Quantitative Recovery Bounds from Vacuum Clustering in Finite-Mode Gaussian States

(A Regularized CCR Blueprint Motivated by Split Inclusions)

Version 2

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January 2026 (v1); July 2026 (v2)

Abstract

We prove a quantitative clustering–recovery bound for centered quasi-free (Gaussian) states in a finite-mode bosonic CCR (Weyl) setting. Motivated by split inclusions in algebraic quantum field theory, we work in a regularized framework where Gaussian states are parametrized by finite covariance matrices and a recovery map admits an explicit covariance block formula. Using a perturbative Gaussian fidelity input and explicit coercivity bounds for inverse covariances, we control the recovery error in terms of a vacuum cross-correlation factor, a cross-correlation perturbation parameter, and a recovery-error matrix norm $\|\Delta\Gamma\|_{\mathrm{HS}}$ with an explicit quadratic+quartic structure. In a distinguished class (Family A, $X = X_0$), this reduces to a bound in terms of the cross-block error $\|\Delta^{(12)}\|_{\mathrm{HS}}$. We include ancillary numerical sanity checks verifying the perturbative regime, a collar-envelope decay model, a dimension sweep $n_1 = n_2 \in \{1, 2, 3\}$, and phase-diagram checks of the perturbative domain. *v2 adds:* a collar-suppressed recovery corollary making the "clustering suppresses recovery error" mechanism a single displayed inequality; an upgraded discussion of the fidelity constants (the local coefficient 1/8 is shown numerically to be a directional benchmark, not a uniform bound, and an empirical constant is certified on the sampled domain); an independent verification suite in pure NumPy/SciPy implementing the Banchi–Braunstein–Pirandola fidelity formula with closed-form anchors (no external Gaussian-optics dependency), with all proved inequalities tested on random draws (zero violations); positioning remarks relative to the companion notes 2512.0060 and 2512.0101; and, aligning with 2512.0060 v2, corrected Petz-identification language (the recovery rule is conditional reattachment, with Petz agreement only in the factorized case) plus an explicit admissibility hypothesis for the recovered covariance.

Contents

1 Scope, positioning, and standing regularization

Remark 1.1 (Scope note). All results are proved in a finite-mode bosonic CCR setting (regularized Weyl algebra on a finite-dimensional symplectic space). The Type III AQFT setting is used only as motivation; continuum statements are formulated as conjectural directions.

Remark 1.2 (Positioning). In finite-mode continuous-variable quantum information, closed formulas for Gaussian fidelity and many norm inequalities are standard. The contribution of this note is an "interface block" that (i) packages vacuum clustering into a cross-correlation parameter η_{vac}, (ii) propagates perturbations through a Gaussian recovery map in covariance block form, and (iii) yields a quantitative recovery bound with explicit dependence on coercivity and correlation parameters.

Remark 1.3 (Relation to the companion notes (added in v2)). This note is the rigorous finite-mode core of the Gaussian clustering–recovery program of ai.viXra:2512.0060 [9]: there, the reattachment/Petz reconstruction is developed for lattice vacua with Bessel/Combes–Thomas collar suppression, and the repaired Proposition 2.14 (v2) restricts the Petz identification to a conditional reattachment with a no-steering admissibility hypothesis. Here, by contrast, everything is finite-mode and unconditional up to one explicit hypothesis (admissibility of the recovered covariance, Remark 3.3): the recovery rule is Definition 3.1 taken as an explicit covariance-level reattachment map – *not* identified with the general Petz map (Remark 3.2), in the same spirit as 0060 v2's repaired Proposition 2.14. The non-Gaussian extension of the same bridge is conjectured via conditional mutual information in ai.viXra:2512.0101 [10]; Conjecture 9.1 below is the Type III counterpart. The three notes are complementary layers of one program: lattice-Gaussian (0060), finite-mode rigorous core (this note), non-Gaussian/CMI (0101).

Table 1: Parameter dictionary (reading aid).

Symbol	Meaning	Where used
ϵ	coercivity margin for A relative to A_0	Theorem 6.8
η_{vac}	reference cross-correlation factor	Definition 4.1
δ	cross-correlation perturbation parameter	Definition 4.2
κ	$\epsilon^{-1/2}(\eta_{\mathrm{vac}}+\delta)$	Theorem 6.8
Φ	prefactor $(1-\kappa)^{-1}\min(\epsilon c_1, c_2)^{-1}$	Theorem 6.8
$\Delta\Gamma$	covariance recovery error $\Gamma-\tilde{\Gamma}$	Theorem 6.8
$\Delta^{(12)}$	cross-block error $X-\tilde{X}$	Definition 6.2

2 Finite-mode CCR Gaussian framework

2.1 Quadrature ordering and symplectic form

Remark 2.1 (Ordering convention)*.* Throughout, we use the standard continuous-variable ordering $(x_1,\dots,x_N,p_1,\dots,p_N)$, so that

$$\Omega=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}.$$

2.2 Finite symplectic space and Weyl algebra

Let $S\simeq\mathbb{R}^{2N}$ be a real symplectic vector space with symplectic form Ω. The finite-mode Weyl algebra is generated by Weyl operators $W(z)$, $z\in S$, satisfying $W(z)W(z')=e^{-\frac{i}{2}\Omega(z,z')}W(z+z')$.

2.3 Quasi-free states and covariance matrices

Definition 2.2 (Covariance and uncertainty condition)**.** *A centered quasi-free state is specified by a real symmetric covariance matrix* $\Gamma\in\mathbb{R}^{2N\times 2N}$ *satisfying* $\Gamma+\frac{i}{2}\Omega\succeq 0$.

Assumption 2.3 (Uniform positivity in the regularized regime)**.** *All covariance blocks under consideration are strictly positive and bounded. In particular, for the reference covariance blocks defined below, there exist* $c_1,c_2>0$ *such that* $A_0\succeq c_1\mathbf{1}$ *and* $B_0\succeq c_2\mathbf{1}$*.*

Remark 2.4 (Norm conventions)*.* For a real matrix M, $\|M\|_{\mathrm{HS}}:=\sqrt{\mathrm{Tr}(M^TM)}$ and $\|M\|_{\mathrm{op}}$ denotes the operator norm (largest singular value).

Definition 2.5 (Uhlmann fidelity)**.** *For density operators* ρ,σ *on the same finite-dimensional Hilbert space, define* $F(\rho,\sigma):=\|\sqrt{\rho}\sqrt{\sigma}\|_1^2$*. We write* $F(\omega_1,\omega_2)$ *for the fidelity of the corresponding density operators.*

Remark 2.6 (Covariance convention and numerical fidelity (updated in v2))*.* We use the convention that physical covariances satisfy $\Gamma+\frac{i\hbar}{2}\Omega\succeq 0$ and work with $\hbar=1$. In v1 all numerical fidelities were evaluated with `thewalrus.quantum.fidelity` (`hbar=1.0`). In v2 the verification suite instead implements the closed Banchi–Braunstein–Pirandola formula [3] directly in NumPy/SciPy (no external Gaussian-optics dependency) and certifies the implementation against exact closed-form anchors: single-mode pairs via the Marian–Marian formula [6], pure-state overlaps ($F(\mathrm{vac},S(r)\mathrm{vac})=1/\cosh r$), the correlated two-mode squeezed vacuum ($F(\mathrm{vac}^{\otimes 2},\mathrm{TMSV}(r))=1/\cosh^2 r$), multiplicativity, and symplectic invariance; all anchors agree to 10^{-10}. When `thewalrus` is installed the suite runs an optional cross-check.

2.4 Bipartition and block form

Fix a bipartition $S = S_1 \oplus S_2$ with dimensions $2n_1$ and $2n_2$. Write

$$\Gamma = \begin{pmatrix} A & X \\ X^T & B \end{pmatrix}, \qquad \Gamma_0 = \begin{pmatrix} A_0 & X_0 \\ X_0^T & B_0 \end{pmatrix}.$$

3 Gaussian recovery in block form

Definition 3.1 (Gaussian recovery map (covariance-level))**.** *Given a centered reference covariance Γ_0 and a centered input covariance Γ as above, define the recovered covariance $\tilde{\Gamma}$ by*

$$\tilde{A} = A, \qquad \tilde{X} = AA_0^{-1}X_0, \qquad \tilde{B} = B_0 + X_0^T A_0^{-1}(A - A_0)A_0^{-1}X_0.$$

Remark 3.2 (Relation to Petz recovery; corrected in v2). Definition 3.1 is the covariance-level *conditional reattachment* rule used in this paper. It coincides with the Petz output in the factorized-reference case $X_0 = 0$, and may approximate the Gaussian Petz output in strongly mixed, weak-correlation regimes, but it is *not* the Petz recovery map for general correlated Gaussian references: for correlated references the Petz map does not preserve the input marginal, and for a pure reference it returns the reference for every input (the failure mode identified and repaired in 2512.0060 v2 [9]). The main results below use Definition 3.1 only as an explicit covariance-level reconstruction rule; see Appendix A.

Remark 3.3 (Admissibility convention (added in v2)). Throughout, Definition 3.1 is used only on inputs for which the recovered matrix is a valid covariance, $\tilde{\Gamma} + \frac{i}{2}\Omega \succeq 0$; equivalently, the reconstruction is a covariance-level rule on its admissible domain. This is the finite-mode counterpart of the admissibility/no-steering hypothesis of 2512.0060 v2 [9]. The verification suite checks this condition on every draw (Section 8).

4 Clustering input: vacuum cross-correlation and perturbations

Definition 4.1 (Vacuum cross-correlation factor)**.** $\eta_{\text{vac}} := \|A_0^{-1/2} X_0 B_0^{-1/2}\|_{\text{op}}$.

Definition 4.2 (Cross-correlation perturbation parameter)**.** $\delta := \|A_0^{-1/2}(X - X_0)B_0^{-1/2}\|_{\text{op}}$.

Remark 4.3 (Bessel-type collar envelope (motivational interface)). In massive relativistic settings, equal-time correlators admit envelopes controlled by modified Bessel functions $K_\nu(mr)$ with large-mr asymptotics $K_\nu(mr) \sim \sqrt{\pi/(2mr)}\, e^{-mr}$. In our finite-mode posture, r is a collar parameter controlling the magnitude of X_0, and our sanity checks use the model envelope $\|X_0(r)\|_{\text{op}} \propto (mr)^p K_\nu(mr)$.

5 Gaussian fidelity control (perturbative regime)

Lemma 5.1 (Perturbative Gaussian fidelity bound (imported, constants on a compact set))**.** *Let ω_1, ω_2 be centered Gaussian states with covariances $\Gamma_1 \succ 0$ and Γ_2. Define $K := \Gamma_1^{-1/2}(\Gamma_1 - \Gamma_2)\Gamma_1^{-1/2}$. Assume $\|K\|_{\text{op}} \le \frac{1}{2}$. Then there exist constants $C_2, C_4 > 0$, depending only on the fixed finite-mode regularization (in particular on the total mode number N) and on coercivity bounds of the form $\Gamma_1 \succeq c\,\mathbf{1}$, such that*

$$1 - F(\omega_1, \omega_2) \le C_2\|K\|_{\text{HS}}^2 + C_4\|K\|_{\text{HS}}^4.$$

In particular, there exists $C > 0$ with the same dependence such that if $\|K\|_{\mathrm{HS}} \leq 1$ then $1 - F(\omega_1, \omega_2) \leq C\|K\|_{\mathrm{HS}}^2$.

Remark 5.2 (Reference and empirical constants (upgraded in v2)). Closed formulas for the fidelity of Gaussian states and perturbative expansions are standard; see [3, 4, 5]. Lemma 5.1 is used as a local perturbative input; we do not optimize constants. *v2 adds two quantitative facts from the verification suite (Section 8).* First, the local Taylor coefficient 1/8 suggested by expansions of the closed fidelity formula is a *directional benchmark, not a uniform bound*: across 450 random Family-A draws ($n_1 = n_2 \in \{1, 2, 3\}$), the ratio $(1 - F)/\big(\frac{1}{8}\|K\|_{\mathrm{HS}}^2\big)$ reaches 1.142 – and the worst case occurs at $\|K\|_{\mathrm{HS}} \approx 2 \times 10^{-3}$, i.e. in the local regime, so the excess is directional rather than higher-order. Second, on the sampled compact domain the suite certifies the empirical constant $C_2^{\mathrm{emp}} \approx 0.143$, i.e. $1 - F \leq C_2^{\mathrm{emp}} \|K\|_{\mathrm{HS}}^2$ across all draws, consistent with (and instantiating) the existence claim of Lemma 5.1.

6 Main theorem: clustering–recovery bridge (Route 1)

Definition 6.1 (Partial local excitation). *We say ω is a partial local excitation (relative to the bipartition) if $B = B_0$.*

Definition 6.2 (Recovery cross-block error). *With $\tilde{X}$ as in Definition 3.1, define $\Delta^{(12)} := X - \tilde{X} = X - AA_0^{-1}X_0$.*

Lemma 6.3 (From relative perturbation to a uniform lower bound). *If $\|A_0^{-1/2}(A - A_0)A_0^{-1/2}\|_{\mathrm{op}} \leq 1 - \epsilon$ with $\epsilon \in (0, 1)$, then $A \succeq \epsilon A_0$.*

Proof. Let $E := A_0^{-1/2}(A - A_0)A_0^{-1/2}$. The hypothesis implies $E \succeq -(1 - \epsilon)\mathbf{1}$, hence $A = A_0^{1/2}(\mathbf{1} + E)A_0^{1/2} \succeq \epsilon A_0$. □

Lemma 6.4 (Block structure of the recovery covariance error). *Assume $B = B_0$ and $\tilde{\Gamma}$ is given by Definition 3.1. Then $\Delta\Gamma := \Gamma - \tilde{\Gamma}$ has block form*

$$\Delta\Gamma = \begin{pmatrix} 0 & \Delta^{(12)} \\ (\Delta^{(12)})^T & \Delta^{(22)} \end{pmatrix}, \qquad \Delta^{(22)} = B_0 - \tilde{B} = -X_0^T A_0^{-1}(A - A_0)A_0^{-1}X_0.$$

In particular, $\|\Delta\Gamma\|_{\mathrm{HS}}^2 = 2\|\Delta^{(12)}\|_{\mathrm{HS}}^2 + \|\Delta^{(22)}\|_{\mathrm{HS}}^2$, and $\|\Delta^{(22)}\|_{\mathrm{HS}} \leq \|A - A_0\|_{\mathrm{HS}}\|A_0^{-1}X_0\|_{\mathrm{op}}^2$.

Proof. The block form is immediate from $\tilde{A} = A$ and $B = B_0$. For the estimate, use $\|UVW\|_{\mathrm{HS}} \leq \|U\|_{\mathrm{op}}\|V\|_{\mathrm{HS}}\|W\|_{\mathrm{op}}$ with $U = X_0^T A_0^{-1}$, $V = A - A_0$, $W = A_0^{-1}X_0$. □

Proposition 6.5 (Coercivity bound for $\|\Gamma^{-1}\|_{\mathrm{op}}$ with no hidden constants). *Let $B_0 \succ 0$ and $\Gamma = \begin{pmatrix} A & X \\ X^T & B_0 \end{pmatrix}$ with $A \succ 0$. Define $\eta_\omega := \|A^{-1/2}XB_0^{-1/2}\|_{\mathrm{op}}$. If $\eta_\omega < 1$, then*

$$\Gamma \succeq (1 - \eta_\omega)\begin{pmatrix} A & 0 \\ 0 & B_0 \end{pmatrix}, \qquad \textit{hence} \qquad \|\Gamma^{-1}\|_{\mathrm{op}} \leq \frac{1}{(1 - \eta_\omega)\min(\lambda_{\min}(A), \lambda_{\min}(B_0))}.$$

In particular, if $A \succeq \epsilon A_0$, $A_0 \succeq c_1\mathbf{1}$ and $B_0 \succeq c_2\mathbf{1}$, then $\|\Gamma^{-1}\|_{\mathrm{op}} \leq \big[(1 - \eta_\omega)\min(\epsilon c_1, c_2)\big]^{-1}$.

Proof. Let $u \in \mathbb{R}^{2n_1}$, $v \in \mathbb{R}^{2n_2}$ and set $a := A^{1/2}u$, $b := B_0^{1/2}v$. Then

$$\begin{pmatrix} u \\ v \end{pmatrix}^T \Gamma \begin{pmatrix} u \\ v \end{pmatrix} = \|a\|_2^2 + 2\, a^T \big(A^{-1/2} X B_0^{-1/2}\big) b + \|b\|_2^2.$$

By Cauchy–Schwarz, $2\, a^T(A^{-1/2}XB_0^{-1/2})b \geq -2\eta_\omega \|a\|_2\|b\|_2 \geq -\eta_\omega(\|a\|_2^2 + \|b\|_2^2)$, which proves the Loewner bound; taking minimal eigenvalues yields the inverse-norm bound. □

Proposition 6.6 (Bounding η_ω by $\eta_{\mathrm{vac}} + \delta$). *Assume $A \succeq \epsilon A_0$ with $\epsilon \in (0,1)$. Then $\eta_\omega \leq \epsilon^{-1/2}\|A_0^{-1/2}XB_0^{-1/2}\|_{\mathrm{op}} \leq \epsilon^{-1/2}(\eta_{\mathrm{vac}} + \delta)$.*

Proof. Since $A \succeq \epsilon A_0$, $\|A^{-1/2}A_0^{1/2}\|_{\mathrm{op}} \leq \epsilon^{-1/2}$; submultiplicativity and the triangle inequality for $X = X_0 + (X - X_0)$ finish the proof. □

Proposition 6.7 (A simple sufficient condition for the perturbative regime). *If $\|\Gamma^{-1}\|_{\mathrm{op}}\|\Delta\Gamma\|_{\mathrm{HS}} \leq \frac{1}{2}$, then $\|K\|_{\mathrm{op}} \leq \frac{1}{2}$ and Lemma 5.1 applies.*

Proof. $\|K\|_{\mathrm{op}} = \|\Gamma^{-1/2}\Delta\Gamma\,\Gamma^{-1/2}\|_{\mathrm{op}} \leq \|\Gamma^{-1}\|_{\mathrm{op}}\|\Delta\Gamma\|_{\mathrm{op}} \leq \|\Gamma^{-1}\|_{\mathrm{op}}\|\Delta\Gamma\|_{\mathrm{HS}}$. □

Theorem 6.8 (Clustering–recovery bridge (finite-mode Gaussian; Route 1, quadratic+quartic)). *Let ω_0 be a centered reference Gaussian state with covariance Γ_0 and blocks (A_0, X_0, B_0) satisfying Assumption 2.3. Let ω be a centered Gaussian state with covariance Γ and blocks (A, X, B) such that ω is a partial local excitation (Definition 6.1), i.e. $B = B_0$. Assume there exists $\epsilon \in (0,1)$ such that $\|A_0^{-1/2}(A - A_0)A_0^{-1/2}\|_{\mathrm{op}} \leq 1 - \epsilon$, and define $\kappa := \epsilon^{-1/2}(\eta_{\mathrm{vac}} + \delta)$. Assume $\kappa < 1$. Let $\tilde{\omega}$ be the recovered centered Gaussian state with covariance $\tilde{\Gamma}$ given by Definition 3.1. Assume in addition that the recovered covariance $\tilde{\Gamma}$ is physical, $\tilde{\Gamma} + \frac{i}{2}\Omega \succeq 0$ (Remark 3.3), and that $\|K\|_{\mathrm{op}} \leq \frac{1}{2}$, where $K := \Gamma^{-1/2}(\Gamma - \tilde{\Gamma})\Gamma^{-1/2}$ (for instance, it suffices that Proposition 6.7 holds). Then, with $\Delta\Gamma := \Gamma - \tilde{\Gamma}$,*

$$1 - F(\omega, \tilde{\omega}) \leq C_2\, \Phi^2 \|\Delta\Gamma\|_{\mathrm{HS}}^2 + C_4\, \Phi^4 \|\Delta\Gamma\|_{\mathrm{HS}}^4, \qquad \Phi := \frac{1}{(1-\kappa)\min(\epsilon c_1, c_2)},$$

where $C_2, C_4 > 0$ are the constants from Lemma 5.1.

Proof. By submultiplicativity and $\|M\|_{\mathrm{op}} \leq \|M\|_{\mathrm{HS}}$, $\|K\|_{\mathrm{HS}} \leq \|\Gamma^{-1}\|_{\mathrm{op}}\|\Delta\Gamma\|_{\mathrm{HS}}$. By Lemma 6.3, $A \succeq \epsilon A_0$, and Proposition 6.6 yields $\eta_\omega \leq \kappa < 1$. Therefore Proposition 6.5 gives $\|\Gamma^{-1}\|_{\mathrm{op}} \leq \Phi$. Insert into Lemma 5.1. □

Remark 6.9 (Coarsenings of the correlation denominator). For $\kappa \in [0,1)$, $1 - \kappa^2 = (1-\kappa)(1+\kappa) \leq 2(1-\kappa)$, hence $(1-\kappa)^{-m} \leq 2^m(1-\kappa^2)^{-m}$ for $m = 1, 2, 4$; prefactors may equivalently be rewritten in terms of $(1-\kappa^2)$ at the cost of numerical constants. We keep the explicit $(1-\kappa)$ form matching Proposition 6.5.

Corollary 6.10 (Purely quadratic regime). *Under the hypotheses of Theorem 6.8, if additionally $\|K\|_{\mathrm{HS}} \leq 1$, then $1 - F(\omega, \tilde{\omega}) \leq C\, \Phi^2 \|\Delta\Gamma\|_{\mathrm{HS}}^2$, with C from Lemma 5.1 and Φ as in Theorem 6.8.*

Corollary 6.11 (Reduction to $\|\Delta^{(12)}\|_{\mathrm{HS}}$ in the Family A case). *Assume the hypotheses of Theorem 6.8 and, in addition, $X = X_0$. Then one may take $C_\Delta := \sqrt{2 + \|X_0^T A_0^{-1}\|_{\mathrm{op}}^2}$ so that $\|\Delta\Gamma\|_{\mathrm{HS}} \leq C_\Delta \|\Delta^{(12)}\|_{\mathrm{HS}}$.*

Proof. If $X = X_0$, then $\Delta^{(12)} = X_0 - AA_0^{-1}X_0 = -(A - A_0)A_0^{-1}X_0$. Lemma 6.4 gives $\Delta^{(22)} = X_0^T A_0^{-1}\Delta^{(12)}$, hence $\|\Delta^{(22)}\|_{\mathrm{HS}} \le \|X_0^T A_0^{-1}\|_{\mathrm{op}}\|\Delta^{(12)}\|_{\mathrm{HS}}$; substitute into $\|\Delta\Gamma\|_{\mathrm{HS}}^2 = 2\|\Delta^{(12)}\|_{\mathrm{HS}}^2 + \|\Delta^{(22)}\|_{\mathrm{HS}}^2$. □

Corollary 6.12 (Collar-suppressed recovery (added in v2))**.** *Assume the hypotheses of Theorem 6.8 with $X = X_0$ (Family A, so $\delta = 0$) and $\|K\|_{\mathrm{HS}} \le 1$. Then*

$$\boxed{1 - F(\omega, \tilde{\omega}) \;\le\; C\,\Phi^2\, C_\Delta^2\; \|A - A_0\|_{\mathrm{HS}}^2\; \left\|A_0^{-1}X_0\right\|_{\mathrm{op}}^2}$$

with C from Lemma 5.1, Φ from Theorem 6.8 and C_Δ from Corollary 6.11. In particular, under a collar envelope $\|X_0(r)\|_{\mathrm{op}} \propto (mr)^p K_\nu(mr)$, the right-hand side inherits the squared envelope e^{-2mr} (up to the polynomial factor) at fixed perturbation strength $\|A - A_0\|_{\mathrm{HS}}$: vacuum clustering suppresses the recovery error quadratically in the correlator envelope.

Proof. $\Delta^{(12)} = -(A - A_0)A_0^{-1}X_0$ gives $\|\Delta^{(12)}\|_{\mathrm{HS}} \le \|A - A_0\|_{\mathrm{HS}}\|A_0^{-1}X_0\|_{\mathrm{op}}$; combine with Corollaries 6.10 and 6.11. For the envelope statement, $\|A_0^{-1}X_0\|_{\mathrm{op}} \le \|A_0^{-1}\|_{\mathrm{op}}\|X_0\|_{\mathrm{op}} \le c_1^{-1}\|X_0(r)\|_{\mathrm{op}}$. □

7 Ancillary numerical sanity-checks

Remark 7.1 (Reproducibility and physicality checks)*.* The numerical experiments check physicality of both Γ and $\tilde{\Gamma}$ by verifying $\lambda_{\min}(\Gamma + \frac{i}{2}\Omega) \ge -\tau$ and $\lambda_{\min}(\tilde{\Gamma} + \frac{i}{2}\Omega) \ge -\tau$ for a fixed tolerance $\tau > 0$.

Remark 7.2 (Numerical constants versus theoretical constants (updated in v2))*.* Lemma 5.1 is stated with constants on a compact set. In the numerical plots we additionally compare against sharper coefficients such as $\frac{1}{8}$ and $\frac{1}{48}$ suggested by local Taylor expansions of closed Gaussian fidelity formulas; per Remark 5.2, v2 quantifies their status: $\frac{1}{8}$ is a directional benchmark exceeded by up to 14% in worst-case directions, and the certified empirical constant on the sampled domain is $C_2^{\mathrm{emp}} \approx 0.143$.

7.1 Representative figures

For readability, we keep the base sanity check figures in the main text and move extended sweeps to Appendix B.

8 Independent verification suite (added in v2)

The repository ships `verification/2601-0007/verify_2601_0007.py`, a pure NumPy/SciPy suite with no Gaussian-optics dependency:

(i) Certified fidelity kernel. The Banchi–Braunstein–Pirandola closed formula [3] is implemented directly ($\hbar = 1$ convention of Remark 2.6) and certified against exact anchors – the single-mode Marian–Marian formula [6], pure-state overlaps $1/\cosh r$, the two-mode squeezed vacuum $1/\cosh^2 r$ (a correlated multimode case), multiplicativity on products, and symplectic invariance – all to 10^{-10}.

(ii) Inequality suite. On 450 random Family-A draws ($n_1 = n_2 \in \{1, 2, 3\}$, 150 each, physicality enforced), the suite verifies with *zero violations*: physicality of the recovered covariance $\tilde{\Gamma} + \frac{i}{2}\Omega \succeq 0$ on every draw (Remark 3.3); the exact block identity and HS decomposition of Lemma 6.4 (machine precision); the coercivity inequality of Proposition 6.5; the bound of Proposition 6.6; Corollary 6.11 with its explicit C_Δ; the norm chain $\|K\|_{\mathrm{HS}} \le \|\Gamma^{-1}\|_{\mathrm{op}}\|\Delta\Gamma\|_{\mathrm{HS}} \le \Phi\|\Delta\Gamma\|_{\mathrm{HS}}$; and the collar-suppressed bound of Corollary 6.12.

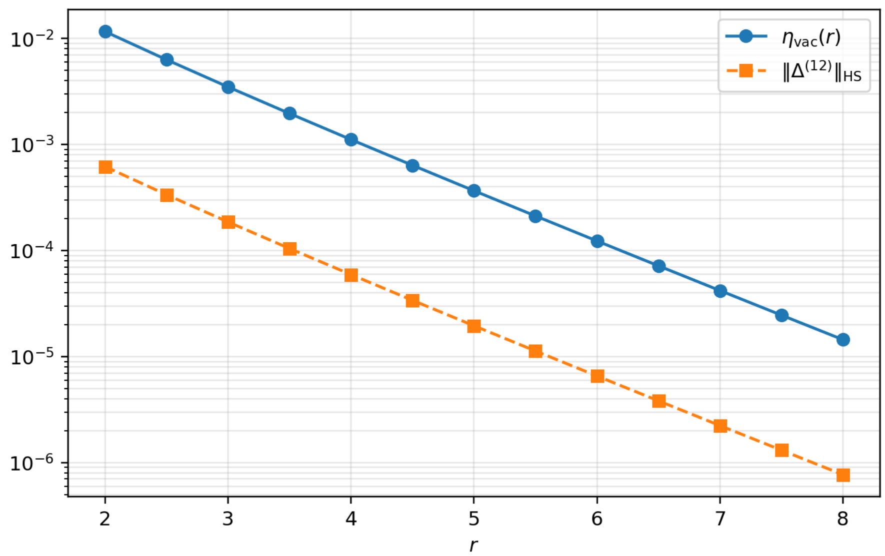

Figure 1: Base sanity check ($n_1 = n_2 = 2$): $\eta_{\mathrm{vac}}(r)$ and $\|\Delta^{(12)}\|_{\mathrm{HS}}$ versus r under a Bessel-type envelope.

(iii) Constants. The suite certifies $C_2^{\mathrm{emp}} \approx 0.143$ on the sampled domain and locates the worst-case direction (ratio 1.142 against the $\frac{1}{8}$ benchmark at $\|K\|_{\mathrm{HS}} \approx 2 \times 10^{-3}$; see Remark 5.2).

(iv) Collar sweep. Under the Bessel envelope of Remark 4.1 with $r \in [2, 8]$, both $1 - F$ (from 3.9×10^{-6} to 5.9×10^{-12}) and the Corollary 6.12 bound (from 1.4×10^{-5} to 1.9×10^{-11}) decay monotonically, with the bound dominating pointwise.

9 Conjectural outlook (Type III / non-Gaussian)

Conjecture 9.1 (Type III clustering–recovery blueprint)**.** *Let ω_0 be the vacuum state of a massive AQFT satisfying a split inclusion for $\mathcal{O}_1 \Subset \mathcal{O}_2$. For suitable classes of states ω that are vacuum-like outside $\mathcal{O}_2$ and have finite Araki relative entropy, the Petz recovery associated with the ω_0-preserving conditional expectation onto a canonical split factor yields an estimate of the form $1 - F(\omega, \tilde{\omega}) \leq C\, \mathcal{C}(\mathcal{O}_1, \mathcal{O}_2)^{\gamma}$, where $\mathcal{C}(\mathcal{O}_1, \mathcal{O}_2)$ is a clustering coefficient controlled by collar width and the mass gap.*

Remark 9.2 (Type III roadmap). The transition to the Type III setting likely requires noncommutative L^p methods (e.g. Haagerup spaces) to express quantitative analogues of density operators and fidelity-type functionals. In this picture, covariance-block control in the finite-mode Gaussian setting is expected to be replaced by bounds on relative modular operators associated with the split factor and the ω_0-preserving conditional expectation. The CMI-based route to the same bridge is developed in [10].

Note added in v2

No theorem of v1 required correction: all v1 bounds, proofs and figures are unchanged. v2 does correct the non-load-bearing *Petz identification* language of v1 (Remark 3.2, Appendix A): the covariance rule of Definition 3.1 is conditional reattachment, not the general Petz map for correlated

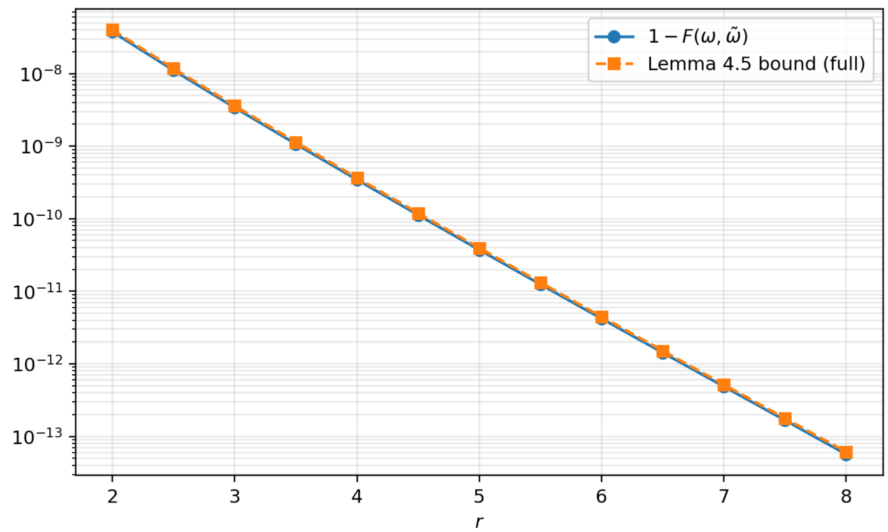

Figure 2: Base sanity check ($n_1 = n_2 = 2$): $1 - F(\omega, \tilde{\omega})$ versus a perturbative Gaussian fidelity benchmark.

references – aligning this note with the repaired Proposition 2.14 of 2512.0060 v2. The theorem now also states explicitly the admissibility hypothesis $\tilde{\Gamma} + \frac{i}{2}\Omega \succeq 0$ (Remark 3.3), previously only enforced in the numerics. v2 further adds: (1) Corollary 6.12 (collar-suppressed recovery), packaging the main mechanism into one displayed inequality with squared-envelope decay; (2) Remark 5.2: the $\frac{1}{8}$ Taylor coefficient is a directional benchmark exceeded by up to 14% in worst-case directions even as $\|K\| \to 0$; an empirical constant $C_2^{\mathrm{emp}} \approx 0.143$ is certified on the sampled domain, instantiating Lemma 5.1; (3) Section 8: an independent pure-NumPy verification suite (BBP fidelity kernel certified to 10^{-10} against closed-form anchors, including a correlated two-mode case; 450 draws, zero violations of any proved inequality; collar sweep), removing the `thewalrus` dependency of the v1 appendix script (kept as optional cross-check); (4) Remark 1.3: positioning relative to 2512.0060 and 2512.0101 (lattice-Gaussian / finite-mode core / non-Gaussian CMI); (5) references added: [6], series companions.

A Appendix: conditional reattachment covariance rule (rewritten in v2)

The v1 version of this appendix presented a "Petz identification sketch" arguing that Definition 3.1 matches the Petz map by requiring that the map (i) fixes the subsystem-1 marginal and (ii) recovers the reference exactly. That heuristic is false in general: conditions (i)–(ii) characterize the *conditional reattachment* rule, not the Petz map, which for correlated references does not satisfy (i), and for pure references collapses to the reference on all inputs (2512.0060 v2 [9]). What survives is the following: Definition 3.1 is the unique covariance-level map that is affine in A, fixes the subsystem-1 marginal ($\tilde{A} = A$), and is exact on the reference ($\Gamma = \Gamma_0 \Rightarrow \tilde{\Gamma} = \Gamma_0$); it agrees with the Petz output when $X_0 = 0$ (factorized reference), and its relation to the Gaussian Petz map beyond that case is an approximation question in strongly mixed regimes (route: Lami–Das–Wilde), not an identity. All results of this paper use only the reattachment rule on its admissible domain (Remark 3.3).

B Appendix: extended figures

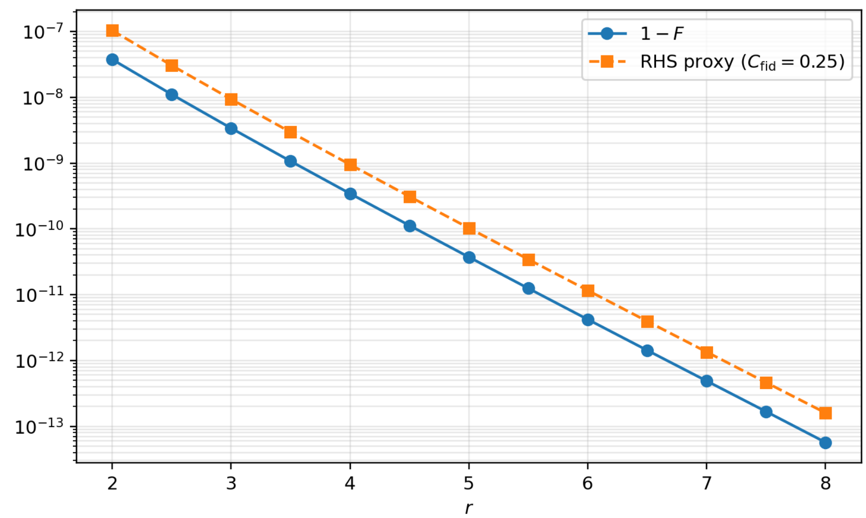

Figure 3: Base sanity check ($n_1 = n_2 = 2$): $1-F$ versus a proxy of the RHS scaling in Theorem 6.8.

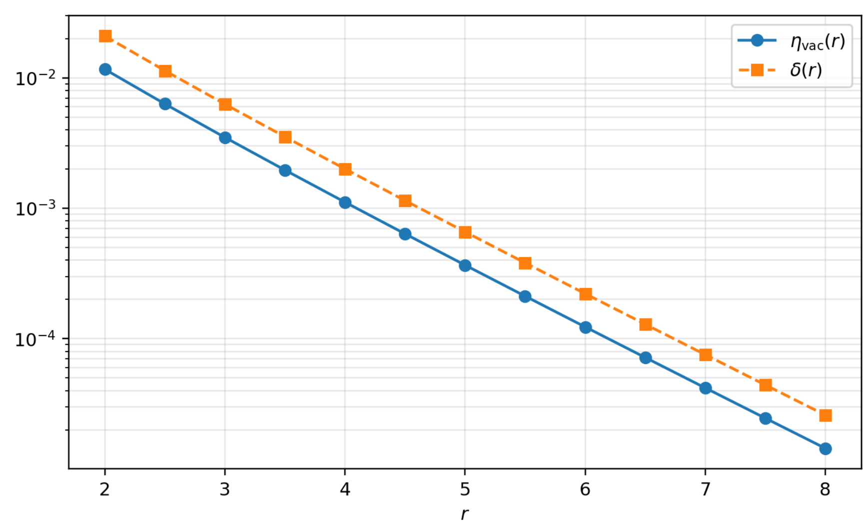

Figure 4: N1 (Family B): engineered perturbation $\delta(r)$ inherits the collar envelope and decays with r, alongside $\eta_{\text{vac}}(r)$.

C Appendix: verification interface (updated in v2)

The v1 appendix carried a minimal runnable script depending on `thewalrus==0.22.0`. In v2 it is superseded by the full suite `verification/2601-0007/` in the series repository, consisting of `gauss_fid.py` (certified pure-NumPy BBP fidelity, Remark 2.6) and `verify_2601_0007.py` (Section 8); `thewalrus`, when installed, is used only as an optional cross-check. The recovery map and

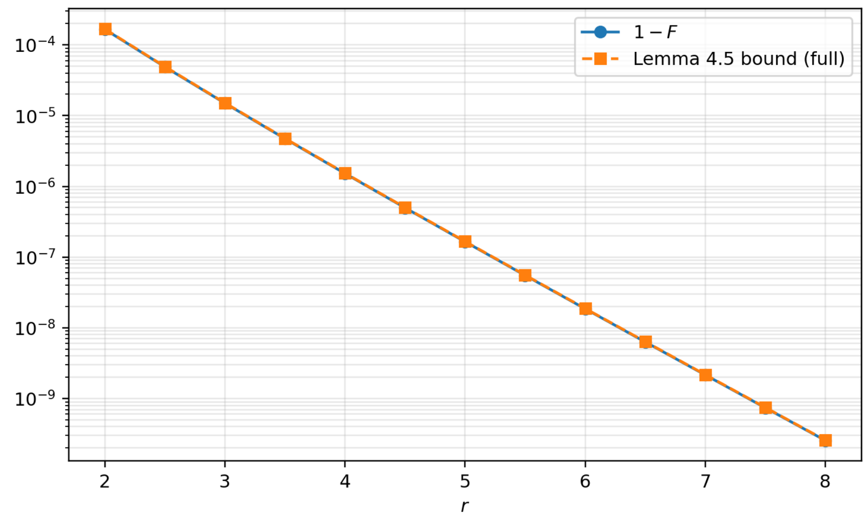

Figure 5: N1: fidelity-benchmark comparison for the Family B construction.

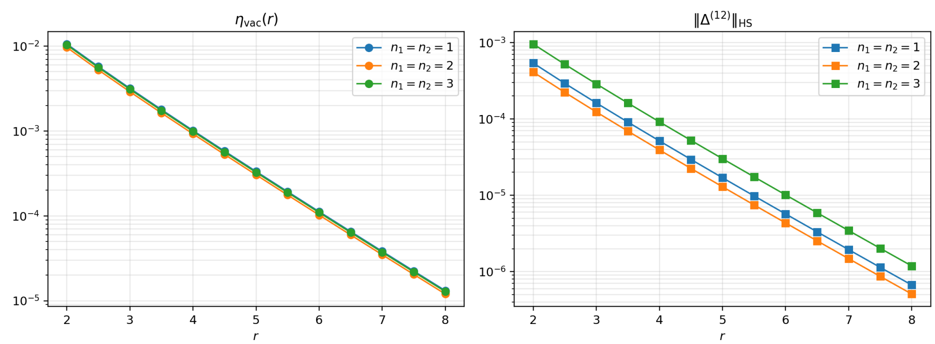

Figure 6: N2: $\eta_{\text{vac}}(r)$ and $\|\Delta^{(12)}\|_{\text{HS}}$ for $n_1 = n_2 \in \{1, 2, 3\}$.

all parameter definitions follow this paper's equations verbatim; the suite exits nonzero if any check fails.

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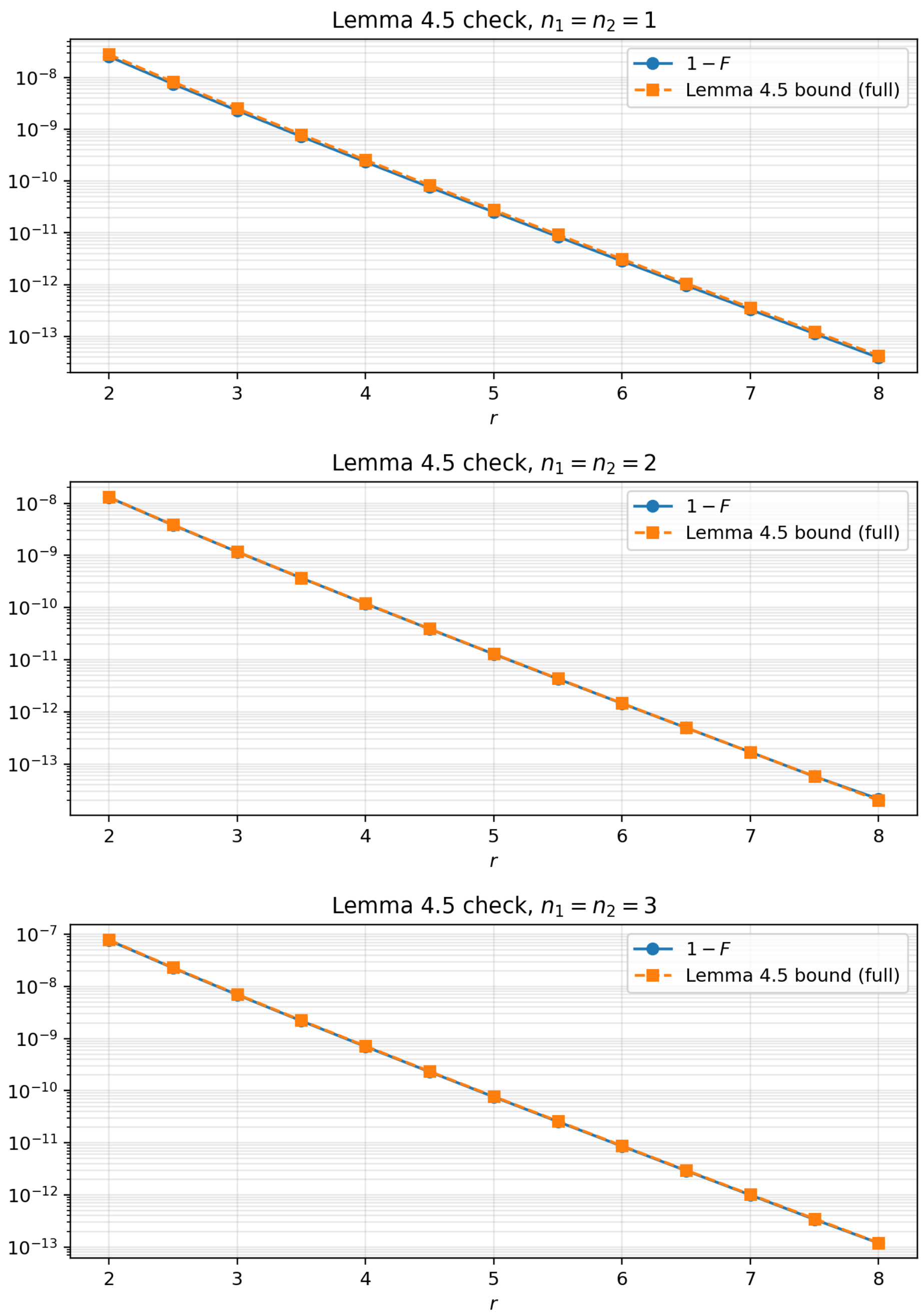

Figure 7: N2: fidelity-benchmark comparison across $n_1 = n_2 \in \{1, 2, 3\}$.

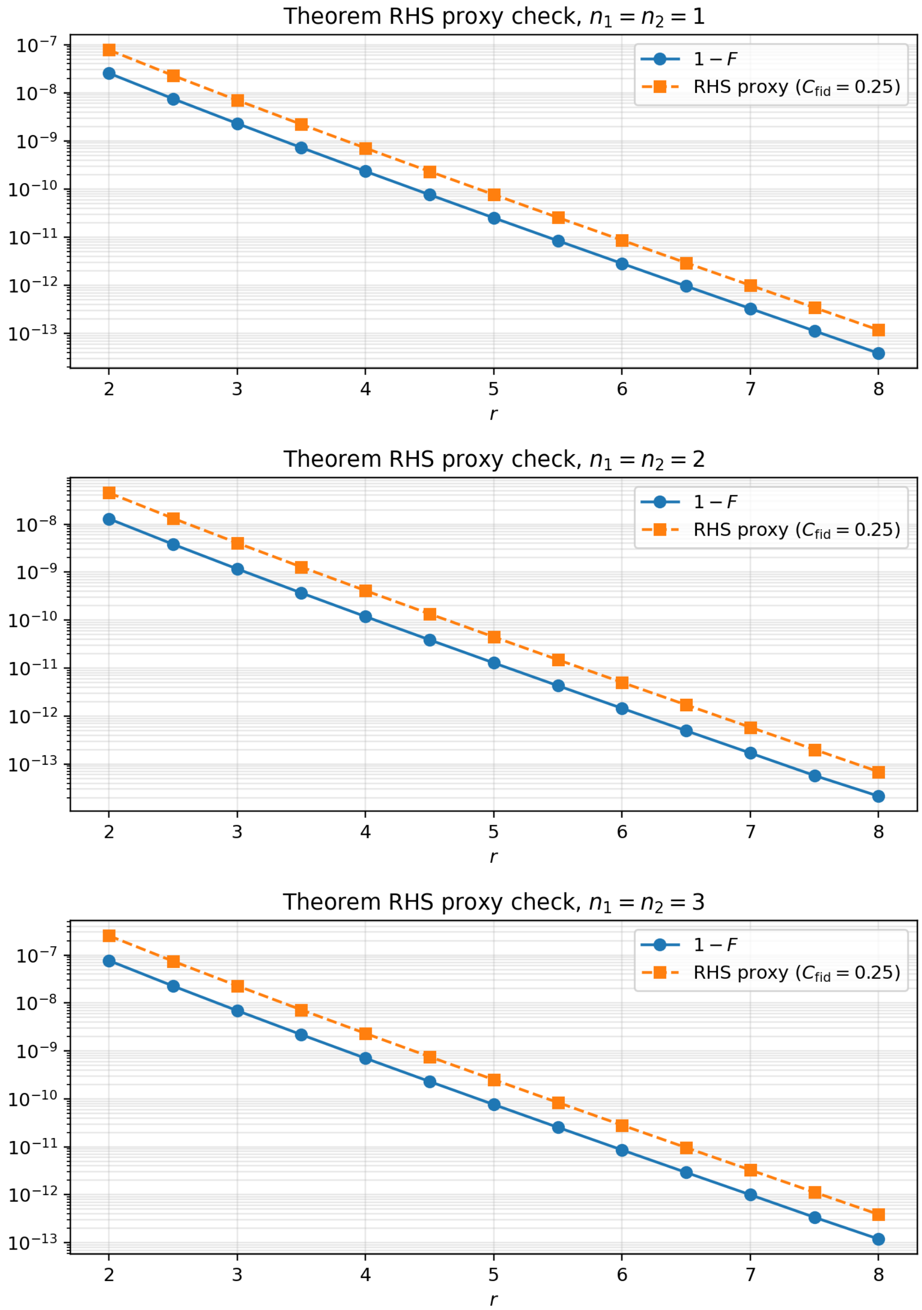

Figure 8: N2: $1 - F$ versus the theorem RHS proxy across $n_1 = n_2 \in \{1, 2, 3\}$.

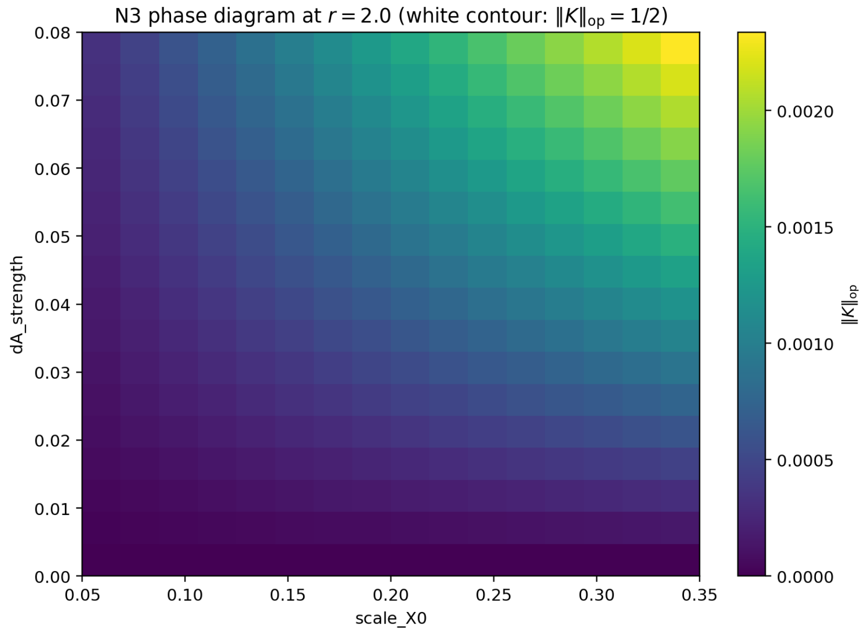

Figure 9: N3 (fixed $r = 2$): heatmap of $\|K\|_{\text{op}}$ in a deep perturbative region.

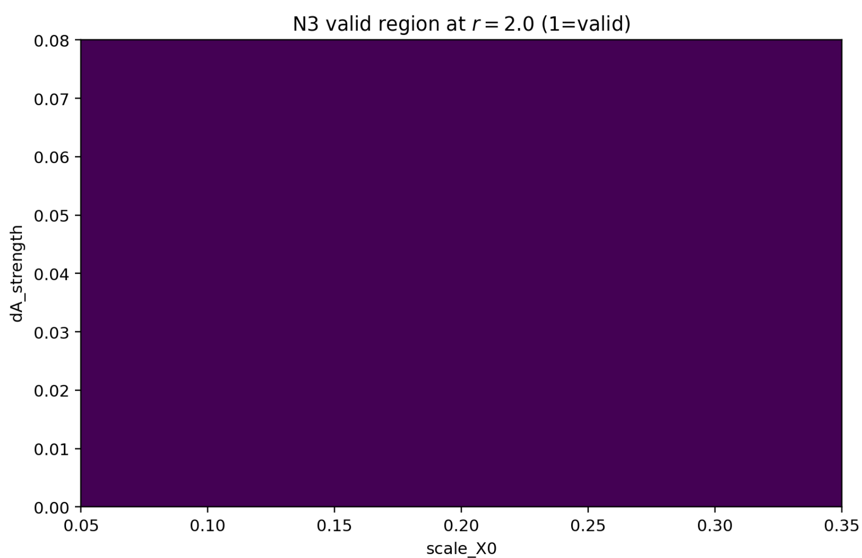

Figure 10: N3 (fixed $r = 2$): validity region for physicality and $\|K\|_{\text{op}} \leq \frac{1}{2}$.

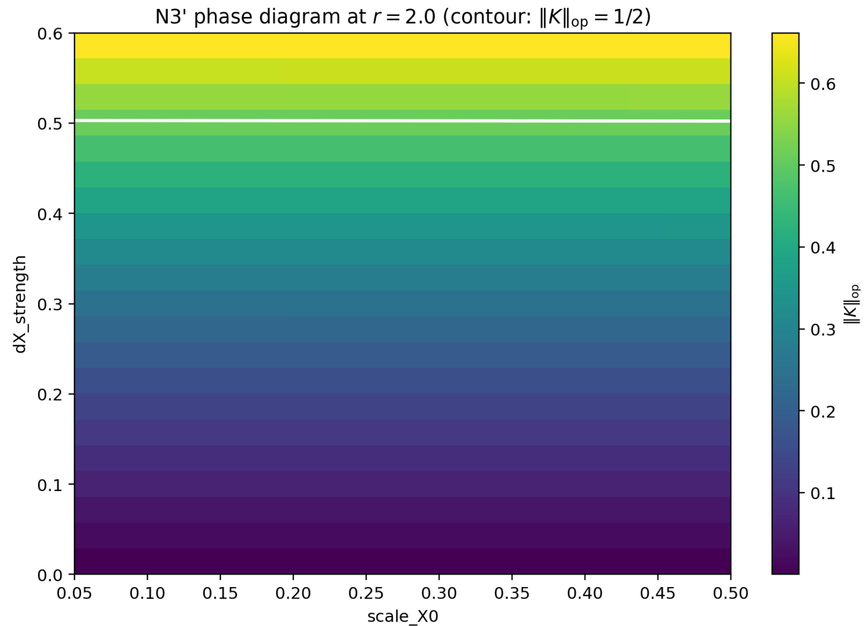

Figure 11: N3' (fixed $r = 2$): phase diagram of $\|K\|_{\mathrm{op}}$ versus scale X_0 and dX strength; contour indicates $\|K\|_{\mathrm{op}} = \frac{1}{2}$.

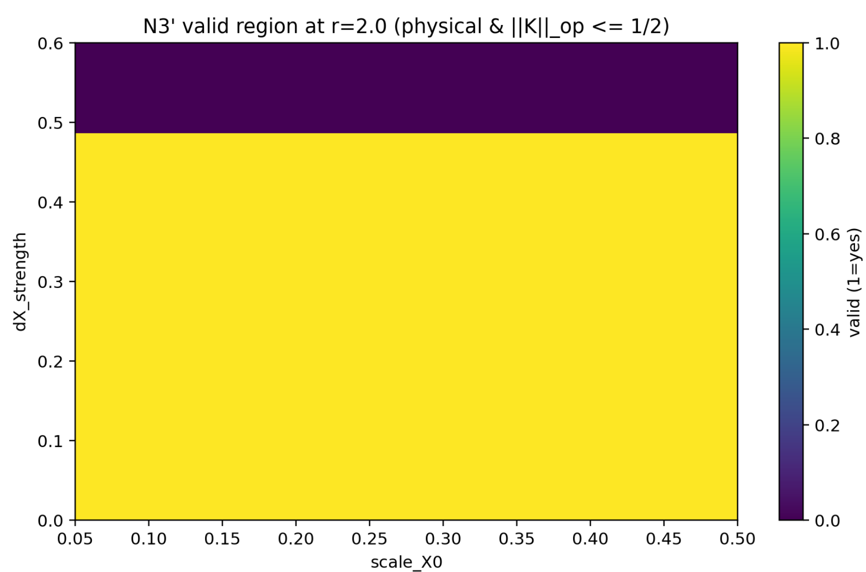

Figure 12: N3' (fixed $r = 2$): validity region where the state remains physical and satisfies $\|K\|_{\mathrm{op}} \leq \frac{1}{2}$.