Geometric Markov Bounds and Rate Inheritance Modulo Fixed Points:
A Scalar Entropic Interface from Static Locality to Davies Dynamics

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Abstract

We propose an entropic interface between locality, recoverability, and dynamical decay rates across a geometric collar. The central scalar invariant is the conditional mutual information (CMI) $I_\rho(A : C|B)$, where B is a buffer separating A and C. In finite dimension (Type I algebras), the Fawzi–Renner theorem implies that small CMI yields a quantitative recovery channel acting on B. We formulate a volume-uniform geometric Markov bound with a boundary prefactor, $I_{\rho_\Lambda}(A : C|B) \leq \sigma(\partial B)\, g(w)$, and summarize recent literature inputs establishing exponential CMI decay in shielded/high-temperature regimes. On the dynamical side, we formulate a Rate Inheritance Principle (RIP) for Davies/KMS-symmetric generators: static Markovness across the collar constrains decay rates on the fast sector $\mathcal{F}^\perp$ modulo the fixed-point algebra $\mathcal{F} = \ker \mathcal{L}$ (the $\omega = 0$ floor), with a dynamical input stated as a Poincaré inequality for a local collar Dirichlet form. The only remaining nontrivial link is isolated as an explicit Dirichlet comparison assumption. We also verify a diagonal (classical) heat-bath comparison and derive a diagonal subsector corollary with an explicit transfer coefficient. Finally, we define a split reduction datum and a split-regularized CMI target quantity for an AQFT lift and include finite-size illustrations/diagnostics.

Contents

1 Introduction

Many hard problems in quantum field theory and mathematical physics are not blocked by "more calculation" but by the lack of a representation in which the controlling structure is manifest. For locality-to-dynamics questions, we propose a convenient such representation in terms of a single scalar entropic quantity: the conditional mutual information across a geometric collar, for a tripartition A–B–C where B separates A and C and has collar width w,

$$I_\rho(A:C|B).$$

In our setting, the relevant global constraint is approximate quantum Markovness across a collar, quantified by small CMI.

In finite dimension (Type I algebras), small CMI implies quantitative recoverability via the Fawzi–Renner theorem. We then ask a dynamical question: does static recoverability constrain decay rates under Davies/KMS-symmetric dynamics? The intrinsic obstruction is the fixed-point algebra $\mathcal{F} = \ker \mathcal{L}$ (the $\omega = 0$ floor). Accordingly, any rate-inheritance statement must be formulated on the fast sector $\mathcal{F}^\perp$ (or assume primitivity).

In AQFT (Type III algebras), von Neumann entropies are not directly available. We therefore introduce a split reduction datum at collar width w and define a split-regularized CMI $I_\omega^{\mathrm{split},w}(A : C|B)$ on the induced Type I model. Quantitative split-stability bounds are deferred.

2 Related work and context

CMI and recovery. The quantitative recovery theorem of Fawzi–Renner connects small CMI to the existence of recovery maps; see [2].

Gibbs states and local Markovness. Recent work constructs quasi-local recovery maps for quantum Gibbs states in shielded geometries and derives exponential decay of CMI with shielding distance; see [4].

High-temperature CMI via belief propagation. Belief-propagation-channel methods yield CMI clustering bounds in high-temperature regimes and clarify limitations of naive cluster-expansion approaches; see [5].

Dissipative Gibbs states and information-theoretic phases. Related results on conditional mutual information in decohered Gibbs settings and "hidden Markov" structures appear in [6].

3 Framework and first lemma (Type I baseline; AQFT lift as a construction)

3.1 Type I baseline: algebras, CMI, fidelity

We work in finite dimension. Let

$$\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C, \qquad \mathcal{A}_{ABC} = \mathcal{B}(\mathcal{H}_{ABC}).$$

Let $\rho_{ABC} > 0$ be a full-rank state. For $X \in \{A, B, C, AB, BC, ABC\}$, write ρ_X for the reduced state.

All logarithms are natural throughout. Define von Neumann entropy $S(\sigma) := -\mathrm{Tr}(\sigma \log \sigma)$. For any full-rank state σ_{ABC} define the conditional mutual information

$$I_\sigma(A:C|B) := S(\sigma_{AB}) + S(\sigma_{BC}) - S(\sigma_B) - S(\sigma_{ABC}).$$

In particular, for the baseline state ρ_{ABC},

$$I_\rho(A:C|B) := S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_B) - S(\rho_{ABC}).$$

We use Uhlmann fidelity

$$F(\sigma,\tau) := \left\|\sqrt{\sigma}\sqrt{\tau}\right\|_1^2.$$

3.2 Recoverability from small CMI (Fawzi–Renner)

Lemma 3.1 (Fawzi–Renner recovery bound (finite dimension))**.** *Let ρ_{ABC} be full-rank. There exists a CPTP map $\mathcal{R}_{B\to BC}$ (acting on B, possibly nonlocal within B) such that*

$$I_\rho(A:C|B) \;\geq\; -2\log F(\rho_{ABC}, (\mathrm{id}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AB}))\,.$$

In particular, if $I_\rho(A:C|B) \leq \delta$, then for some CPTP $\mathcal{R}_{B\to BC}$,

$$F(\rho_{ABC}, (\mathrm{id}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AB})) \;\geq\; e^{-\delta/2}, \qquad 1-F \;\leq\; 1-e^{-\delta/2} \;\leq\; \delta/2.$$

3.3 Static interface axiom (geometric Markov bound)

Axiom 3.2 (Uniform geometric Markov bound with boundary prefactor)**.** *Fix a model class $\mathcal{C} = \{\rho_\Lambda\}_\Lambda$. Fix a boundary functional $\sigma(\partial B) \geq 0$ depending only on the chosen notion of boundary and the ambient lattice geometry (e.g. on $\mathbb{Z}^d$ one may take $\sigma(\partial B) = |\partial B|$, the edge boundary size). There exists a function $g : \mathbb{N} \to [0,\infty)$ such that for every finite volume Λ and every admissible triple $(A,B,C) \subset \Lambda$ with collar width w,*

$$I_{\rho_\Lambda}(A:C|B) \;\leq\; \sigma(\partial B)\, g(w),$$

where g depends only on microscopic parameters and not on $|\Lambda|$.

Corollary 3.3 (Geometric recovery bound)**.** *Under Axiom 3.2, for $\rho_{ABC} := (\rho_\Lambda)_{ABC}$ there exists a CPTP map $\mathcal{R}_{B\to BC}$ acting on B such that*

$$F(\rho_{ABC}, (\mathrm{id}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AB})) \;\geq\; \exp\!\big(-\tfrac{1}{2}\,\sigma(\partial B)\, g(w)\big)\,.$$

In particular,

$$1 - F(\rho_{ABC}, (\mathrm{id}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AB})) \;\leq\; \tfrac{1}{2}\,\sigma(\partial B)\, g(w).$$

3.4 AQFT lift: split reduction datum and split-regularized CMI

Definition 3.4 (Split reduction datum at width w)**.** Let ω be a normal state on the AQFT local algebra associated with ABC. A split reduction datum at collar width w for (A,B,C) consists of an injective $*$-homomorphism

$$\iota_w :\; \mathcal{A}^{(w)}_{ABC} \hookrightarrow \mathcal{A}(ABC), \qquad \mathcal{A}^{(w)}_{ABC} \cong \mathcal{B}(\mathcal{H}^{(w)}_A \otimes \mathcal{H}^{(w)}_B \otimes \mathcal{H}^{(w)}_C),$$

together with the induced Type I state $\rho^{(w)}_{ABC} := \omega \circ \iota_w$. We assume $\rho^{(w)}_{ABC}$ is faithful; if not, we replace it by

$$\rho^{(w)}_{ABC,\varepsilon} := (1-\varepsilon)\rho^{(w)}_{ABC} + \varepsilon\,\frac{\mathbb{I}}{d_w}, \qquad d_w := \dim(\mathcal{H}^{(w)}_A \otimes \mathcal{H}^{(w)}_B \otimes \mathcal{H}^{(w)}_C),$$

for a fixed $\varepsilon \in (0,1)$. We do not establish split-existence or split-stability bounds here.

Definition 3.5 (Split-regularized CMI)**.** Define

$$I_{\omega}^{\text{split},w}(A:C|B) := I_{\rho^{(w)}}(A:C|B),$$

where the right-hand side is the Type I CMI computed in the induced split model $\mathcal{A}_{ABC}^{(w)}$. Optionally, one may work with an operational variant such as $\inf_{\iota_w} I_{\rho^{(w)}}(A:C|B)$.

4 Static input from the literature: collar-CMI bounds for Gibbs states

We separate the abstract interface (Section 3–5) from model-dependent verification of static collar-CMI decay. We record the following inputs from recent literature and keep Axiom 3.2 as the static interface.

Proposition 4.1 (Literature input I: shielded-region CMI decay for Gibbs states)**.** *Let ρ_Λ be a quantum Gibbs state of a local Hamiltonian with bounded interaction degree. For shielded geometries in which A is a fixed-size local region separated from C by a collar B of width w, one has exponential decay*

$$I_{\rho_\Lambda}(A:C|B) \;\leq\; K_0\, e^{-mw},$$

for constants $K_0 < \infty$ and $m > 0$ depending only on microscopic parameters (and not on $|\Lambda|$). See [4].

Proposition 4.2 (Literature input II: high-temperature CMI decay via belief-propagation channels)**.** *In high-temperature regimes (and, more generally, under additional uniform dynamical/static assumptions such as rapid mixing or clustering), belief-propagation-channel methods yield CMI clustering bounds with exponential decay in distance. See [5].*

Remark 4.3. The precise prefactor dependence (e.g. whether one can take $\sigma(\partial B) = |\partial B|$ for arbitrarily large regions) is geometry- and regime-dependent. In this paper we keep Axiom 3.2 as the static interface assumption and use the above results as concrete evidence in regimes where the required prefactor and uniformity are established.

5 From Static Recovery to Dynamic Rates: the $\omega = 0$ Obstruction

5.1 Davies/KMS-symmetric setup

Let $\mathcal{L}$ be a Davies generator acting on $\mathcal{A}_{ABC}$ with a full-rank stationary state ρ_∞ and satisfying detailed balance (KMS symmetry) with respect to ρ_∞. We work with the KMS inner product

$$\langle X, Y\rangle_{\rho_\infty} := \mathrm{Tr}(\rho_\infty^{1/2} X^\dagger \rho_\infty^{1/2} Y), \qquad \|X\|_{2,\rho_\infty}^2 := \langle X, X\rangle_{\rho_\infty}.$$

Here B^{+r} denotes the r-neighborhood of the collar, where r is the interaction range of the local decomposition $\mathcal{L} = \sum_Z \mathcal{L}_Z$.

5.2 Fixed-point algebra (the $\omega = 0$ floor)

Definition 5.1 (Fixed-point algebra)**.** Define the fixed-point algebra $\mathcal{F} := \ker(\mathcal{L})$. Under detailed balance, let $\mathcal{E}_0$ denote the orthogonal projection onto $\mathcal{F}$ with respect to the KMS inner product. We define the **fast sector** as

$$\mathcal{F}^\perp := \{X \in \mathcal{A}_{ABC} : \mathcal{E}_0(X) = 0\}.$$

Definition 5.2 (Support of observables (finite dimension)). We say that an observable $X \in \mathcal{A}_{ABC}$ is supported in A if it can be written as

$$X = X_A \otimes \mathbb{I}_{(BC)} \quad \text{for some } X_A \in \mathcal{B}(\mathcal{H}_A),$$

and analogously for B and C.

Proposition 5.3 (Dirichlet lower bound on an invariant subspace $\Rightarrow$ exponential decay). *Let $\mathcal{L}$ be KMS-symmetric with respect to ρ_∞. Let $\mathcal{K} \subseteq \mathcal{A}_{ABC}$ be a linear subspace that is invariant under the semigroup, i.e. $e^{t\mathcal{L}}(\mathcal{K}) \subseteq \mathcal{K}$ for all $t \geq 0$. Assume $\mathcal{K} \subseteq \mathcal{F}^\perp$ and that there exists $\lambda > 0$ such that*

$$-\langle Y, \mathcal{L}(Y)\rangle_{\rho_\infty} \;\geq\; \lambda\, \langle Y, Y\rangle_{\rho_\infty} \qquad \text{for all } Y \in \mathcal{K}.$$

Then for all $X \in \mathcal{K}$ and all $t \geq 0$,

$$\left\|e^{t\mathcal{L}}(X)\right\|_{2,\rho_\infty} \;\leq\; e^{-\lambda t}\, \|X\|_{2,\rho_\infty}.$$

Proof. Fix $X \in \mathcal{K}$ and set $X_t := e^{t\mathcal{L}}(X)$. By invariance, $X_t \in \mathcal{K}$ for all $t \geq 0$. By KMS symmetry,

$$\frac{d}{dt}\|X_t\|^2_{2,\rho_\infty} = \langle \mathcal{L}(X_t), X_t\rangle_{\rho_\infty} + \langle X_t, \mathcal{L}(X_t)\rangle_{\rho_\infty} = 2\,\langle X_t, \mathcal{L}(X_t)\rangle_{\rho_\infty}.$$

Using the assumed inequality on $\mathcal{K}$ gives

$$\frac{d}{dt}\|X_t\|^2_{2,\rho_\infty} \;\leq\; -2\lambda\,\|X_t\|^2_{2,\rho_\infty}.$$

Grönwall yields $\|X_t\|^2_{2,\rho_\infty} \leq e^{-2\lambda t}\|X\|^2_{2,\rho_\infty}$. ◻

5.3 Target theorem: RIP modulo fixed points (conditional)

Reduction viewpoint. Theorem 5.4 isolates the only model-dependent analytic step as the bulk–collar Dirichlet comparison (Assumption A.7). We verify the relevant Dirichlet comparison in a commuting heat-bath diagonal subsector in Appendix A (Proposition A.4), yielding a diagonal subsector corollary there (Corollary A.5).

Theorem 5.4 (RIP as a reduction to a bulk–collar Dirichlet comparison (conditional)). *Assume the detailed-balanced Davies setup. Suppose:*

(i) ***Static Markov input:***

$$I_{\rho_\infty}(A:C|B) \leq \varepsilon(w) = \sigma(\partial B)\, g(w),$$

as per Axiom 3.2.

(ii) ***Collar gap input (fast sector):*** *Let $\mathcal{D}_B$ be the collar Dirichlet contribution (Definition A.1). Assume a Poincaré inequality holds on the collar fast sector:*

$$\mathcal{D}_B(X_B) \geq \lambda_B \|X_B\|^2_{2,\rho_\infty}$$

for all X_B supported on B with $\mathcal{E}_0(X_B) = 0$.

(iii) ***Comparison assumption:*** *Assume Assumption A.7.*

Then the following instantaneous *rate inheritance bound holds:*

$$-\langle X,\mathcal{L}(X)\rangle_{\rho_\infty} \;\geq\; \big(\lambda_B - C_0(1+\lambda_B)\varepsilon(w)\big)\,\langle X,X\rangle_{\rho_\infty} \quad \textit{for all } X\in\mathcal{F}^\perp \textit{ supported in } A.$$

In particular, the bound is nontrivial whenever $\lambda_B > C_0(1+\lambda_B)\varepsilon(w)$*.*

Proof of Theorem 5.4. Let $X\in\mathcal{F}^\perp$ supported in A. By Assumption A.7, there exists $\widetilde{X}$ supported in B^{+r} with $\mathcal{E}_0(\widetilde{X})=0$ such that

$$-\langle X,\mathcal{L}(X)\rangle_{\rho_\infty} \;\geq\; \mathcal{D}_B(\widetilde{X}) \;-\; C_0\,\varepsilon(w)\,\langle X,X\rangle_{\rho_\infty}, \qquad \left\langle \widetilde{X},\widetilde{X}\right\rangle_{\rho_\infty} \;\geq\; (1-C_0\varepsilon(w))\,\langle X,X\rangle_{\rho_\infty}.$$

By the collar Poincaré inequality, $\mathcal{D}_B(\widetilde{X}) \geq \lambda_B \left\langle \widetilde{X},\widetilde{X}\right\rangle_{\rho_\infty}$, hence

$$-\langle X,\mathcal{L}(X)\rangle_{\rho_\infty} \;\geq\; \big(\lambda_B(1-C_0\varepsilon(w)) - C_0\varepsilon(w)\big)\,\langle X,X\rangle_{\rho_\infty} = \big(\lambda_B - C_0(1+\lambda_B)\varepsilon(w)\big)\,\langle X,X\rangle_{\rho_\infty}.$$

This matches the stated coefficient $\lambda_B - C_0(1+\lambda_B)\varepsilon(w)$. □

Corollary 5.5 (Exponential decay under invariant-subspace extension)**.** *Assume the setting of Theorem 5.4. Assume moreover that* $\lambda_B - C_0(1+\lambda_B)\varepsilon(w) > 0$*. Let*

$$\mathcal{K}_A := \overline{\mathrm{span}}\{\,\mathcal{L}^n(X) \;:\; X\in\mathcal{F}^\perp \textit{ supported in } A,\;\; n\geq 0\,\},$$

where the closure is taken in $\|\cdot\|_{2,\rho_\infty}$*. If the inequality*

$$-\langle Y,\mathcal{L}(Y)\rangle_{\rho_\infty} \;\geq\; \big(\lambda_B - C_0(1+\lambda_B)\varepsilon(w)\big)\,\langle Y,Y\rangle_{\rho_\infty}$$

holds for all $Y\in\mathcal{K}_A$*, then for all* $X\in\mathcal{K}_A$ *and all* $t\geq 0$*,*

$$\left\|e^{t\mathcal{L}}(X)\right\|_{2,\rho_\infty} \;\leq\; e^{-(\lambda_B - C_0(1+\lambda_B)\varepsilon(w))t}\,\|X\|_{2,\rho_\infty}.$$

Proof. In finite dimension, $\mathcal{K}_A$ is invariant under $e^{t\mathcal{L}}$ since it is closed and invariant under $\mathcal{L}$, and $e^{t\mathcal{L}}$ is given by a convergent power series in $\mathcal{L}$. Apply Proposition 5.3 with $\mathcal{K} = \mathcal{K}_A$ and $\lambda = \lambda_B - C_0(1+\lambda_B)\varepsilon(w)$. □

6 Outlook and limitations

This manuscript is structured as an interface/reduction. The RIP is conditional in general quantum settings: proving Assumption A.7 remains the main model-dependent step.

6.1 What remains to strengthen the manuscript (roadmap)

1. **Operational verification routes for Assumption A.7.** Provide sufficient conditions or a short proof-map (even conjectural templates).
2. **Constants.** Keep constants explicit to avoid "absorbing constants" ambiguity.
3. **Reproducibility.** Include a minimal script/notebook and a precise definition of the finite test sets used in the proxy plots.

A Davies/Dirichlet interface lemmas (finite dimension)

A.1 Collar Dirichlet contribution

Definition A.1 (Collar Dirichlet contribution)**.** Assume $\mathcal{L}$ admits a finite-range decomposition $\mathcal{L} = \sum_Z \mathcal{L}_Z$ (diameter at most r). Let B^{+r} denote the r-neighborhood of the collar B. Define

$$\mathcal{D}_B(X) := -\sum_{Z \subseteq B^{+r}} \langle X, \mathcal{L}_Z(X) \rangle_{\rho_\infty}.$$

Thus $\mathcal{D}_B$ depends on the collar neighborhood B^{+r} via the restriction $Z \subseteq B^{+r}$. In applications we primarily use $\mathcal{D}_B$ on observables supported in B^{+r}, for which it represents the collar-neighborhood Dirichlet contribution.

A.2 A classical heat-bath (commuting) case where the comparison holds

Commuting reduction. Assume H is diagonal in a product basis, hence ρ_∞ is diagonal in that basis. Let $\mathcal{A}_{\mathrm{diag}} \subset \mathcal{A}_{ABC}$ be the diagonal (commutative) $*$-subalgebra, identified with real functions on a finite configuration space Ω. Write μ for the induced classical Gibbs measure so that for $f, g \in \mathcal{A}_{\mathrm{diag}}$,

$$\langle f, g \rangle_{\rho_\infty} = \mathbb{E}_\mu[f\, g].$$

Assume $\mathcal{L}$ leaves $\mathcal{A}_{\mathrm{diag}}$ invariant; denote by L its restriction to $\mathcal{A}_{\mathrm{diag}}$.

Heat-bath block dynamics. Assume $L = \sum_Z L_Z$ (diameter at most r), where each L_Z is a heat-bath update on Z: there exists a conditional expectation $\mathbb{E}_Z[\cdot]$ resampling variables in Z from $\mu(\cdot \,|\, \sigma(Z^c))$ such that

$$L_Z f = \mathbb{E}_Z[f] - f, \qquad -\langle f, L_Z f \rangle_{\rho_\infty} = \mathbb{E}_\mu[{}_Z(f)],$$

where $\sigma(Z^c)$ denotes the σ-algebra generated by the configuration variables outside Z, $\mathbb{E}_Z[\cdot]$ denotes conditional expectation with respect to $\mu(\cdot \,|\, \sigma(Z^c))$, and ${}_Z(f) := \mathbb{E}_Z[f^2] - \mathbb{E}_Z[f]^2$.

Definition A.2 (Collar conditional expectation extension (diagonal case))**.** For $X \in \mathcal{A}_{\mathrm{diag}}$ supported in A, define its collar extension by

$$\widetilde{X} := \mathbb{E}_\mu[X \,|\, \sigma(B^{+r})],$$

i.e. the μ-orthogonal projection of X onto the commutative algebra generated by B^{+r}.

Definition A.3 (Norm-transfer coefficient from A to the collar)**.** Define

$$c_{A\to B} := \inf_{\substack{X \in \mathcal{A}_{\mathrm{diag}} \cap \mathcal{F}^\perp \\ X \neq 0,\ X \text{ supported in } A}} \frac{\left\langle \widetilde{X}, \widetilde{X} \right\rangle_{\rho_\infty}}{\langle X, X \rangle_{\rho_\infty}}, \qquad \widetilde{X} := \mathbb{E}_\mu[X \,|\, \sigma(B^{+r})].$$

Proposition A.4 (Bulk–collar Dirichlet comparison for diagonal observables (heat-bath))**.** *For every $X \in \mathcal{A}_{\mathrm{diag}}$ supported in A, with $\widetilde{X}$ as in Definition A.2,*

$$-\langle X, \mathcal{L}(X) \rangle_{\rho_\infty} \;\geq\; \mathcal{D}_B(\widetilde{X}).$$

Corollary A.5 (Diagonal bulk inheritance (instantaneous) with a transfer coefficient). *Assume Proposition A.4. Assume moreover that a collar Poincaré inequality holds on diagonal observables supported in B^{+r}: there exists $\lambda_B > 0$ such that*

$$\mathcal{D}_B(Y) \;\geq\; \lambda_B \,\langle Y, Y\rangle_{\rho_\infty} \quad \textit{for all } Y \in \mathcal{A}_{\rm diag} \textit{ supported in } B^{+r} \textit{ with } \mathcal{E}_0(Y) = 0.$$

Then for every diagonal observable $X \in \mathcal{A}_{\rm diag} \cap \mathcal{F}^\perp$ supported in A,

$$-\langle X, \mathcal{L}(X)\rangle_{\rho_\infty} \;\geq\; c_{A\to B}\lambda_B \,\langle X, X\rangle_{\rho_\infty},$$

where $c_{A\to B}$ is defined in Definition A.3.

Moreover, if the same inequality holds for all Y in the invariant subspace

$$\mathcal{K}_{A,\rm diag} := \overline{\rm span}\{\, \mathcal{L}^n(X) \,:\, X \in \mathcal{A}_{\rm diag} \cap \mathcal{F}^\perp \textit{ supported in } A,\; n \geq 0\,\}$$

(closure taken in $\|\cdot\|_{2,\rho_\infty}$), then for all $X \in \mathcal{K}_{A,\rm diag}$ and all $t \geq 0$,

$$\left\|e^{t\mathcal{L}}(X)\right\|_{2,\rho_\infty} \;\leq\; e^{-c_{A\to B}\lambda_B t}\,\|X\|_{2,\rho_\infty}.$$

Remark A.6 (Degeneracy of $c_{A\to B}$). If $A \cap B^{+r} = \emptyset$, then for centered diagonal observables the conditional expectation $\widetilde{X}$ may vanish, so $c_{A\to B} = 0$ and the bound becomes trivial.

A.3 The missing link (isolated)

Assumption A.7 (Bulk–collar Dirichlet comparison). For every $X \in \mathcal{F}^\perp$ supported in A, there exists an observable $\widetilde{X}$ supported in B^{+r} with $\mathcal{E}_0(\widetilde{X}) = 0$ such that

$$-\langle X, \mathcal{L}(X)\rangle_{\rho_\infty} \;\geq\; \mathcal{D}_B(\widetilde{X}) \;-\; C_0\,\varepsilon(w)\,\langle X, X\rangle_{\rho_\infty}, \qquad \left\langle \widetilde{X}, \widetilde{X}\right\rangle_{\rho_\infty} \;\geq\; (1 - C_0\varepsilon(w))\,\langle X, X\rangle_{\rho_\infty}.$$

B Numerical diagnostics (commuting IsingZ + heat-bath embedded Lindbladian)

Files. If present in the project root, the following PNG files will be included: `cmi_vs_w.png` and `collar_rate_diagnostics.png`.

Code availability. No script/notebook is included with this manuscript version. The figure-generation procedure is specified in-text via the test sets and parameters stated in the captions; releasing a minimal script is left as a reproducibility upgrade.

Dirichlet ratio used in the diagnostics. For an observable O, we use the (global) $L^2(\rho_\infty)$ Dirichlet ratio

$$\frac{\mathcal{D}(O)}{\|O\|^2_{2,\rho_\infty}}, \qquad \mathcal{D}(O) := -\Re\,\langle O, \mathcal{L}(O)\rangle_{\rho_\infty}.$$

In the proxy plots, observables are centered by subtracting their ρ_∞-expectation. For the finite test sets, we use centered one-site and adjacent two-site Pauli observables restricted to the collar neighborhood B^{+r}. Let $\sigma_i^x, \sigma_i^y, \sigma_i^z$ denote the usual Pauli operators on site i. Define

$$\mathcal{T}_1(B^{+r}) = \left\{\, \sigma_i^\alpha - {\rm Tr}(\rho_\infty\sigma_i^\alpha)\mathbb{I} \,:\, i \in B^{+r},\; \alpha \in \{x, y, z\}\,\right\},$$

and

$$\mathcal{T}_2(B^{+r}) = \left\{ \sigma_i^\alpha \sigma_{i+1}^\beta - \mathrm{Tr}(\rho_\infty \sigma_i^\alpha \sigma_{i+1}^\beta)\mathbb{I} \,:\, i, i+1 \subseteq B^{+r},\ \alpha, \beta \in \{x, y, z\} \right\}.$$

We minimize $\mathcal{D}(O)/\|O\|_{2,\rho_\infty}^2$ over $\mathcal{T}_1(B^{+r}) \cup \mathcal{T}_2(B^{+r})$. We include also off-diagonal Pauli observables (i.e. $\alpha \in \{x, y\}$) since the embedded Lindbladian acts on the full matrix algebra even when the stationary state is diagonal.

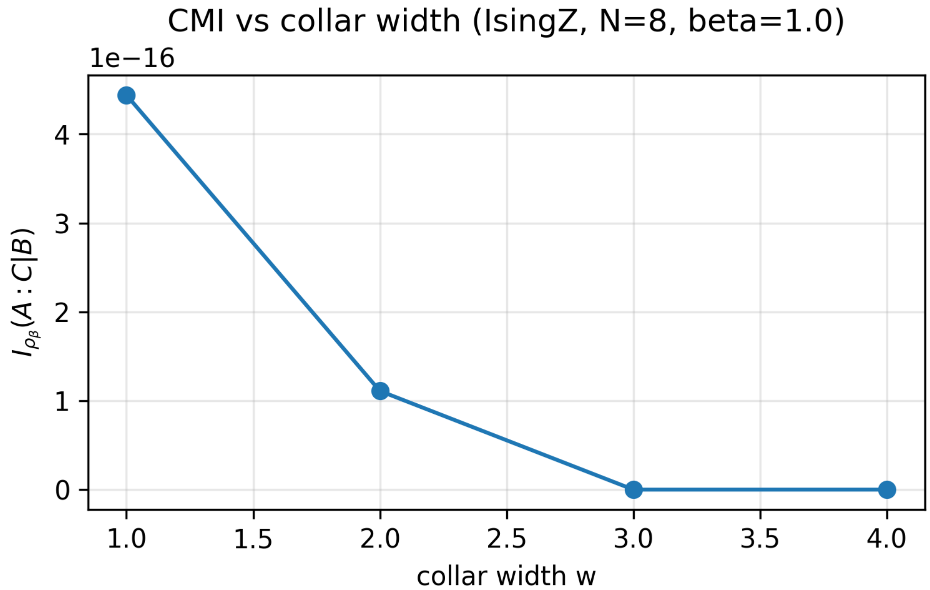

Figure 1: CMI diagnostic: $I_{\rho_\infty}(A : C|B)(A : C|B)$ vs collar width w (commuting Ising-Z, $N = 8$, $\beta = 1$). The 10^{-16} level reflects floating-point noise.

C Numerical illustrations (finite-size TFIM)

Note. These numerical results are illustrative only. No theorem in the main text depends on numerical results.

C.1 Operational influence proxy (TEBD/MCWF)

We use the one-site trace-distance influence proxy

$$D_{\mathrm{tr}}(t; \epsilon) := \frac{1}{2} \left\| \bar{\rho}_S^{\mathrm{noisy}}(t; \epsilon) - \rho_S^{\mathrm{uni}}(t) \right\|_1.$$

C.2 $\omega = 0$ witness stress test (ED upgrade)

References

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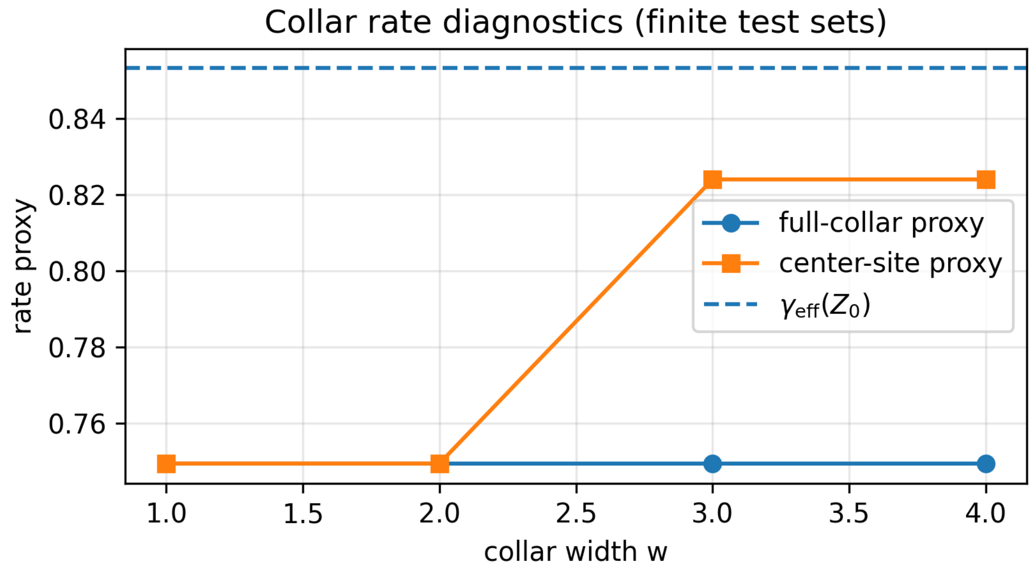

Figure 2: Collar rate diagnostics (finite test sets) for the commuting Ising-Z heat-bath embedded Lindbladian ($N = 8$, $\beta = 1$). The full-collar proxy is boundary dominated (minimizer at the first collar site for all w), while a center-site proxy increases when the collar center shifts inward.

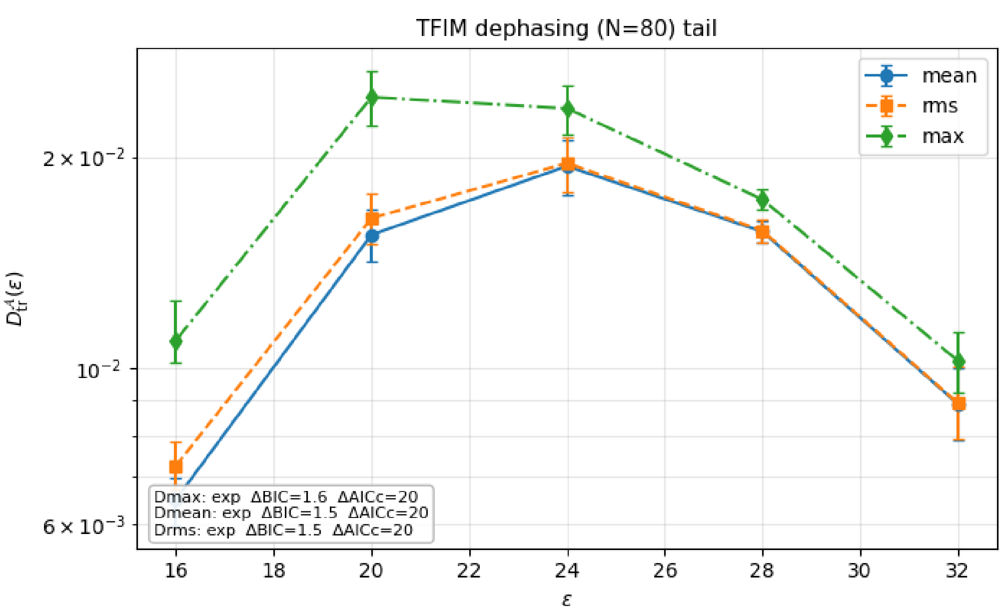

Figure 3: TFIM dephasing tail ($N = 80$): windowed influence proxy (illustrative).

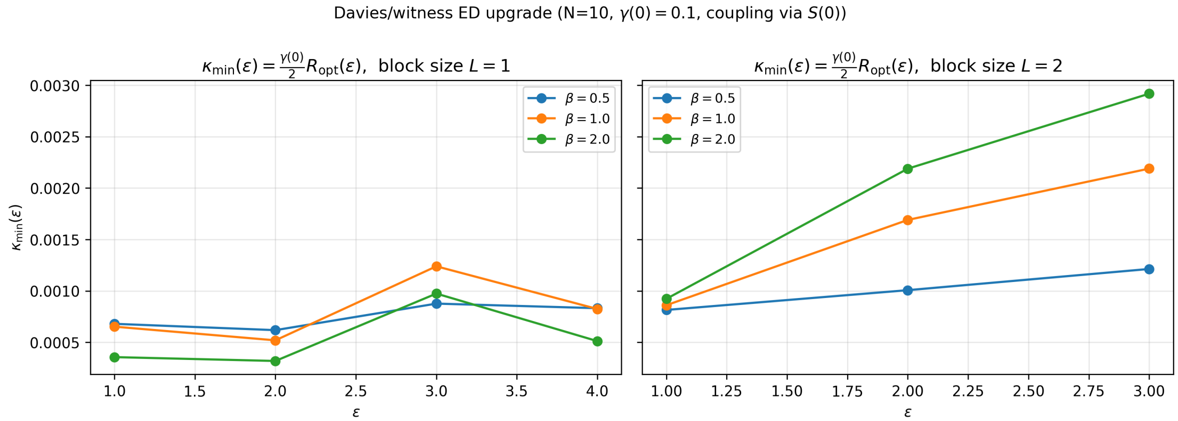

Figure 4: Witness-based lower bound plot (illustrative).

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