Quantitative Recovery Bounds from Vacuum Clustering in Finite-Mode Gaussian States (A Regularized CCR Blueprint Motivated by Split Inclusions)

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January 2026

Abstract

We prove a quantitative clustering–recovery bound for centered quasi-free (Gaussian) states in a finite-mode bosonic CCR (Weyl) setting. Motivated by split inclusions in algebraic quantum field theory, we work in a regularized framework where Gaussian states are parametrized by finite covariance matrices and a recovery map admits an explicit covariance block formula. Using a perturbative Gaussian fidelity input and explicit coercivity bounds for inverse covariances, we control the recovery error in terms of a vacuum cross-correlation factor, a cross-correlation perturbation parameter, and a recovery-error matrix norm $\|\Delta\Gamma\|_{\mathrm{HS}}$ with an explicit quadratic+quartic structure. In a distinguished class (Family A, $X = X_0$), this reduces to a bound in terms of the cross-block error $\|\Delta^{(12)}\|_{\mathrm{HS}}$. We include ancillary numerical sanity checks verifying the perturbative regime, a collar-envelope decay model, a dimension sweep $n_1 = n_2 \in \{1, 2, 3\}$, and phase-diagram checks of the perturbative domain.

Contents

1 Scope, positioning, and standing regularization

Remark 1.1 (Scope note)**.** All results are proved in a finite-mode bosonic CCR setting (regularized Weyl algebra on a finite-dimensional symplectic space). The Type III AQFT setting is used only as motivation; continuum statements are formulated as conjectural directions.

Remark 1.2 (Positioning)**.** In finite-mode continuous-variable quantum information, closed formulas for Gaussian fidelity and many norm inequalities are standard. The contribution of this note is an "interface block" that (i) packages vacuum clustering into a cross-correlation parameter η_{vac}, (ii) propagates perturbations through a Gaussian recovery map in covariance block form, and (iii) yields a quantitative recovery bound with explicit dependence on coercivity and correlation parameters.

Table 1: Parameter dictionary (reading aid).

Symbol	Meaning	Where used
ϵ	coercivity margin for A relative to A_0	Theorem 6.1
η_{vac}	reference cross-correlation factor	Definition 4.1
δ	cross-correlation perturbation parameter	Definition 4.2
κ	$\epsilon^{-1/2}(\eta_{\text{vac}} + \delta)$	Theorem 6.1
Φ	prefactor $\big((1-\kappa)\min(\epsilon c_1, c_2)\big)^{-1}$	Theorem 6.1
$\Delta\Gamma$	covariance recovery error $\Gamma - \tilde{\Gamma}$	Theorem 6.1
$\Delta^{(12)}$	cross-block error $X - \tilde{X}$	Definition 6.2

2 Finite-mode CCR Gaussian framework

2.1 Quadrature ordering and symplectic form

Remark 2.1 (Ordering convention)**.** Throughout, we use the standard continuous-variable ordering

$$(x_1, \dots, x_N, p_1, \dots, p_N),$$

so that

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

This ordering matches the convention used by `thewalrus` in our numerical fidelity evaluations.

2.2 Finite symplectic space and Weyl algebra

Let $S \simeq \mathbb{R}^{2N}$ be a real symplectic vector space with symplectic form Ω. The finite-mode Weyl algebra is generated by Weyl operators $W(z)$, $z \in S$, satisfying

$$W(z)W(z') = e^{-\frac{i}{2}\Omega(z,z')}\, W(z+z').$$

2.3 Quasi-free states and covariance matrices

Definition 2.1 (Covariance and uncertainty condition)**.** A centered quasi-free state is specified by a real symmetric covariance matrix $\Gamma \in \mathbb{R}^{2N\times 2N}$ satisfying

$$\Gamma + \frac{i}{2}\Omega \succeq 0.$$

Assumption 2.1 (Uniform positivity in the regularized regime)**.** All covariance blocks under consideration are strictly positive and bounded. In particular, for the reference covariance blocks defined below, there exist $c_1, c_2 > 0$ such that

$$A_0 \succeq c_1 \mathbf{1}, \qquad B_0 \succeq c_2 \mathbf{1}.$$

Remark 2.2 (Norm conventions)**.** For a real matrix M, the Hilbert–Schmidt norm is

$$\|M\|_{\mathrm{HS}} := \sqrt{\mathrm{Tr}(M^T M)},$$

and $\|M\|_{\mathrm{op}}$ denotes the operator norm (largest singular value).

Definition 2.2 (Uhlmann fidelity)**.** For density operators ρ, σ on the same finite-dimensional Hilbert space, define

$$F(\rho, \sigma) := \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1^2.$$

We write $F(\omega_1, \omega_2)$ for the fidelity of the corresponding density operators representing states ω_1, ω_2.

Remark 2.3 (Covariance convention and numerical fidelity)**.** We use the convention that physical covariances satisfy

$$\Gamma + \frac{i\hbar}{2}\Omega \succeq 0,$$

and in the numerical experiments we work with $\hbar = 1$ (i.e. $\Gamma + \frac{i}{2}\Omega \succeq 0$). All numerical fidelities are evaluated using `thewalrus.quantum.fidelity` with parameter `hbar=1.0` in the same convention.

2.4 Bipartition and block form

Fix a bipartition $S = S_1 \oplus S_2$ with dimensions $2n_1$ and $2n_2$. Write

$$\Gamma = \begin{pmatrix} A & X \\ X^T & B \end{pmatrix}, \qquad \Gamma_0 = \begin{pmatrix} A_0 & X_0 \\ X_0^T & B_0 \end{pmatrix}.$$

3 Gaussian recovery in block form

Definition 3.1 (Gaussian recovery map (covariance-level))**.** Given a centered reference covariance $\Gamma_0 = \left(\begin{smallmatrix} A_0 & X_0 \\ X_0^T & B_0 \end{smallmatrix}\right)$ and a centered input covariance $\Gamma = \left(\begin{smallmatrix} A & X \\ X^T & B \end{smallmatrix}\right)$, define the recovered covariance $\tilde{\Gamma}$ by

$$\tilde{A} = A, \qquad \tilde{X} = A A_0^{-1} X_0, \qquad \tilde{B} = B_0 + X_0^T A_0^{-1}(A - A_0) A_0^{-1} X_0.$$

Remark 3.1 (Relation to Petz recovery)**.** Within the centered Gaussian sector and for the partial-trace channel discarding subsystem 2, Definition 3.1 matches the Petz recovery map associated with the reference state ω_0. A covariance-level derivation sketch and precise assumptions are given in Appendix A.

4 Clustering input: vacuum cross-correlation and perturbations

Definition 4.1 (Vacuum cross-correlation factor)**.** Define

$$\eta_{\mathrm{vac}} := \left\|A_0^{-1/2} X_0 B_0^{-1/2}\right\|_{\mathrm{op}}.$$

Definition 4.2 (Cross-correlation perturbation parameter)**.** Define

$$\delta := \left\| A_0^{-1/2}(X - X_0)B_0^{-1/2} \right\|_{\mathrm{op}}.$$

Remark 4.1 (Bessel-type collar envelope (motivational interface))**.** In massive relativistic settings, equal-time correlators admit envelopes controlled by modified Bessel functions $K_\nu(mr)$ with large-mr asymptotics

$$K_\nu(mr) \sim \sqrt{\frac{\pi}{2mr}}\, e^{-mr}.$$

In our finite-mode posture, r is a collar parameter controlling the magnitude of X_0, and our sanity checks use the model envelope

$$\|X_0(r)\|_{\mathrm{op}} \propto (mr)^p K_\nu(mr).$$

5 Gaussian fidelity control (perturbative regime)

Lemma 5.1 (Perturbative Gaussian fidelity bound (imported, constants on a compact set))**.** *Let ω_1, ω_2 be centered Gaussian states with covariances $\Gamma_1 \succ 0$ and Γ_2. Define*

$$K := \Gamma_1^{-1/2}(\Gamma_1 - \Gamma_2)\Gamma_1^{-1/2}.$$

Assume $\|K\|_{\mathrm{op}} \le \frac{1}{2}$. Then there exist constants $C_2, C_4 > 0$, depending only on the fixed finite-mode regularization (in particular on the total mode number N) and on coercivity bounds of the form $\Gamma_1 \succeq c\,\mathbf{1}$, such that

$$1 - F(\omega_1, \omega_2) \le C_2 \|K\|_{\mathrm{HS}}^2 + C_4 \|K\|_{\mathrm{HS}}^4.$$

In particular, there exists a constant $C > 0$ with the same dependence such that if $\|K\|_{\mathrm{HS}} \le 1$ then

$$1 - F(\omega_1, \omega_2) \le C\,\|K\|_{\mathrm{HS}}^2.$$

Remark 5.1 (Reference for Lemma 5.1)**.** Closed formulas for the fidelity of Gaussian states and perturbative expansions are standard; see [3, 5, 4]. Lemma 5.1 is used as a local perturbative input; we do not optimize constants.

6 Main theorem: clustering–recovery bridge (Route 1)

Definition 6.1 (Partial local excitation)**.** We say ω is a partial local excitation (relative to the bipartition) if $B = B_0$.

Definition 6.2 (Recovery cross-block error)**.** With $\tilde{X}$ as in Definition 3.1, define

$$\Delta^{(12)} := X - \tilde{X} = X - AA_0^{-1}X_0.$$

Lemma 6.1 (From relative perturbation to a uniform lower bound)**.** *If $\|A_0^{-1/2}(A - A_0)A_0^{-1/2}\|_{\mathrm{op}} \le 1 - \epsilon$ with $\epsilon \in (0, 1)$, then*

$$A \succeq \epsilon A_0.$$

Proof. Let $E := A_0^{-1/2}(A - A_0)A_0^{-1/2}$. The hypothesis implies $E \succeq -(1 - \epsilon)\mathbf{1}$, hence

$$A = A_0^{1/2}(\mathbf{1} + E)A_0^{1/2} \succeq \epsilon A_0.$$

□

Lemma 6.2 (Block structure of the recovery covariance error)**.** *Assume $B = B_0$ and $\tilde{\Gamma}$ is given by Definition 3.1. Then $\Delta\Gamma := \Gamma - \tilde{\Gamma}$ has block form*

$$\Delta\Gamma = \begin{pmatrix} 0 & \Delta^{(12)} \\ (\Delta^{(12)})^T & \Delta^{(22)} \end{pmatrix}, \qquad \Delta^{(22)} = B_0 - \tilde{B} = -X_0^T A_0^{-1}(A - A_0)A_0^{-1}X_0.$$

In particular,

$$\|\Delta\Gamma\|_{\mathrm{HS}}^2 = 2\|\Delta^{(12)}\|_{\mathrm{HS}}^2 + \|\Delta^{(22)}\|_{\mathrm{HS}}^2,$$

and

$$\|\Delta^{(22)}\|_{\mathrm{HS}} \le \|A - A_0\|_{\mathrm{HS}}\, \|A_0^{-1}X_0\|_{\mathrm{op}}^2.$$

Proof. The block form is immediate from $\tilde{A} = A$ and $B = B_0$. For the estimate, use $\|UVW\|_{\mathrm{HS}} \le \|U\|_{\mathrm{op}}\|V\|_{\mathrm{HS}}\|W\|_{\mathrm{op}}$ with $U = X_0^T A_0^{-1}$, $V = (A - A_0)$, $W = A_0^{-1}X_0$. □

Proposition 6.1 (Coercivity bound for $\|\Gamma^{-1}\|_{\mathrm{op}}$ with no hidden constants)**.** *Let $B_0 \succ 0$ and $\Gamma = \left(\begin{smallmatrix} A & X \\ X^T & B_0 \end{smallmatrix}\right)$ with $A \succ 0$. Define*

$$\eta_\omega := \big\|A^{-1/2}XB_0^{-1/2}\big\|_{\mathrm{op}}.$$

If $\eta_\omega < 1$, then

$$\Gamma \succeq (1 - \eta_\omega)\begin{pmatrix} A & 0 \\ 0 & B_0 \end{pmatrix},$$

and consequently

$$\|\Gamma^{-1}\|_{\mathrm{op}} \le \frac{1}{(1 - \eta_\omega)\min(\lambda_{\min}(A), \lambda_{\min}(B_0))}.$$

In particular, if $A \succeq \epsilon A_0$ and $A_0 \succeq c_1\mathbf{1}$ and $B_0 \succeq c_2\mathbf{1}$, then

$$\|\Gamma^{-1}\|_{\mathrm{op}} \le \frac{1}{(1 - \eta_\omega)\min(\epsilon c_1, c_2)}.$$

Proof. Let $u \in \mathbb{R}^{2n_1}$ and $v \in \mathbb{R}^{2n_2}$ and set $a := A^{1/2}u$, $b := B_0^{1/2}v$. Then

$$\begin{pmatrix} u \\ v \end{pmatrix}^T \Gamma \begin{pmatrix} u \\ v \end{pmatrix} = \|a\|_2^2 + 2\, a^T\big(A^{-1/2}XB_0^{-1/2}\big)b + \|b\|_2^2.$$

By Cauchy–Schwarz,

$$2\, a^T\big(A^{-1/2}XB_0^{-1/2}\big)b \ge -2\eta_\omega\|a\|_2\|b\|_2 \ge -\eta_\omega\big(\|a\|_2^2 + \|b\|_2^2\big),$$

so

$$\begin{pmatrix} u \\ v \end{pmatrix}^T \Gamma \begin{pmatrix} u \\ v \end{pmatrix} \ge (1 - \eta_\omega)\big(\|a\|_2^2 + \|b\|_2^2\big) = (1 - \eta_\omega)\begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} A & 0 \\ 0 & B_0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

This proves the Loewner bound. Taking minimal eigenvalues yields the inverse norm bound. □

Proposition 6.2 (Bounding η_ω by $\eta_{\mathrm{vac}} + \delta$)**.** *Assume $A \succeq \epsilon A_0$ with $\epsilon \in (0, 1)$. Then*

$$\eta_\omega := \|A^{-1/2}XB_0^{-1/2}\|_{\mathrm{op}} \le \epsilon^{-1/2}\,\|A_0^{-1/2}XB_0^{-1/2}\|_{\mathrm{op}} \le \epsilon^{-1/2}(\eta_{\mathrm{vac}} + \delta).$$

Proof. Since $A \succeq \epsilon A_0$, we have $\|A^{-1/2}A_0^{1/2}\|_{\mathrm{op}} \le \epsilon^{-1/2}$. Thus

$$\|A^{-1/2}XB_0^{-1/2}\|_{\mathrm{op}} \le \|A^{-1/2}A_0^{1/2}\|_{\mathrm{op}}\,\|A_0^{-1/2}XB_0^{-1/2}\|_{\mathrm{op}}.$$

Finally,

$$\|A_0^{-1/2}XB_0^{-1/2}\|_{\mathrm{op}} \le \|A_0^{-1/2}X_0B_0^{-1/2}\|_{\mathrm{op}} + \|A_0^{-1/2}(X - X_0)B_0^{-1/2}\|_{\mathrm{op}} = \eta_{\mathrm{vac}} + \delta.$$

□

Proposition 6.3 (A simple sufficient condition for the perturbative regime)**.** *If*

$$\|\Gamma^{-1}\|_{\mathrm{op}}\,\|\Delta\Gamma\|_{\mathrm{HS}} \le \frac{1}{2},$$

then $\|K\|_{\mathrm{op}} \le \frac{1}{2}$ *and Lemma 5.1 applies.*

Proof. $\|K\|_{\mathrm{op}} = \|\Gamma^{-1/2}\Delta\Gamma\,\Gamma^{-1/2}\|_{\mathrm{op}} \le \|\Gamma^{-1}\|_{\mathrm{op}}\|\Delta\Gamma\|_{\mathrm{op}} \le \|\Gamma^{-1}\|_{\mathrm{op}}\|\Delta\Gamma\|_{\mathrm{HS}}$. □

Theorem 6.1 (Clustering–recovery bridge (finite-mode Gaussian; Route 1, quadratic+quartic))**.** *Let* ω_0 *be a centered reference Gaussian state with covariance* Γ_0 *and blocks* (A_0, X_0, B_0) *satisfying Assumption 2.1. Let* ω *be a centered Gaussian state with covariance* Γ *and blocks* (A, X, B) *such that* ω *is a partial local excitation (Definition 6.1), i.e.* $B = B_0$*.*

Assume there exists $\epsilon \in (0, 1)$ *such that*

$$\left\|A_0^{-1/2}(A - A_0)A_0^{-1/2}\right\|_{\mathrm{op}} \le 1 - \epsilon,$$

and define

$$\kappa := \epsilon^{-1/2}\big(\eta_{\mathrm{vac}} + \delta\big).$$

Assume $\kappa < 1$*. Let* $\tilde{\omega}$ *be the recovered centered Gaussian state with covariance* $\tilde{\Gamma}$ *given by Definition 3.1.*

Assume in addition that $\|K\|_{\mathrm{op}} \le \frac{1}{2}$*, where* $K := \Gamma^{-1/2}(\Gamma - \tilde{\Gamma})\Gamma^{-1/2}$ *(for instance, it suffices that Proposition 6.3 holds).*

Then, with $\Delta\Gamma := \Gamma - \tilde{\Gamma}$*,*

$$1 - F(\omega, \tilde{\omega}) \le C_2\,\Phi^2\,\|\Delta\Gamma\|_{\mathrm{HS}}^2 + C_4\,\Phi^4\,\|\Delta\Gamma\|_{\mathrm{HS}}^4,$$

where

$$\Phi := \frac{1}{(1 - \kappa)\min(\epsilon c_1, c_2)},$$

and $C_2, C_4 > 0$ *are the constants from Lemma 5.1.*

Proof. By submultiplicativity and $\|M\|_{\mathrm{op}} \le \|M\|_{\mathrm{HS}}$,

$$\|K\|_{\mathrm{HS}} = \|\Gamma^{-1/2}\Delta\Gamma\,\Gamma^{-1/2}\|_{\mathrm{HS}} \le \|\Gamma^{-1}\|_{\mathrm{op}}\,\|\Delta\Gamma\|_{\mathrm{HS}}.$$

By Lemma 6.1, $A \succeq \epsilon A_0$, and Proposition 6.2 yields $\eta_\omega \le \kappa < 1$. Therefore Proposition 6.1 gives

$$\|\Gamma^{-1}\|_{\mathrm{op}} \le \frac{1}{(1 - \eta_\omega)\min(\epsilon c_1, c_2)} \le \frac{1}{(1 - \kappa)\min(\epsilon c_1, c_2)} = \Phi.$$

Insert $\|K\|_{\mathrm{HS}} \le \|\Gamma^{-1}\|_{\mathrm{op}}\|\Delta\Gamma\|_{\mathrm{HS}}$ into Lemma 5.1. □

Remark 6.1 (Coarsenings of the correlation denominator)**.** For $\kappa \in [0, 1)$ one has $1 - \kappa^2 = (1 - \kappa)(1 + \kappa) \le 2(1 - \kappa)$, hence

$$\frac{1}{(1 - \kappa)^m} \le \frac{2^m}{(1 - \kappa^2)^m} \qquad (m = 1, 2, 4).$$

Thus one may equivalently rewrite prefactors in terms of $(1 - \kappa^2)$ at the cost of adjusting numerical constants. We keep the explicit $(1 - \kappa)$ form matching Proposition 6.1.

Corollary 6.1 (Purely quadratic regime)**.** *Under the hypotheses of Theorem 6.1, if additionally* $\|K\|_{\mathrm{HS}} \le 1$*, then*

$$1 - F(\omega, \tilde{\omega}) \le C\,\Phi^2\,\|\Delta\Gamma\|_{\mathrm{HS}}^2,$$

with C *from Lemma 5.1 and* Φ *as in Theorem 6.1.*

Corollary 6.2 (Reduction to $\|\Delta^{(12)}\|_{\mathrm{HS}}$ in the Family A case)**.** *Assume the hypotheses of Theorem 6.1 and, in addition, $X = X_0$. Then one may take*

$$C_\Delta := \sqrt{2 + \|X_0^T A_0^{-1}\|_{\mathrm{op}}^2}$$

so that

$$\|\Delta\Gamma\|_{\mathrm{HS}} \le C_\Delta \, \|\Delta^{(12)}\|_{\mathrm{HS}}.$$

Proof. If $X = X_0$, then $\Delta^{(12)} = X_0 - AA_0^{-1}X_0 = -(A - A_0)A_0^{-1}X_0$. Lemma 6.2 gives $\Delta^{(22)} = X_0^T A_0^{-1}\Delta^{(12)}$, hence

$$\|\Delta^{(22)}\|_{\mathrm{HS}} \le \|X_0^T A_0^{-1}\|_{\mathrm{op}} \, \|\Delta^{(12)}\|_{\mathrm{HS}}.$$

Substitute into $\|\Delta\Gamma\|_{\mathrm{HS}}^2 = 2\|\Delta^{(12)}\|_{\mathrm{HS}}^2 + \|\Delta^{(22)}\|_{\mathrm{HS}}^2$. □

7 Ancillary numerical sanity-checks

Remark 7.1 (Reproducibility and physicality checks)**.** The numerical experiments check physicality of both Γ and $\tilde{\Gamma}$ by verifying

$$\lambda_{\min}\Big(\Gamma + \frac{i}{2}\Omega\Big) \ge -\tau, \qquad \lambda_{\min}\Big(\tilde{\Gamma} + \frac{i}{2}\Omega\Big) \ge -\tau,$$

for a fixed tolerance $\tau > 0$.

Remark 7.2 (Numerical constants versus theoretical constants)**.** Lemma 5.1 is stated with constants on a compact set. In the numerical plots we additionally compare against sharper coefficients such as $\frac{1}{8}$ and $\frac{1}{48}$ suggested by local Taylor expansions of closed Gaussian fidelity formulas; these coefficients are used only as a tight benchmark and are not claimed as universal bounds in the present note.

7.1 Representative figures

For readability, we keep the base sanity check figures in the main text and move extended sweeps to Appendix B.

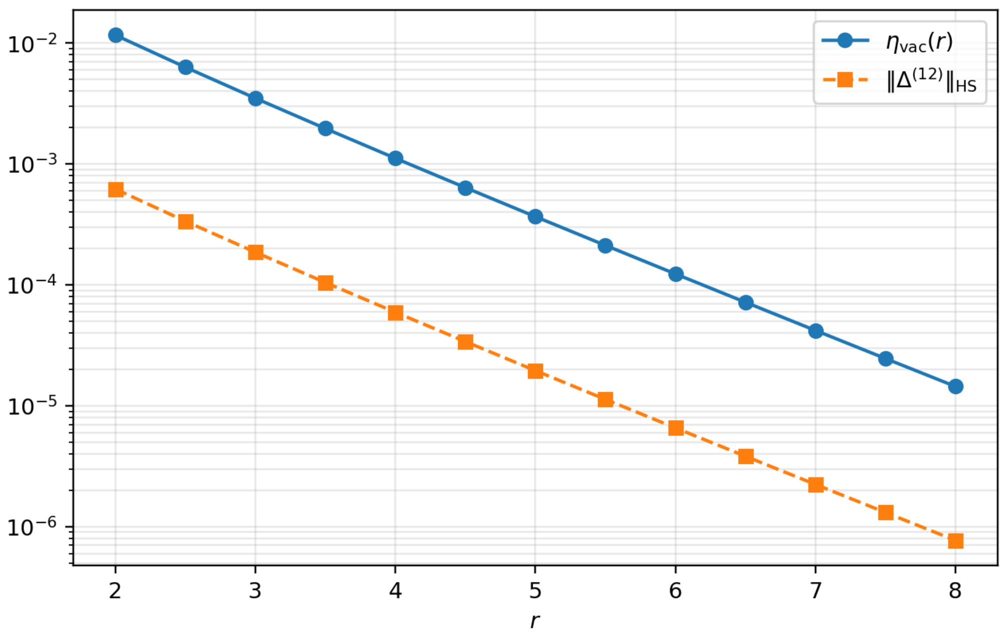

Figure 1: Base sanity check ($n_1 = n_2 = 2$): $\eta_{\mathrm{vac}}(r)$ and $\|\Delta^{(12)}\|_{\mathrm{HS}}$ versus r under a Bessel-type envelope.

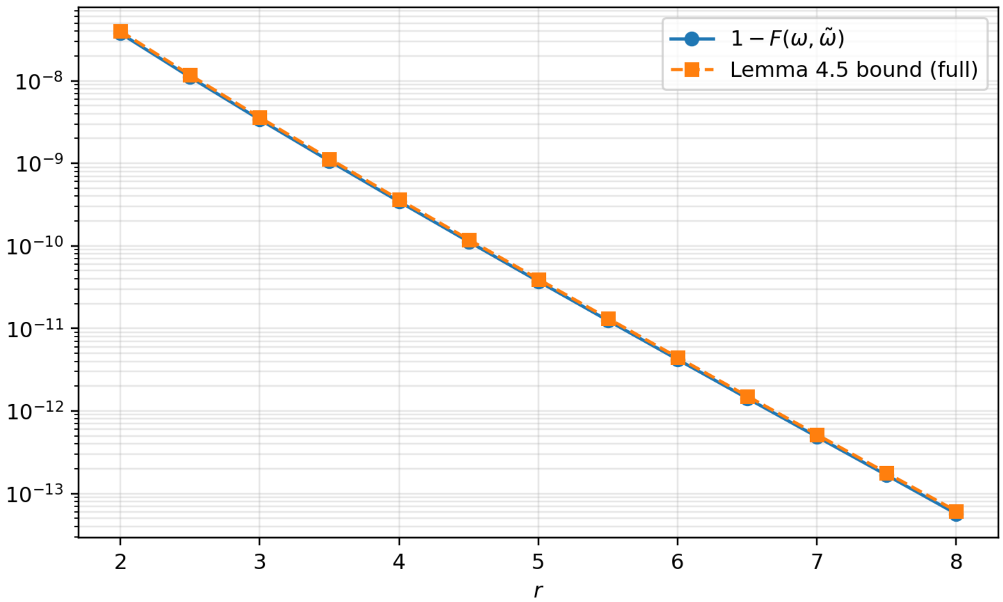

Figure 2: Base sanity check ($n_1 = n_2 = 2$): $1 - F(\omega, \tilde{\omega})$ versus a perturbative Gaussian fidelity benchmark.

8 Conjectural outlook (Type III / non-Gaussian)

Conjecture 8.1 (Type III clustering–recovery blueprint)**.** *Let ω_0 be the vacuum state of a massive AQFT satisfying a split inclusion for $O_1 \Subset O_2$. For suitable classes of states ω that are vacuum-like outside O_2 and have finite Araki relative entropy, the Petz recovery associated with the ω_0-preserving conditional expectation onto a canonical split factor yields an estimate of the form*

$$1 - F(\omega, \tilde{\omega}) \leq C\, \mathcal{C}(O_1, O_2)^{\gamma},$$

where $\mathcal{C}(O_1, O_2)$ is a clustering coefficient controlled by collar width and the mass gap.

Remark 8.1 (Type III roadmap)**.** The transition to the Type III setting likely requires non-commutative L_p methods (e.g. Haagerup spaces) to express quantitative analogues of density operators and fidelity-type functionals. In this picture, covariance-block control in the finite-mode Gaussian setting is expected to be replaced by bounds on relative modular operators associated with the split factor and the ω_0-preserving conditional expectation.

A Appendix: Petz identification sketch

Remark A.1. This appendix provides a covariance-level derivation sketch for the claim in Remark 3.1. We emphasize that the main results of the paper only use Definition 3.1 as an explicit recovery rule; the "Petz identification" is additional structural context.

Sketch. For centered Gaussian states, all second moments are encoded by the covariance matrix Γ, and the reduced state on subsystem 1 is Gaussian with covariance A. The Petz recovery map for the partial-trace channel is the transpose channel relative to the reference state ω_0. Within the centered Gaussian sector, the transpose channel is Gaussian and is therefore determined by its action on covariances. Requiring that it (i) fixes the subsystem-1 marginal ($\tilde{A} = A$) and (ii) recovers the reference exactly ($\Gamma = \Gamma_0 \Rightarrow \tilde{\Gamma} = \Gamma_0$), and imposing covariance-level affinity in A yields the block update in Definition 3.1. A full phase-space derivation can be obtained by tracking the Petz map on Weyl operators and comparing characteristic functions. $\square$

B Appendix: extended figures

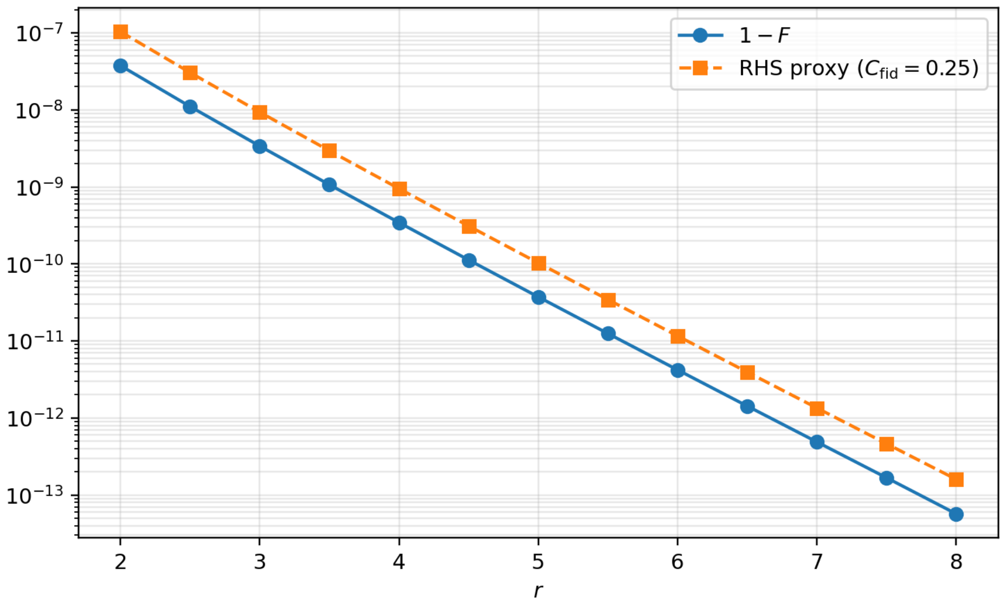

Figure 3: Base sanity check ($n_1 = n_2 = 2$): $1 - F$ versus a proxy of the RHS scaling in Theorem 6.1.

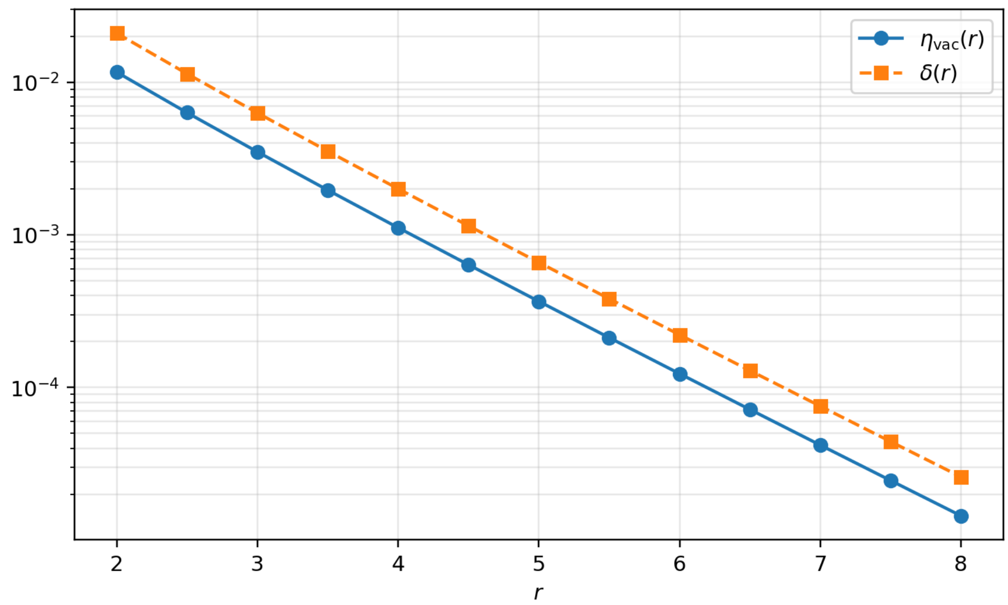

Figure 4: N1 (Family B): engineered perturbation $\delta(r)$ inherits the collar envelope and decays with r, alongside $\eta_{\text{vac}}(r)$.

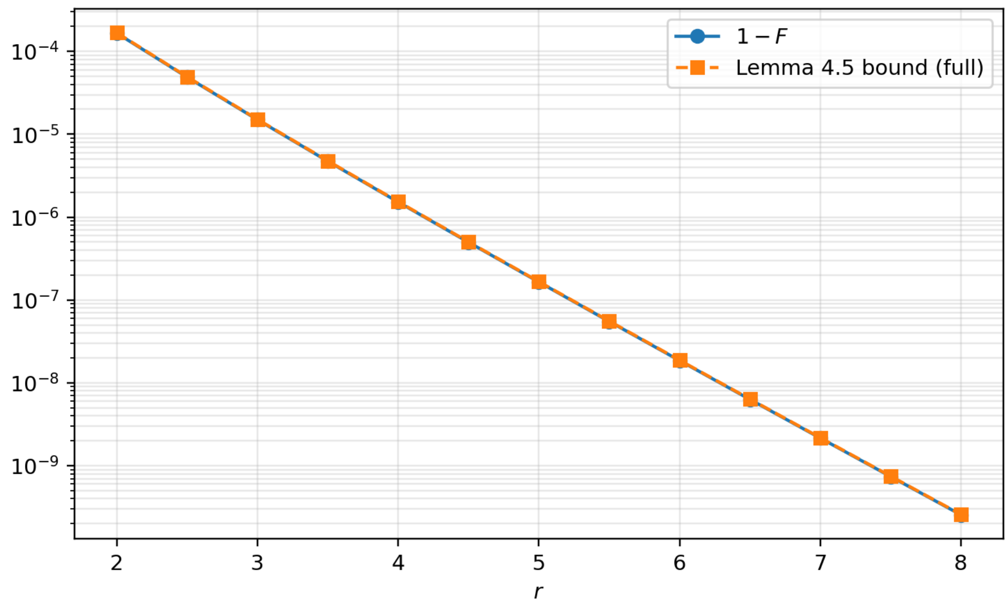

Figure 5: N1: fidelity-benchmark comparison for the Family B construction.

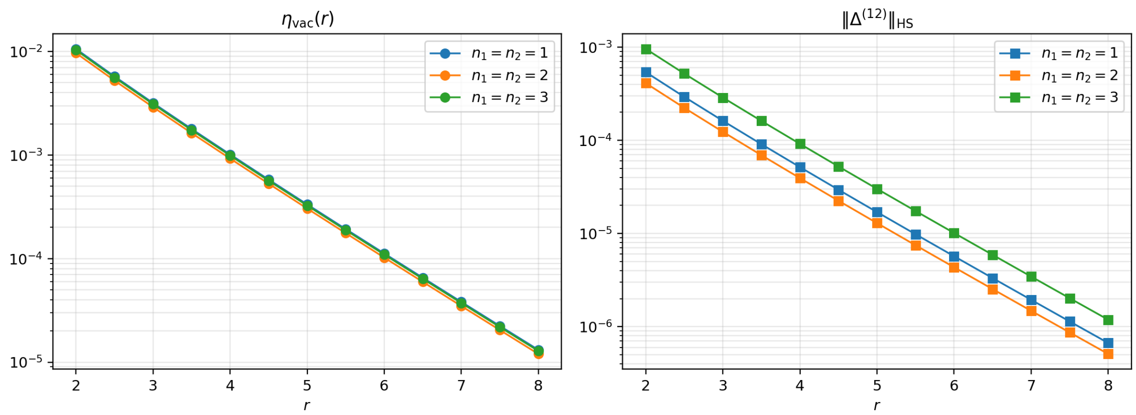

Figure 6: N2: $\eta_{\text{vac}}(r)$ and $\|\Delta^{(12)}\|_{\text{HS}}$ for $n_1 = n_2 \in \{1, 2, 3\}$.

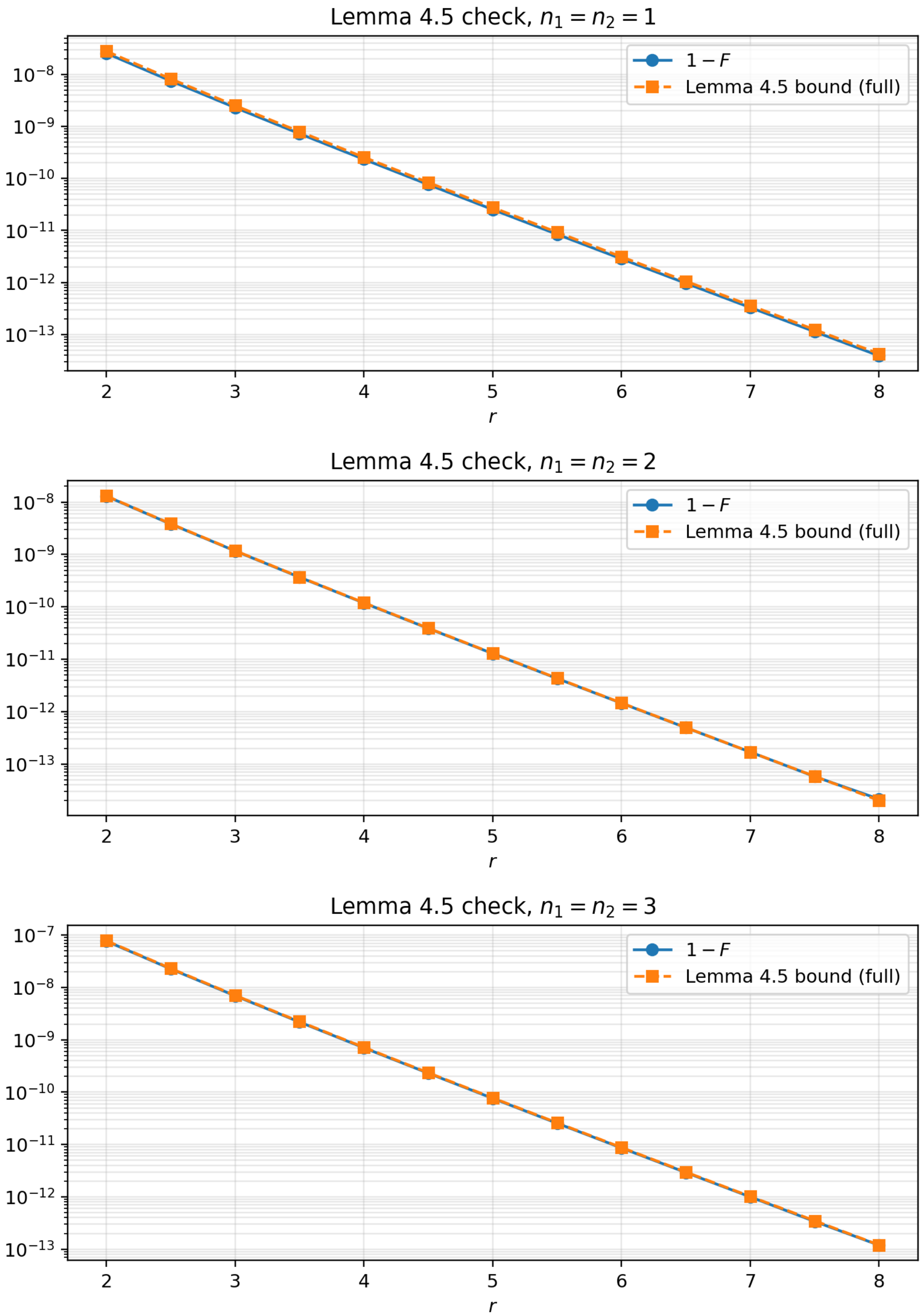

Figure 7: N2: fidelity-benchmark comparison across $n_1 = n_2 \in \{1, 2, 3\}$.

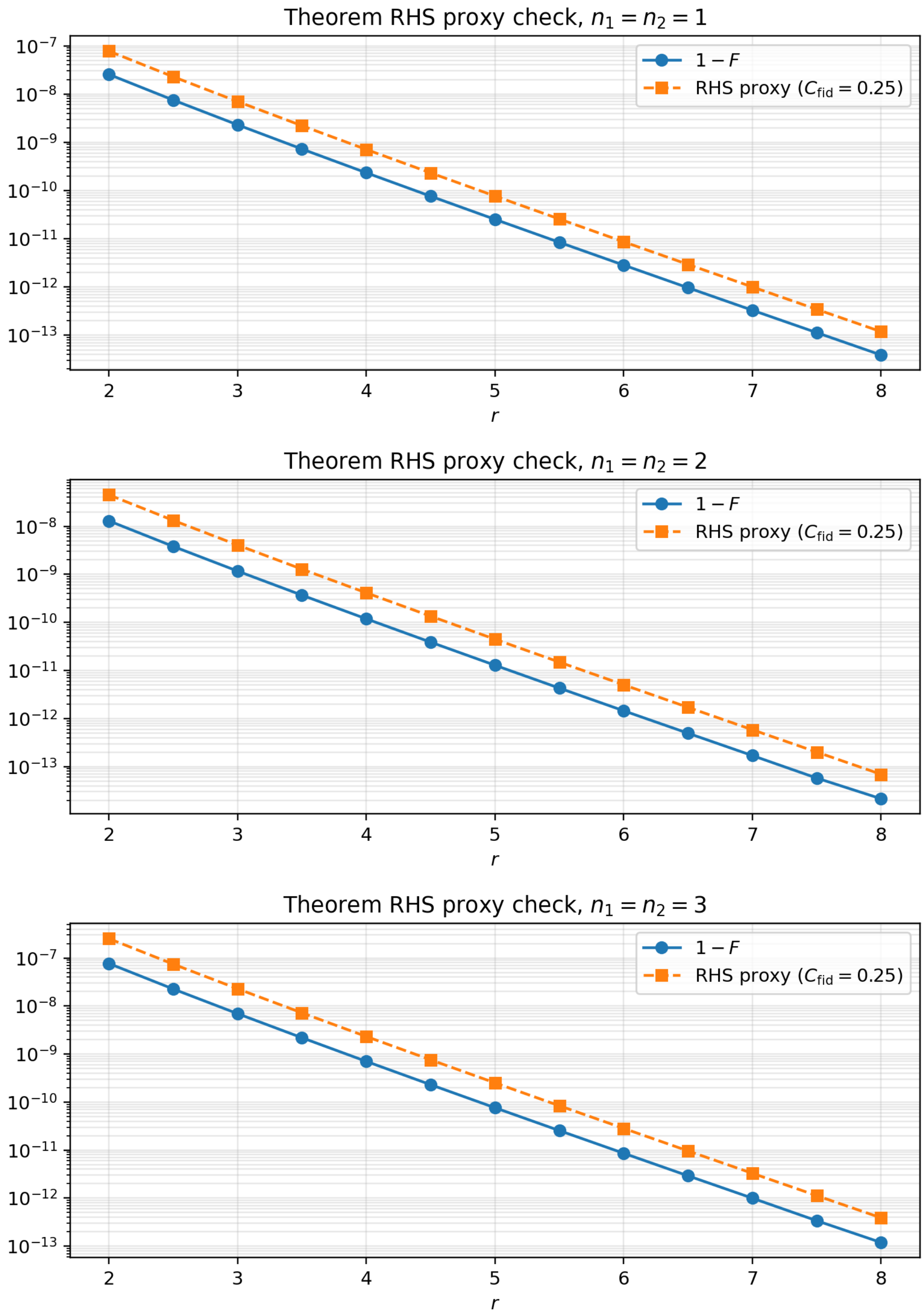

Figure 8: N2: $1 - F$ versus the theorem RHS proxy across $n_1 = n_2 \in \{1, 2, 3\}$.

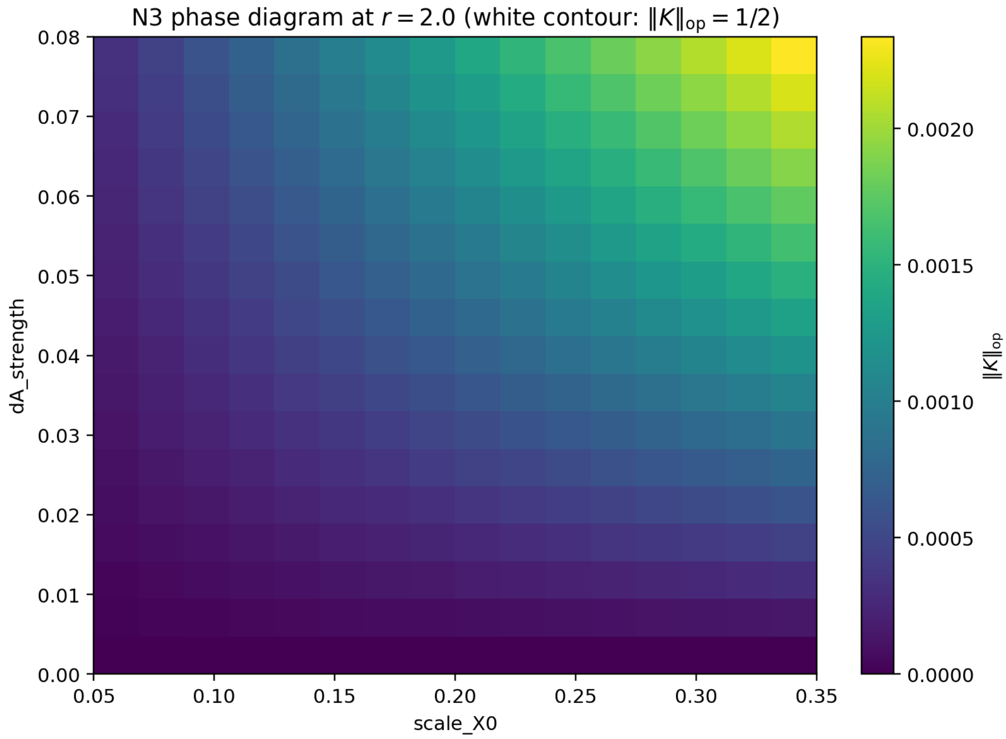

Figure 9: N3 (fixed $r = 2$): heatmap of $\|K\|_{\text{op}}$ in a deep perturbative region.

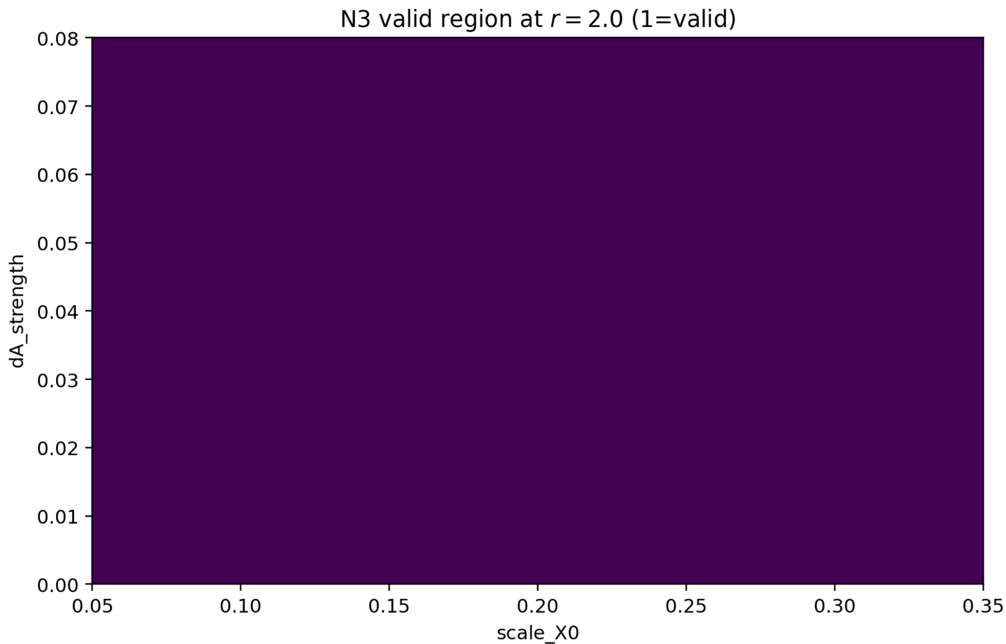

Figure 10: N3 (fixed $r = 2$): validity region for physicality and $\|K\|_{\text{op}} \leq \frac{1}{2}$ in the corresponding parameter range.

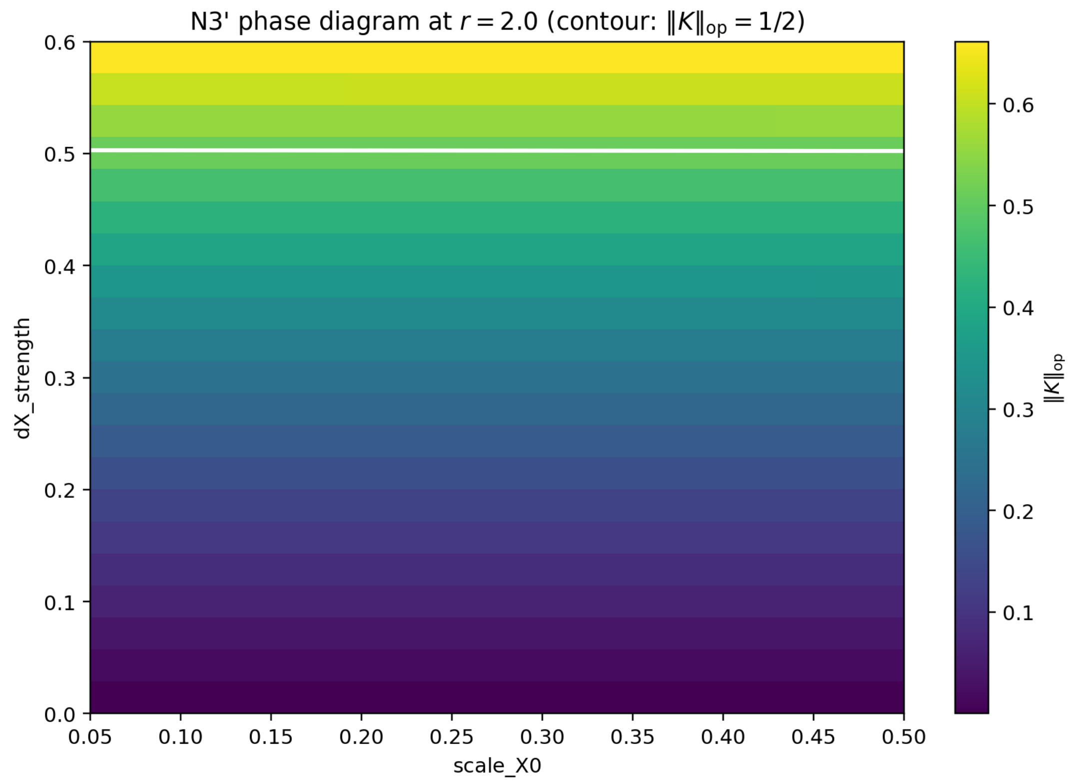

Figure 11: N3' (fixed $r = 2$): phase diagram showing $\|K\|_{\mathrm{op}}$ versus scale_X0 and dX_strength; contour indicates $\|K\|_{\mathrm{op}} = \frac{1}{2}$.

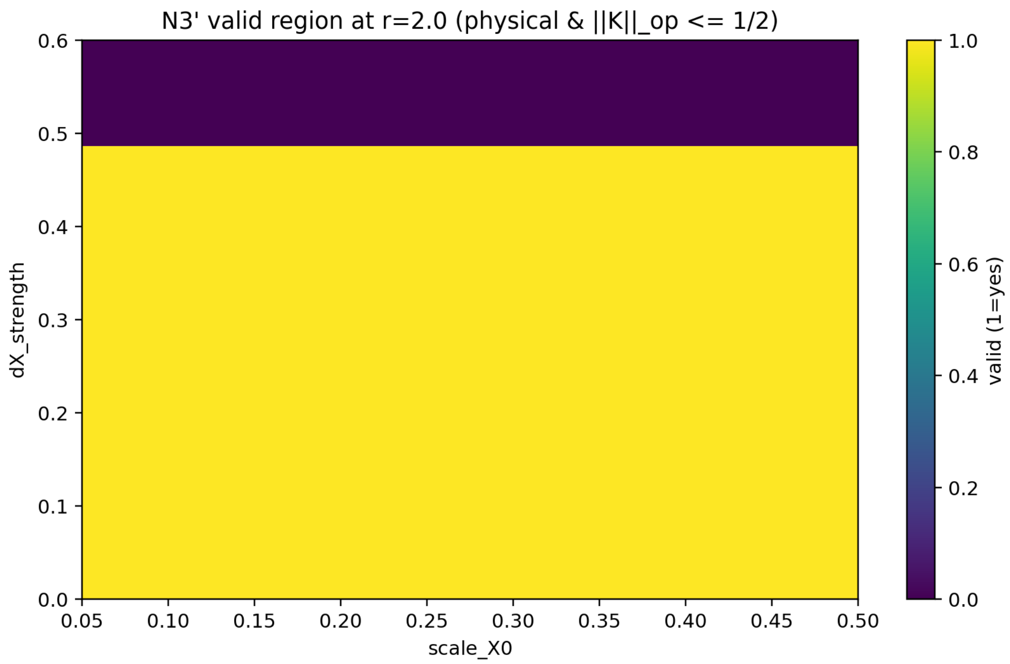

Figure 12: N3' (fixed $r = 2$): validity region where the state remains physical and satisfies $\|K\|_{\mathrm{op}} \leq \frac{1}{2}$.

C Appendix: minimal runnable script

Remark C.1. The following compact script is intended as a minimal, runnable reference (finite-mode, centered Gaussian, $\hbar = 1$). It reproduces the base figure $\eta_{\text{vac}}(r)$ vs $\|\Delta^{(12)}\|_{\text{HS}}$ and demonstrates how numerical fidelity is evaluated.

```
# Minimal runnable script (finite-mode Gaussian sanity slice).
# Dependencies: numpy, scipy, matplotlib, thewalrus==0.22.0
import os, json
import numpy as np
import scipy.linalg as la
from scipy.special import kv
import matplotlib.pyplot as plt
from thewalrus.quantum import fidelity

CONFIG = {
  "seed": 0,
  "hbar": 1.0,
  "m": 1.0, "nu": 0.5, "p": 0.0,
  "r_list": list(np.linspace(2.0, 8.0, 13)),
  "scale_X0": 0.15,
  "dA_strength": 0.03,
  "c1": 1.2, "c2": 1.4,
  "tol_phys": 1e-9,
  "n1": 2, "n2": 2
}

def symplectic_form(n_modes):
  I = np.eye(n_modes); Z = np.zeros((n_modes,n_modes))
  return np.block([[Z, I], [-I, Z]])

def hs_norm(M):
  return float(np.sqrt(np.real(np.trace(M.T @ M))))

def op_norm(M):
  return float(np.linalg.norm(M, ord=2))

def project_spd(M, lam_min=1e-9):
  M = 0.5*(M + M.T)
  w, U = la.eigh(M)
  w = np.maximum(w, lam_min)
  return (U * w) @ U.T

def is_physical_cov(V, Omega, hbar=1.0, tol=1e-9):
  M = V + 0.5j*hbar*Omega
  ev = la.eigvalsh(M)
  return (np.min(ev.real) >= -tol), float(np.min(ev.real))

def f_bessel(r, m=1.0, nu=0.5, p=0.0):
  z = m*r
  return (z**p) * kv(nu, z)

def recovery_blocks(A, A0, X0, B0):
  A0_inv = la.inv(A0)
  Xt = A @ A0_inv @ X0
  Bt = B0 + X0.T @ A0_inv @ (A - A0) @ A0_inv @ X0
  return Xt, Bt
```

```python
def main():
  os.makedirs("../figures", exist_ok=True)
  rng = np.random.default_rng(CONFIG["seed"])

  n1, n2 = CONFIG["n1"], CONFIG["n2"]
  n_total = n1 + n2
  Omega = symplectic_form(n_total)

  # ordering: (x1,...,xn,p1,...,pn)
  A0 = CONFIG["c1"] * np.eye(2*n1)
  B0 = CONFIG["c2"] * np.eye(2*n2)

  X_base = rng.normal(size=(2*n1, 2*n2))
  X_base = X_base / np.linalg.norm(X_base, ord=2)

  rs, etas, d12s = [], [], []

  for r in CONFIG["r_list"]:
    X0 = CONFIG["scale_X0"] * f_bessel(r, CONFIG["m"], CONFIG["nu"],
        CONFIG["p"]) * X_base
    V0 = np.block([[A0, X0],[X0.T, B0]])
    ok0,_ = is_physical_cov(V0, Omega, CONFIG["hbar"], CONFIG["tol_phys
        "])
    if not ok0:
      continue

    # Family A: change A only, keep X=X0, B=B0
    M = rng.normal(size=A0.shape)
    dA = CONFIG["dA_strength"] * 0.5*(M + M.T)
    A  = project_spd(A0 + dA, lam_min=1e-9)
    X  = X0
    V  = np.block([[A, X],[X.T, B0]])
    okV,_ = is_physical_cov(V, Omega, CONFIG["hbar"], CONFIG["tol_phys"
        ])
    if not okV:
      continue

    Xt, Bt = recovery_blocks(A, A0, X0, B0)
    Vt = np.block([[A, Xt],[Xt.T, Bt]])
    okT,_ = is_physical_cov(Vt, Omega, CONFIG["hbar"], CONFIG["tol_phys
        "])
    if not okT:
      continue

    mu = np.zeros(2*n_total)
    _ = fidelity(mu, V, mu, Vt, hbar=CONFIG["hbar"])

    A0_mhalf = la.inv(la.sqrtm(A0))
    B0_mhalf = la.inv(la.sqrtm(B0))
    eta = op_norm(A0_mhalf @ X0 @ B0_mhalf)

    rs.append(r)
    etas.append(eta)
    d12s.append(hs_norm(X - Xt))

  rs = np.array(rs); etas = np.array(etas); d12s = np.array(d12s)

  plt.figure(figsize=(6.5,4.2))
```

```
  plt.semilogy(rs, etas, "o-", label=r"$\eta_{\mathrm{vac}}(r)$")
  plt.semilogy(rs, d12s, "s--", label=r"$\|\Delta^{(12)}\|_{\mathrm{HS
     }}$")
  plt.grid(True, which="both", alpha=0.3)
  plt.xlabel(r"$r$")
  plt.legend()
  plt.tight_layout()
  plt.savefig("../figures/fig_eta_and_delta12_vs_r.png", dpi=220)
  plt.close()

  with open("../figures/config_minimal.json","w") as f:
    json.dump(CONFIG, f, indent=2)

if __name__ == "__main__":
  main()
```

References

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