Heat Kernel Methods and the Sign of Induced Gravity

Resolving Conventions via the Laplacian–Lichnerowicz Identity

Version 2

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Abstract

We derive the local one-loop contribution proportional to the scalar curvature R in the Euclidean effective action obtained by integrating out matter fields on a curved background. Using a Schwinger proper-time cutoff $\varepsilon = \Lambda^{-2}$ and the Seeley–DeWitt coefficient a_1, we extract the quadratically divergent term multiplying $\int d^4x\sqrt{g}\,R$. We fix a single Euclidean convention for the Einstein–Hilbert action, state an explicit Laplacian convention, and write the Laplacian–Lichnerowicz identity in a sign-robust form so that the fermionic contribution is unambiguous. We provide a unified bookkeeping coefficient $A_1^{(\mathrm{eff})}$ such that

$$W^{(E)}_{R,\mathrm{total}} = -\frac{A_1^{(\mathrm{eff})}}{32\pi^2}\,\Lambda^2\int d^4x\sqrt{g}\,R,$$

and hence an induced Newton coupling G_{ind} via comparison with the Euclidean Einstein–Hilbert action. We also include the minimal gauge+ghost package in background Feynman gauge, a species table, and a reproducible verification suite. *v2 adds:* the equivalence of the species table with the classic counting $G_{\mathrm{ind}}^{-1} = (\Lambda^2/12\pi)(N_0 + 2N_{1/2} - 4N_1)$, gauge- and scheme-dependence caveats for the vector sector, a Weyl/Majorana caveat, corrected Wick-rotation wording, and an exact-spectrum numerical verification of every a_1 entry against closed-form spectra on S^4.

Contents

Relation to the companion paper 2512.0091. The series companion 2512.0091 (*Heat Kernel Methods and Induced Gravity*) contains the compact bookkeeping of the induced-gravity sign used elsewhere in the series. The present note is its expanded conventions-and-verification companion: it fixes every sign convention explicitly, derives the species table from scratch, anchors it to the classic counting of Corollary 1, states the gauge/scheme caveats, and verifies each heat-kernel entry against closed-form spectra on S^4 (Section 8.2). The two documents are complementary, not redundant: 0091 uses these coefficients; 0102 is where they are pinned down and checked.

1 Conventions and notation

1.1 Lorentzian geometry and curvature sign

We adopt Lorentzian signature $(-,+,+,+)$ and the curvature convention

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ce}\Gamma^e{}_{bd} - \Gamma^a{}_{de}\Gamma^e{}_{bc}, \qquad R_{bd} = R^a{}_{bad}, \qquad R = g^{bd}R_{bd}.$$

1.2 Euclidean formulation and Einstein–Hilbert action

Heat-kernel calculations are performed in $D = 4$ Euclidean signature. We fix the Euclidean Einstein–Hilbert action as

$$S_{\text{EH}}^{(E)} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\,R, \tag{1}$$

obtained by Wick rotation $t = -i\tau$ from the standard Lorentzian action

$$S^{(L)} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R,$$

under which the path-integral weight rotates as $e^{iS^{(L)}} \to e^{-S^{(E)}}$, i.e. $iS^{(L)} \to -S^{(E)}$ (v1 wrote this loosely as $S^{(E)} = -iS^{(L)}$; the weight-level statement is the convention actually used). Phases from topological/eta invariants are irrelevant for the local divergent coefficient we extract.

1.3 Laplacian convention, Laplace-type operators, and Schwinger cutoff

Define the covariant Laplacian as $\nabla^2 \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$, so that in flat Euclidean space $\nabla^2 = \partial^2$ and $\partial^2 e^{ik\cdot x} = -k^2 e^{ik\cdot x}$. Hence the principal operator $-\nabla^2$ is positive. A Laplace-type operator is taken as

$$P = -\nabla^2 + X(x),$$

where $X(x)$ is a local endomorphism (a scalar function for scalar fields). We use Schwinger proper-time regularization with cutoff $\varepsilon = \Lambda^{-2}$:

$$\ln\det P = -\int_{\varepsilon}^{\infty} \frac{ds}{s}\, \mathrm{Tr}\, e^{-sP}. \tag{2}$$

1.4 Fermionic hinge: commutator and Laplacian–Lichnerowicz identity

For spinors we take $D_E \equiv \gamma^\mu \nabla_\mu$ with Euclidean gamma matrices $(\gamma^\mu)^\dagger = \gamma^\mu$ and Dirac operator anti-Hermitian, $D_E^\dagger = -D_E$. The covariant derivative on spinors includes the spin connection. We fix the curvature term via

$$[\nabla_\mu, \nabla_\nu]\psi = \tfrac{1}{4} R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\psi, \qquad \gamma^{\rho\sigma} \equiv \tfrac{1}{2}[\gamma^\rho, \gamma^\sigma].$$

With the conventions above, the Laplacian–Lichnerowicz identity can be written in the sign-robust form

$$-D_E^2 = -\nabla^2 + \frac{R}{4}. \tag{3}$$

Remark 1. *In the fermionic sector $\nabla^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the connection Laplacian acting on spinors.*

2 Heat kernel expansion and the coefficient a_1

Let $K(s; x, y) = \langle x|e^{-sP}|y\rangle$ be the heat kernel. The trace has the asymptotic expansion as $s \to 0^+$:

$$\mathrm{Tr}\, e^{-sP} \sim \frac{1}{(4\pi s)^2} \int d^4x \sqrt{g}\, \big[a_0(x) + a_1(x)\, s + a_2(x)\, s^2 + \cdots\big]. \tag{4}$$

For a Laplace-type operator $P = -\nabla^2 + X$ the first coefficients are

$$a_0(x) = \mathrm{tr}\,\mathbf{1}, \qquad a_1(x) = \mathrm{tr}\Big(\frac{1}{6}R - X\Big). \tag{5}$$

For a real scalar with $X = m^2 + \xi R$:

$$a_1(x) = \Big(\frac{1}{6} - \xi\Big) R - m^2, \qquad a_{1,R}(x) = \tilde{a}_1 R, \qquad \tilde{a}_1 = \frac{1}{6} - \xi.$$

The term $-m^2$ contributes to $\int d^4x\sqrt{g}$ (cosmological constant renormalization) and not to $\int d^4x \sqrt{g}\, R$.

3 Scalars: extraction of the $\int \sqrt{g}\, R$ term

For a real bosonic scalar,

$$W^{(E)}_{\text{scalar}} = \frac{1}{2}\ln\det P = -\frac{1}{2}\int_{\varepsilon}^{\infty} \frac{ds}{s}\, \mathrm{Tr}\, e^{-sP}.$$

Keeping only the piece proportional to R:

$$W^{(E)}_{R,\text{scalar}} = -\frac{1}{2}\frac{1}{(4\pi)^2} \int d^4x \sqrt{g}\, a_{1,R}(x) \int_{\varepsilon}^{\infty} ds\, s^{-2}.$$

Since $\int_{\varepsilon}^{\infty} ds\, s^{-2} = 1/\varepsilon = \Lambda^2$, we obtain

$$W^{(E)}_{R,\text{scalar}} = -\frac{\tilde{a}_1}{32\pi^2}\, \Lambda^2 \int d^4x \sqrt{g}\, R. \tag{6}$$

4 Dirac fermions: reduction to Laplace type and the R term

4.1 Determinant reduction

For a Euclidean Dirac fermion with action $S_F = \int d^4x\sqrt{g}\,\bar{\psi}(D_E + m)\psi$, the functional integral yields

$$W^{(E)}_{\rm ferm} = -\ln\det(D_E + m).$$

Using the adjoint product:

$$\ln\det(D_E + m) = \frac{1}{2}\ln\det\big[(D_E + m)^\dagger(D_E + m)\big] + i\Theta,$$

where the phase $i\Theta$ does not affect the local divergent coefficient multiplying $\int\sqrt{g}\,R$ for the present purposes. Define the positive operator $P_D \equiv (D_E + m)^\dagger(D_E + m)$. Since $D_E^\dagger = -D_E$ and m is constant:

$$P_D = (-D_E + m)(D_E + m) = -D_E^2 + m^2.$$

With Laplacian–Lichnerowicz (3):

$$P_D = -\nabla^2 + m^2 + \frac{R}{4}. \tag{7}$$

4.2 a_1 coefficient and spin trace

Here $X_D = m^2 + \frac{1}{4}R$, so per spinor component

$$a_1^{(P_D)}(x) = \frac{1}{6}R - X_D = -\frac{1}{12}R - m^2.$$

Tracing over Dirac spin indices in $D = 4$ gives $\mathrm{tr}\,\mathbf{1} = 4$, hence

$$\mathrm{tr}\,a_{1,R}^{(P_D)}(x) = 4\Big(-\frac{1}{12}R\Big) = -\frac{1}{3}R. \tag{8}$$

4.3 Schwinger insertion

Because $W^{(E)}_{\rm ferm} = -\frac{1}{2}\ln\det P_D$ and $\ln\det P_D = -\int ds\,s^{-1}\ \mathrm{Tr}\,e^{-sP_D}$:

$$W^{(E)}_{\rm ferm} = \frac{1}{2}\int_\varepsilon^\infty \frac{ds}{s}\ \mathrm{Tr}\,e^{-sP_D}.$$

Thus the R-term is

$$W^{(E)}_{R,\rm ferm} = \frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\ \mathrm{tr}\,a_{1,R}^{(P_D)}(x)\int_\varepsilon^\infty ds\,s^{-2} = -\frac{1}{96\pi^2}\,\Lambda^2\int d^4x\sqrt{g}\,R. \tag{9}$$

Remark 2. *Phases and zero modes matter in chiral/topological settings (eta invariants, anomalies). For the present local $\Lambda^2\int\sqrt{g}\,R$ coefficient, they do not modify the result.*

5 Gauge fields and ghosts (spin-1): minimal package in Feynman gauge

For a gauge field (per generator) in background Feynman gauge, one may take the 1-form operator

$$(P_1)^\mu{}_\nu = -\delta^\mu{}_\nu \nabla^2 + R^\mu{}_\nu,$$

and the Faddeev–Popov ghost operator (complex Grassmann scalar) $P_{\rm gh} = -\nabla^2$. The gauge-fixed one-loop contribution has the standard structure

$$W^{(E)}_{\rm gauge} = \frac{1}{2}\ln\det P_1 - \ln\det P_{\rm gh}.$$

5.1 Short derivation of $A_1^{(\rm eff)}$(vector+ghosts) $= -\frac{2}{3}$ per generator

For a Laplace-type operator on 1-forms, $P_1 = -\nabla^2 + E$ with $E^\mu{}_\nu = R^\mu{}_\nu$. Then

$$\operatorname{tr} a^{(\text{1-form})}_{1,R} = \Big(\frac{\operatorname{tr}\mathbf{1}}{6} - \frac{\operatorname{tr}E}{R}\Big)R = \Big(\frac{4}{6} - 1\Big)R = -\frac{1}{3}R.$$

The ghost is a complex Grassmann scalar, so it contributes with the opposite sign of a complex bosonic scalar. A complex bosonic scalar gives $\operatorname{tr} a_{1,R} = 2\cdot\frac{1}{6}R = \frac{1}{3}R$, hence the ghost gives $\operatorname{tr} a^{(\rm gh)}_{1,R} = -\frac{1}{3}R$. The ghost entry in Table 1 is thus the *effective* one-loop contribution, including the Grassmann sign and the determinant power – not a raw bosonic heat-kernel trace. In the unified convention of Section 6, this yields (per generator)

$$A_1^{(\rm eff)}(\text{vector+ghosts}) = -\frac{2}{3}. \tag{10}$$

Remark 3 (Gauge and scheme dependence of the vector coefficient)**.** *Eq. (10) is computed in background Feynman gauge. Off shell (i.e. on a background that does not satisfy the matter-driven equations of motion), quadratically divergent coefficients of this type are known to be gauge-parameter and parametrization dependent; a gauge-invariant off-shell treatment requires the Vilkovisky–DeWitt unique effective action [4, 9]. The negative sign of the vector contribution is nevertheless robust within the fixed scheme used here, and coincides with the well-known negative gauge-field contribution to the quadratically divergent part of the induced/entanglement Newton constant identified by Kabat [8] (the "contact term"), whose physical interpretation is still discussed in the entanglement-entropy literature. What is unambiguous in this paper is the bookkeeping: fixed convention, fixed gauge, fixed regulator, all species treated identically.*

6 Unified convention and induced Newton constant

Define the unified coefficient $A_1^{(\rm eff)}$ by

$$W^{(E)}_{R,\rm total} = -\frac{A_1^{(\rm eff)}}{32\pi^2}\,\Lambda^2 \int d^4x\sqrt{g}\,R. \tag{11}$$

Comparing with the Euclidean Einstein–Hilbert term (1) and identifying $S^{(E)}_{\rm EH,induced} = W^{(E)}_{R,\rm total}$ gives

$$\frac{1}{16\pi G_{\rm ind}} = \frac{A_1^{(\rm eff)}}{32\pi^2}\,\Lambda^2, \qquad G_{\rm ind} = \frac{2\pi}{A_1^{(\rm eff)}}\,\Lambda^{-2}. \tag{12}$$

With this convention, $G_{\rm ind} > 0$ if and only if $A_1^{(\rm eff)} > 0$.

Corollary 1 (Equivalence with the classic species counting)**.** *For N_0 minimal real scalars, $N_{1/2}$ Dirac fermions and N_1 gauge generators, Table 1 and Eq. (12) give*

$$\frac{1}{G_{\text{ind}}} = \frac{A_1^{(\text{eff})}\Lambda^2}{2\pi} = \frac{\Lambda^2}{12\pi}\big(N_0 + 2N_{1/2} - 4N_1\big), \tag{13}$$

i.e. the unified coefficients $\{\frac{1}{6}, \frac{1}{3}, -\frac{2}{3}\}$ are exactly $\frac{1}{6}\times\{1, 2, -4\}$, reproducing the standard counting in the induced-gravity and entanglement-entropy literature [6, 7, 8]. This provides an external anchor for the sign conventions fixed in Section 1.

7 Species table (unified convention)

Table 1: Bookkeeping coefficients $A_1^{(\text{eff})}$ in the convention of Section 6.

Species	$A_1^{(\text{eff})}$	Comment
Real scalar (general ξ)	$\frac{1}{6} - \xi$	$X = m^2 + \xi R$
Real scalar (minimal $\xi = 0$)	$\frac{1}{6}$	
Complex scalar	$2(\frac{1}{6} - \xi)$	two real d.o.f.
Dirac fermion (4 components)	$\frac{1}{3}$	derived above
Weyl fermion (2 components)	$\frac{1}{6}$	half of Dirac $^{(*)}$
Majorana fermion	$\frac{1}{6}$	half of Dirac $^{(*)}$
Gauge vector + ghosts (per generator)	$-\frac{2}{3}$	background Feynman gauge

$(*)$ *Weyl/Majorana caveat (added in v2).* The halving is a statement about the *local* coefficient a_1 only: a Weyl fermion is treated via the standard Dirac doubling, $\ln\det$ of the chiral operator being defined through the doubled Laplace-type operator, and Majorana via the Pfaffian, $\ln \mathrm{Pf} = \frac{1}{2}\ln\det$. Phases of the chiral determinant (eta invariants, global anomalies) are *not* halved by this bookkeeping; they do not contribute to the local $\Lambda^2 \int \sqrt{g}\, R$ term but must be tracked separately in any application sensitive to them.

8 Examples and reproducible verification

8.1 Sign examples

- Minimal real scalar: $A_1^{(\text{eff})} = \frac{1}{6} > 0$ implies $G_{\text{ind}} > 0$.
- Real scalar with $\xi = \frac{1}{4}$: $A_1^{(\text{eff})} = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12} < 0$ implies $G_{\text{ind}} < 0$.
- Gauge vector (per generator): $A_1^{(\text{eff})} = -\frac{2}{3} < 0$ implies $G_{\text{ind}} < 0$ for that sector alone.

8.2 Exact-spectrum numerical verification on S^4 (added in v2)

Every entry of Table 1 that comes from a heat-kernel computation is verified in the companion script `verification/0102/verify_0102.py` (repository of the series) against *exact* eigenvalue sums on the round unit S^4 ($V = 8\pi^2/3$, $R = 12$), where all three relevant spectra are known in closed form: scalar Laplacian $\lambda_l = l(l+3)$ with multiplicity $(2l+3)(l+1)(l+2)/6$; Dirac $-D^2$ with $\lambda_l = (l+2)^2$ and total multiplicity $\frac{4}{3}(l+1)(l+2)(l+3)$; and the Hodge–de Rham operator on 1-forms $\Delta_1 = -\nabla^2 + \mathrm{Ric}$ (the Feynman-gauge P_1), whose exact part inherits the scalar spectrum and whose co-exact part has $\lambda_l = (l+1)(l+2)$ with multiplicity $l(l+3)(2l+3)/2$ (anchor: $\lambda = 6$ with multiplicity $10 = \dim SO(5)$, the Killing 1-forms). The spectra are exact; the extraction of a_1 is numerical (truncated eigenvalue sums, $l \leq 4000$, and Richardson extrapolation of $16\pi^2 s^2\ \mathrm{Tr}\, e^{-sP}/V$ as $s \to 0^+$), and reproduces

$$\mathrm{tr}\, a_1(\text{scalar}) = +\tfrac{R}{6}, \qquad \mathrm{tr}\, a_1(\text{Dirac}) = -\tfrac{R}{3}, \qquad \mathrm{tr}\, a_1(\text{1-form}) = -\tfrac{R}{3},$$

each to better than 10^{-5}, and the ghost-subtracted combination gives $A_1^{(\text{eff})}(\text{vector+ghosts}) = -0.666666$. The script also checks Corollary 1 and the sign examples to machine precision (11/11 checks pass).

9 Regularization note

The term proportional to Λ^2 is regulator-dependent: it appears explicitly with a Schwinger cutoff, while in dimensional regularization power divergences do not appear as poles; there the induced Newton constant must be extracted from finite, calculable pieces, which is the core of Adler's program [6]. What is robust here is the local tensorial structure (controlled by a_1) and relative normalizations between species within a fixed scheme – cf. Corollary 1, whose ratios $1 : 2 : -4$ are scheme-independent statements about a_1. The physical Newton constant G_{phys} requires a counterterm and a renormalization condition; and for the vector sector the off-shell gauge dependence of Remark 3 applies. See Visser [7] for a modern assessment of what Sakharov-style induced gravity [5] can and cannot fix.

Note added in v2

Changes with respect to v1, all verified in `verification/0102/verify_0102.py`: (1) Corollary 1: the species table is anchored to the classic counting $(\Lambda^2/12\pi)(N_0 + 2N_{1/2} - 4N_1)$ [6, 7, 8]. (2) Remark 3: gauge/parametrization dependence of the quadratically divergent vector coefficient off shell, with the Vilkovisky–DeWitt caveat and the connection to Kabat's contact term. (3) Weyl/Majorana footnote: the halving concerns the local a_1 only; determinant phases are tracked separately. (4) Wick-rotation wording corrected to the weight-level statement $iS^{(L)} \to -S^{(E)}$. (5) Section 8.2: exact-spectrum numerical verification on S^4 of all three a_1 entries and of the ghost-subtracted vector combination, replacing the merely pedagogical snippet of v1 (which is retained in the script as its Part 4 bookkeeping). No result of v1 changes; the derivations and Table 1 are unchanged.

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