a split-regularized CMI $I_\omega^{\mathrm{split},w}(A : C|B)$ on the induced Type I model. Quantitative split-stability bounds are deferred.

2 Related work and context

CMI and recovery. The quantitative recovery theorem of Fawzi–Renner connects small CMI to the existence of recovery maps; see [2]. **Gibbs states and local Markovness.** Recent work constructs quasi-local recovery maps for quantum Gibbs states in shielded geometries and derives exponential decay of CMI with shielding distance; see [4]. **High-temperature CMI via belief propagation.** Belief-propagation-channel methods yield CMI clustering bounds in high-temperature regimes; see [5]. **Dissipative Gibbs states.** Related results on CMI in decohered Gibbs settings appear in [6].

Series context (added in v2). This note is the bridge between the two blocks of the coherence-maintenance series: the static/entropic side (clustering–recovery: 2512.0060 [7], its finite-mode Gaussian core 2601.0007 [9], and the non-Gaussian CMI route 2512.0101 [8], whose v2 fixes the same FR factor corrected here) and the dynamical side (RIP: formulated in 2512.0072 [10], stress-tested in the Davies class in 2512.0070 [11], proved exactly in the quasi-free local-sink class in 2512.0064 [12]). The $\omega = 0$ floor of Section 5 is the same fixed-point obstruction that drives the maintenance floors of that block.

3 Framework and first lemma (Type I baseline; AQFT lift as a construction)

3.1 Type I baseline: algebras, CMI, fidelity

We work in finite dimension. Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, $\mathcal{A}_{ABC} = \mathcal{B}(\mathcal{H}_{ABC})$. Let $\rho_{ABC} > 0$ be a full-rank state. All logarithms are natural. With $S(\sigma) := -\operatorname{Tr}(\sigma \log \sigma)$, define

$$I_\sigma(A : C|B) := S(\sigma_{AB}) + S(\sigma_{BC}) - S(\sigma_B) - S(\sigma_{ABC}).$$

We use Uhlmann fidelity in the squared convention $F(\sigma, \tau) := \|\sqrt{\sigma}\sqrt{\tau}\|_1^2$.

3.2 Recoverability from small CMI (Fawzi–Renner)

Lemma 3.1 (Fawzi–Renner recovery bound (finite dimension; factor corrected in v2))**.** *Let ρ_{ABC} be full-rank. There exists a CPTP map $\mathcal{R}_{B\to BC}$ (acting on B, possibly nonlocal within B) such that*

$$I_\rho(A : C|B) \;\geq\; -\log F\big(\rho_{ABC}, (\mathrm{id}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AB})\big).$$

In particular, if $I_\rho(A : C|B) \leq \delta$, then for some CPTP $\mathcal{R}_{B\to BC}$,

$$F \geq e^{-\delta}, \qquad 1 - F \leq 1 - e^{-\delta} \leq \delta.$$

Remark 3.2 (Correction note). v1 stated $I \geq -2\log F$ with F in the squared convention; the factor 2 belongs to the root fidelity $f = \sqrt{F}$ ($I \geq -2\log f$). This is the same correction applied in 2512.0101 v2 [8]. The verification suite (Section 7) checks on random tripartite states that the corrected form holds for the Petz recovery (40/40 draws) while the v1 factor fails on 39/40 draws.

3.3 Static interface axiom (geometric Markov bound)

Axiom 3.3 (Uniform geometric Markov bound with boundary prefactor)**.** *Fix a model class $\mathcal{C} = \{\rho_\Lambda\}_\Lambda$ and a boundary functional $\sigma(\partial B) \geq 0$ (e.g. on $\mathbb{Z}^d$, $\sigma(\partial B) = |\partial B|$). There exists $g : \mathbb{N} \to [0, \infty)$ such that for every finite volume Λ and admissible triple $(A, B, C) \subset \Lambda$ with collar width w, $I_{\rho_\Lambda}(A : C|B) \leq \sigma(\partial B)\, g(w)$, with g depending only on microscopic parameters and not on $|\Lambda|$.*

Corollary 3.4 (Geometric recovery bound (constant corrected in v2)). *Under Axiom 3.3, for $\rho_{ABC} := (\rho_\Lambda)_{ABC}$ there exists a CPTP map $\mathcal{R}_{B\to BC}$ acting on B such that*

$$F\big(\rho_{ABC}, (\mathrm{id}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AB})\big) \geq \exp\big(-\sigma(\partial B)\, g(w)\big), \qquad 1 - F \leq \sigma(\partial B)\, g(w).$$

3.4 AQFT lift: split reduction datum and split-regularized CMI

Definition 3.5 (Split reduction datum at width w). *Let ω be a normal state on the AQFT local algebra associated with ABC. A split reduction datum at collar width w for (A, B, C) consists of an injective $*$-homomorphism $\iota_w : \mathcal{A}^{(w)}_{ABC} \hookrightarrow \mathcal{A}(ABC)$, with $\mathcal{A}^{(w)}_{ABC} \cong \mathcal{B}(\mathcal{H}^{(w)}_A \otimes \mathcal{H}^{(w)}_B \otimes \mathcal{H}^{(w)}_C)$, together with the induced Type I state $\rho^{(w)}_{ABC} := \omega \circ \iota_w$. If $\rho^{(w)}_{ABC}$ is not faithful we replace it by $(1-\varepsilon)\rho^{(w)}_{ABC} + \varepsilon I/d_w$ for fixed $\varepsilon \in (0, 1)$. We do not establish split-existence or split-stability bounds here.*

Definition 3.6 (Split-regularized CMI). $I^{\mathrm{split},w}_\omega(A : C|B) := I_{\rho^{(w)}}(A : C|B)$*, computed in the induced split model; optionally* $\inf_{\iota_w} I_{\rho^{(w)}}(A : C|B)$.

4 Static input from the literature: collar-CMI bounds for Gibbs states

Proposition 4.1 (Literature input I: shielded-region CMI decay). *For quantum Gibbs states of local Hamiltonians with bounded interaction degree, in shielded geometries, $I_{\rho_\Lambda}(A : C|B) \leq K_0 e^{-mw}$ with K_0, m depending only on microscopic parameters; see [4].*

Proposition 4.2 (Literature input II: high-temperature CMI decay). *In high-temperature regimes, belief-propagation-channel methods yield CMI clustering bounds with exponential decay in distance; see [5].*

Remark 4.3. The precise prefactor dependence is geometry- and regime-dependent; we keep Axiom 3.3 as the static interface and use the above results as concrete evidence where the required uniformity is established.

5 From Static Recovery to Dynamic Rates: the $\omega = 0$ Obstruction

5.1 Davies/KMS-symmetric setup

Let $\mathcal{L}$ be a Davies generator acting on $\mathcal{A}_{ABC}$ with full-rank stationary state ρ_∞, satisfying detailed balance (KMS symmetry) with respect to ρ_∞, with KMS inner product $\langle X, Y\rangle_{\rho_\infty} := \mathrm{Tr}(\rho_\infty^{1/2} X^\dagger \rho_\infty^{1/2} Y)$ and $\|X\|^2_{2,\rho_\infty} := \langle X, X\rangle_{\rho_\infty}$. B^{+r} denotes the r-neighborhood of the collar, r the interaction range of the local decomposition $\mathcal{L} = \sum_Z \mathcal{L}_Z$.

5.2 Fixed-point algebra (the $\omega = 0$ floor)

Definition 5.1 (Fixed-point algebra). $\mathcal{F} := \ker(\mathcal{L})$*; under detailed balance, E_0 denotes the KMS-orthogonal projection onto $\mathcal{F}$, and $\mathcal{F}^\perp := \{X : E_0(X) = 0\}$.*

Definition 5.2 (Support of observables). *X is supported in A if $X = X_A \otimes I_{BC}$, and analogously for B, C.*

Proposition 5.3 (Dirichlet lower bound on an invariant subspace $\Rightarrow$ exponential decay)**.** *Let $\mathcal{L}$ be KMS-symmetric with respect to ρ_∞. Let $\mathcal{K} \subseteq \mathcal{A}_{ABC}$ be a linear subspace invariant under the semigroup, $\mathcal{K} \subseteq \mathcal{F}^\perp$, and suppose $-\langle Y, \mathcal{L}(Y)\rangle_{\rho_\infty} \geq \lambda \langle Y, Y\rangle_{\rho_\infty}$ for all $Y \in \mathcal{K}$ with $\lambda > 0$. Then $\|e^{t\mathcal{L}}(X)\|_{2,\rho_\infty} \leq e^{-\lambda t}\|X\|_{2,\rho_\infty}$ for all $X \in \mathcal{K}$, $t \geq 0$.*

Proof. With $X_t := e^{t\mathcal{L}}(X) \in \mathcal{K}$, KMS symmetry gives $\frac{d}{dt}\|X_t\|^2 = 2\langle X_t, \mathcal{L}(X_t)\rangle \leq -2\lambda\|X_t\|^2$; Grönwall concludes. ☐

5.3 Target theorem: RIP modulo fixed points (conditional)

Reduction viewpoint. Theorem 5.4 isolates the only model-dependent analytic step as the bulk–collar Dirichlet comparison (Assumption A.8). The diagonal heat-bath case is now *proved* in Appendix A (Proposition A.4, corrected in v2), yielding the diagonal subsector corollary with explicit constants.

Theorem 5.4 (RIP as a reduction to a bulk–collar Dirichlet comparison (conditional))**.** *Assume the detailed-balanced Davies setup. Suppose: (i)* Static Markov input: *$I_{\rho_\infty}(A : C|B) \leq \varepsilon(w) = \sigma(\partial B)g(w)$ (Axiom 3.3); (ii)* Collar gap input: *the collar Dirichlet contribution D_B (Definition A.1) satisfies a Poincaré inequality $D_B(X_B) \geq \lambda_B\|X_B\|^2_{2,\rho_\infty}$ for all X_B supported on B with $E_0(X_B) = 0$; (iii) Assumption A.8. Then*

$$-\langle X, \mathcal{L}(X)\rangle_{\rho_\infty} \geq \big(\lambda_B - C_0(1+\lambda_B)\varepsilon(w)\big)\langle X, X\rangle_{\rho_\infty} \quad \textit{for all } X \in \mathcal{F}^\perp \textit{ supported in } A,$$

nontrivial whenever $\lambda_B > C_0(1+\lambda_B)\varepsilon(w)$.

Proof. Let $X \in \mathcal{F}^\perp$ be supported in A. By Assumption A.8 there exists $\tilde{X}$ supported in B^{+r} with $E_0(\tilde{X}) = 0$ such that $-\langle X, \mathcal{L}(X)\rangle \geq D_B(\tilde{X}) - C_0\varepsilon(w)\langle X, X\rangle$ and $\langle \tilde{X}, \tilde{X}\rangle \geq (1 - C_0\varepsilon(w))\langle X, X\rangle$. The collar Poincaré inequality gives $D_B(\tilde{X}) \geq \lambda_B\langle \tilde{X}, \tilde{X}\rangle$, hence $-\langle X, \mathcal{L}(X)\rangle \geq (\lambda_B(1 - C_0\varepsilon) - C_0\varepsilon)\langle X, X\rangle = (\lambda_B - C_0(1+\lambda_B)\varepsilon)\langle X, X\rangle$. ☐

Corollary 5.5 (Exponential decay under invariant-subspace extension)**.** *Assume the setting of Theorem 5.4 with $\lambda_B - C_0(1+\lambda_B)\varepsilon(w) > 0$, and let $\mathcal{K}_A := \overline{\mathrm{span}}\{\mathcal{L}^n(X) : X \in \mathcal{F}^\perp$ supported in A, $n \geq 0\}$. If the Dirichlet inequality holds on all of $\mathcal{K}_A$, then $\|e^{t\mathcal{L}}(X)\|_{2,\rho_\infty} \leq e^{-(\lambda_B - C_0(1+\lambda_B)\varepsilon(w))t}\|X\|_{2,\rho_\infty}$ on $\mathcal{K}_A$.*

Proof. $\mathcal{K}_A$ is closed and $\mathcal{L}$-invariant, hence $e^{t\mathcal{L}}$-invariant in finite dimension; apply Proposition 5.3. ☐

6 Outlook and limitations

This manuscript is structured as an interface/reduction. The RIP is conditional in general quantum settings: proving Assumption A.8 remains the main model-dependent step.

6.1 Roadmap status (updated in v2)

Of the three items listed in v1: (1) operational routes to Assumption A.8 – the diagonal heat-bath case is now proved with corrected geometry (Proposition A.4) and its v1 form is refuted by explicit counterexample, which sharpens what any general proof must respect; (2) constants – all constants in Appendix A are now explicit and numerically computed in the suite (e.g. $\lambda'_B = 0.266$, $c'_{A\to B} = 1$ in the visible geometry $A = \{2\}$, $B = \{3, 4\}$; $\lambda'_B = 0$ in bulk-separated 1D geometries, honestly flagged); (3) reproducibility – delivered: `verification/2601-0020/` (Section 7).

7 Verification suite (added in v2)

`verify_2601_0020.py` (pure NumPy/SciPy, exact enumeration, $N = 9$ Ising-Z chain, $J = 1$, $\beta = 1$, open boundaries, single-site heat-bath, $A = \{0, 1, 2\}$, $B = \{3, 4, 5\}$, $C = \{6, 7, 8\}$) checks: (1) $I(A : C|B) = 0$ to 10^{-15} for the classical 1D Markov field at $w = 1, 2, 3$ (the analytic statement behind Figure 1); (2) the v1 diagonal comparison is *violated* on random diagonal observables supported in A (2/200 draws in this geometry, excess up to 0.18; 18/200 in a tighter geometry), while the corrected Proposition A.4 shows *0/200 violations*; (3) the transfer constants over the image space V_A in three geometries – $\lambda'_B = 0$ for bulk-separated A (1D structural degeneracy, see the sharpened remark), $\lambda'_B = 0.266$ with $c' = 1$ for $A = \{2\}$, $B = \{3, 4\}$ – with the end-to-end diagonal corollary holding on 100/100 draws in each geometry; (4) the corrected FR factor on random 3-qubit states (Petz recovery: $I \geq -\log F$ on 40/40 draws; the v1 factor $-2 \log F$ fails on 39/40). Exit code 0 iff all mandatory checks pass.

A Davies/Dirichlet interface lemmas (finite dimension)

A.1 Collar Dirichlet contribution

Definition A.1 (Collar Dirichlet contribution)**.** *Assume $\mathcal{L} = \sum_Z \mathcal{L}_Z$ with blocks of diameter at most r. Define $D_B(X) := -\sum_{Z \subseteq B^{+r}} \langle X, \mathcal{L}_Z(X)\rangle_{\rho_\infty}$.*

A.2 A classical heat-bath (commuting) case: corrected comparison

Commuting reduction. Assume H diagonal in a product basis, so ρ_∞ is diagonal; let $\mathcal{A}_{\text{diag}}$ be the diagonal subalgebra, identified with functions on a finite configuration space with Gibbs measure μ, and $\langle f, g\rangle_{\rho_\infty} = \mathbb{E}_\mu[fg]$. Assume $\mathcal{L}$ preserves $\mathcal{A}_{\text{diag}}$; each $\mathcal{L}_Z f = E_Z[f] - f$ is a heat-bath update resampling Z from $\mu(\cdot \mid \sigma(Z^c))$, so $-\langle f, \mathcal{L}_Z f\rangle = \mathbb{E}_\mu[\text{Var}_Z(f)]$.

Remark A.2 (The v1 statement is false as stated (corrected in v2)). v1 asserted the comparison $-\langle X, \mathcal{L}(X)\rangle \geq D_B(\tilde{X})$ with the collar extension $\tilde{X} = \mathbb{E}_\mu[X \mid \sigma(B^{+r})]$, without proof. This is *false in general*: for blocks $Z \subseteq B^{+r}$ touching the boundary of B^{+r}, the heat-bath conditional $\mu(\cdot_Z \mid \sigma(Z^c))$ depends on the configuration in $Z^{+r} \not\subseteq B^{+r}$, so E_Z does not commute with the projection onto $\sigma(B^{+r})$ and the variance-contraction step fails. The verification suite exhibits explicit counterexamples (Section 7: violations with excess up to 0.18–0.21, far above numerical noise, localized at the C-side boundary block). The corrected statement below enlarges the extension neighborhood to B^{+2r}, for which the commutation argument goes through.

Definition A.3 (Collar conditional-expectation extension (corrected))**.** *For $X \in \mathcal{A}_{\text{diag}}$ supported in A, define $\tilde{X} := \mathbb{E}_\mu[X \mid \sigma(B^{+2r})]$.*

Proposition A.4 (Bulk–collar Dirichlet comparison, diagonal heat-bath (proved in v2))**.** *For every $X \in \mathcal{A}_{\text{diag}}$, with $\tilde{X} = \mathbb{E}_\mu[X \mid \sigma(B^{+2r})]$,*

$$-\langle X, \mathcal{L}(X)\rangle_{\rho_\infty} \;\geq\; D_B(\tilde{X}).$$

Proof. First, $-\langle X, \mathcal{L}(X)\rangle = \sum_Z \mathbb{E}_\mu[\text{Var}_Z(X)] \geq \sum_{Z \subseteq B^{+r}} \mathbb{E}_\mu[\text{Var}_Z(X)]$. Fix $Z \subseteq B^{+r}$ and let $P := \mathbb{E}_\mu[\,\cdot \mid \sigma(B^{+2r})]$ and E_Z be the heat-bath conditional expectation; both are orthogonal projections on $L^2(\mu)$. Since the interaction has range r, the conditional $\mu(\cdot_Z \mid \sigma(Z^c))$ depends only on $\sigma(Z^{+r})$, and $Z^{+r} \subseteq B^{+2r}$; hence E_Z maps $\sigma(B^{+2r})$-measurable functions to $\sigma(B^{+2r})$-measurable functions. For any test function g that is $\sigma(B^{+2r})$-measurable, $\langle g, E_Z P f\rangle = \langle E_Z g, P f\rangle = \langle E_Z g, f\rangle =$

$\langle g, PE_Z f\rangle$ (using that $E_Z g$ is $\sigma(B^{+2r})$-measurable and P is self-adjoint), and both $E_Z Pf$ and $PE_Z f$ are $\sigma(B^{+2r})$-measurable; hence $E_Z P = PE_Z$. Consequently $(1 - E_Z)$ and P are commuting orthogonal projections, so

$$\mathbb{E}_\mu[\mathrm{Var}_Z(\tilde{X})] = \|(1 - E_Z)PX\|^2_{L^2(\mu)} = \|P(1 - E_Z)X\|^2_{L^2(\mu)} \leq \|(1 - E_Z)X\|^2_{L^2(\mu)} = \mathbb{E}_\mu[\mathrm{Var}_Z(X)].$$

Summing over $Z \subseteq B^{+r}$ gives $D_B(\tilde{X}) \leq \sum_{Z \subseteq B^{+r}} \mathbb{E}_\mu[\mathrm{Var}_Z(X)] \leq -\langle X, \mathcal{L}(X)\rangle$. □

Definition A.5 (Norm-transfer coefficient (corrected extension)). $c'_{A\to B} := \inf\{\langle \tilde{X}, \tilde{X}\rangle / \langle X, X\rangle : X \in \mathcal{A}_{\mathrm{diag}} \cap \mathcal{F}^\perp$ *supported in* $A,\ X \neq 0\}$*, with* $\tilde{X} = \mathbb{E}_\mu[X \mid \sigma(B^{+2r})]$.

Corollary A.6 (Diagonal bulk inheritance with explicit constants). *Let* $V_A := \overline{\mathrm{span}}\{\mathbb{E}_\mu[X \mid \sigma(B^{+2r})] : X \in \mathcal{A}_{\mathrm{diag}}$ *supported in* $A,\ \mathbb{E}_\mu[X] = 0\}$ *and* $\lambda'_B := \inf\{D_B(Y)/\langle Y, Y\rangle : Y \in V_A,\ Y \neq 0\} \geq 0$ *(with* D_B *summing over* $Z \subseteq B^{+r}$ *as in Definition A.1). Then for every* $X \in \mathcal{A}_{\mathrm{diag}} \cap \mathcal{F}^\perp$ *supported in* A,

$$-\langle X, \mathcal{L}(X)\rangle_{\rho_\infty} \geq c'_{A\to B}\, \lambda'_B\, \langle X, X\rangle_{\rho_\infty},$$

and under the invariant-subspace extension of Proposition 5.3, $\|e^{t\mathcal{L}}(X)\|_{2,\rho_\infty} \leq e^{-c'_{A\to B}\lambda'_B t}\|X\|_{2,\rho_\infty}$*. The bound is nontrivial only when* $\lambda'_B > 0$*; see the next remark.*

Remark A.7 (Degeneracy, sharpened in v2). v1 noted that $c_{A\to B} = 0$ when $A \cap B^{+r} = \emptyset$. The verification suite reveals a stronger structural fact in one dimension: by the Markov property of the 1D Gibbs measure, V_A always contains functions of the boundary spin between A and the collar, which are invisible to the collar Dirichlet form ($D_B(Y) = 0$), so $\lambda'_B = 0$ – and the corollary is vacuous – unless A lies inside the collar neighborhood visible to D_B. In the suite's benchmark chain: $\lambda'_B = 0$ for $A = \{0, 1, 2\}$ with $B = \{3, 4, 5\}$ or $\{3, 4\}$, while $A = \{2\}$, $B = \{3, 4\}$ gives $\lambda'_B = 0.266$, $c'_{A\to B} = 1$, and the end-to-end bound holds on 100/100 draws. The diagonal route is thus a genuinely boundary-overlap statement; rate transfer across a bulk separation requires the (open) quantum comparison of Assumption A.8, in higher dimension or with block updates reaching across the collar.

A.3 The missing link (isolated)

Assumption A.8 (Bulk–collar Dirichlet comparison). *For every* $X \in \mathcal{F}^\perp$ *supported in* A*, there exists* $\tilde{X}$ *supported in* B^{+r} *with* $E_0(\tilde{X}) = 0$ *such that* $-\langle X, \mathcal{L}(X)\rangle_{\rho_\infty} \geq D_B(\tilde{X}) - C_0\varepsilon(w)\langle X, X\rangle_{\rho_\infty}$ *and* $\langle \tilde{X}, \tilde{X}\rangle_{\rho_\infty} \geq (1 - C_0\varepsilon(w))\langle X, X\rangle_{\rho_\infty}$.

Remark A.9 (Guidance from the diagonal case (added in v2)). The refutation of the v1 diagonal comparison shows that any proof of Assumption A.8 must respect the range mismatch at the boundary of B^{+r}: extensions built by conditioning on B^{+r} alone cannot work; the natural candidates live on B^{+2r} (or the Dirichlet sum must be restricted to blocks Z with $Z^{+r} \subseteq B^{+r}$).

B Numerical diagnostics (commuting Ising-Z + heat-bath)

C Numerical illustrations (finite-size TFIM)

These results are illustrative only; no theorem depends on them.

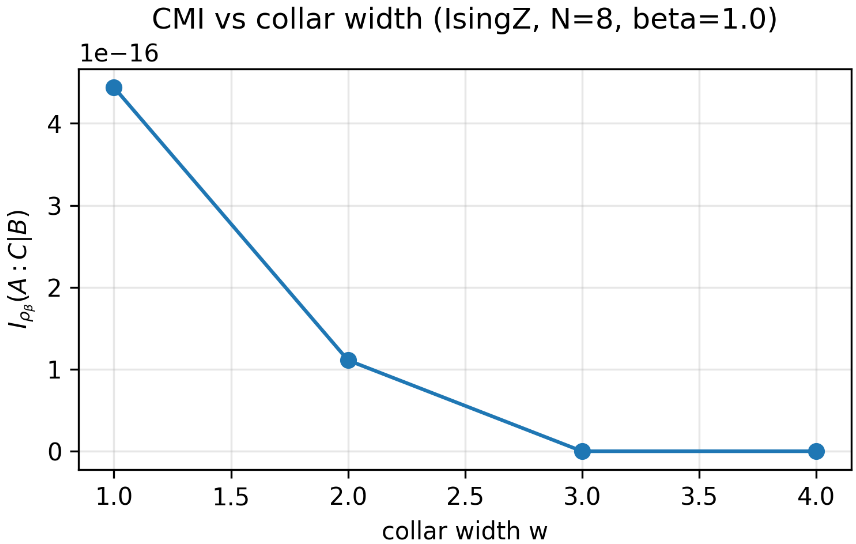

Figure 1: CMI diagnostic: $I_{\rho_\infty}(A : C|B)$ vs collar width w (commuting Ising-Z, $N = 8$, $\beta = 1$). The 10^{-16} level reflects floating-point noise; the suite confirms the exact statement $I(A : C|B) = 0$ for the 1D Markov field.

Note added in v2

Two corrections and one delivery. *Corrections:* (1) the Fawzi–Renner factor under the squared-fidelity convention is $-\log F$ (Lemma 3.1, Remark 3.2); v1's $-2\log F$ overclaimed, and Corollary 3.4 now reads $1-F \le \sigma(\partial B)g(w)$ (not $\frac{1}{2}\sigma g$). Same correction as 2512.0101 v2. (2) v1's diagonal bulk–collar comparison (its Proposition A.4) is false as stated – exact counterexamples in the suite – and is replaced by the proved Proposition A.4 with extension on B^{+2r} (commuting-projection argument), with Corollary A.6 restated over the image space V_A (and the 1D boundary-spin degeneracy $\lambda'_B = 0$ for bulk-separated A identified and flagged); the failure mechanism (range mismatch at the B^{+r} boundary) is recorded as guidance for Assumption A.8. *Delivery:* the reproducibility item of v1's own roadmap: full verification suite (Section 7), series cross-references (Section 2), and the Figure 1 caption typo fixed.

References

[1] D. Petz, Commun. Math. Phys. **105** (1986) 123–131.

[2] O. Fawzi and R. Renner, Commun. Math. Phys. **340** (2015) 575–611.

[3] E. B. Davies, Commun. Math. Phys. **39** (1974) 91–110.

[4] C.-F. Chen and C. Rouzé, *Quantum Gibbs states are locally Markovian*, arXiv:2504.02208 (2025).

[5] K. Kato and T. Kuwahara, *Clustering of Conditional Mutual Information via Quantum Belief-Propagation Channels*, arXiv:2504.02235 (2025).

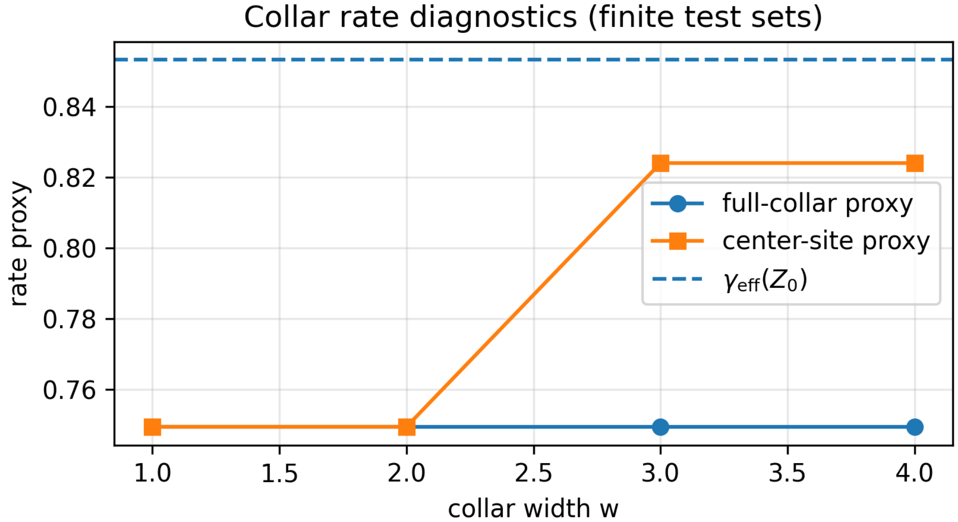

Figure 2: Collar rate diagnostics (finite test sets) for the commuting Ising-Z heat-bath embedded Lindbladian ($N = 8$, $\beta = 1$).

[6] Y. Zhang and S. Gopalakrishnan, *Conditional Mutual Information and Information-Theoretic Phases of Decohered Gibbs States*, arXiv:2502.13210 (2025).

[7] L. Eriksson, ai.viXra:2512.0060 (2025; v2 2026).

[8] L. Eriksson, ai.viXra:2512.0101 (2025; v2 2026).

[9] L. Eriksson, ai.viXra:2601.0007 (2026; v2 2026).

[10] L. Eriksson, ai.viXra:2512.0072 (2025; v2 2026).

[11] L. Eriksson, ai.viXra:2512.0070 (2025; v2 2026).

[12] L. Eriksson, ai.viXra:2512.0064 (2025; v2 2026).

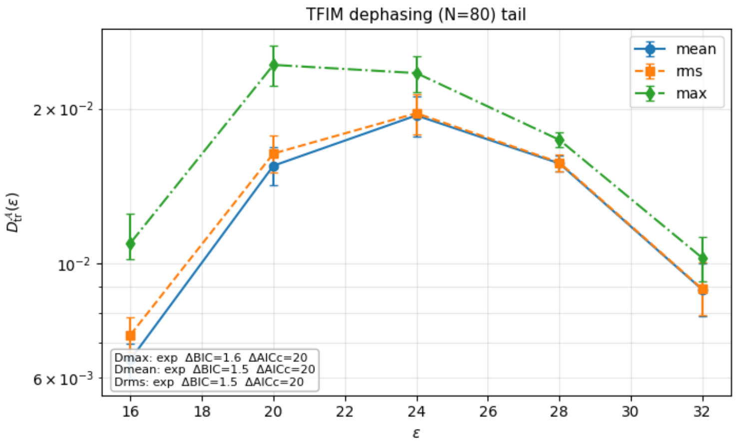

Figure 3: TFIM dephasing tail ($N = 80$): windowed influence proxy (illustrative).

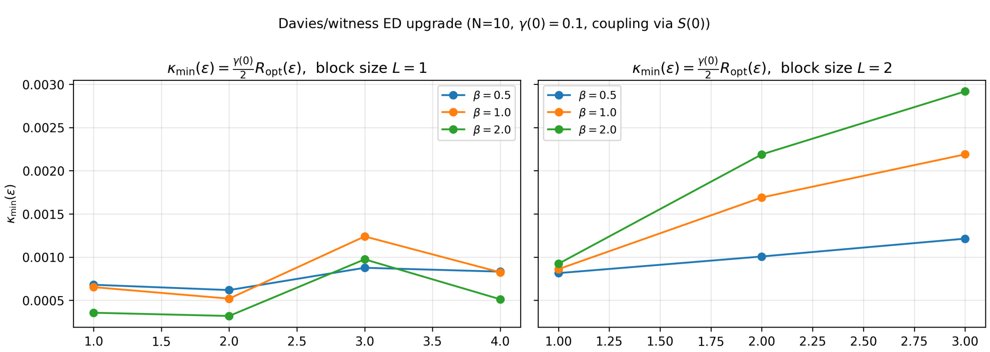

Figure 4: Witness-based lower bound plot (illustrative).