

Beyond Gaussianity: Extending the Clustering–Recovery Bridge

(v2: normalized Markov-product identity corrected;

independent verification suite added)

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July 2026 (v2); v1: December 31, 2025

Abstract

We formulate a non-Gaussian, finite-volume and uniform-in- Λ version of the Clustering–Recovery bridge for interacting lattice systems. We introduce an explicit collar geometry, a CMI formulation via Fawzi–Renner [1], and an operational (Heisenberg-picture) quasi-locality strengthening for the recovery map. *Version 2* corrects one identity: v1’s Eq. (15) equated $I(A : C|B)$ with the relative entropy to the *normalized* Markov product state; the exact identity holds for the unnormalized product, and the normalized version underestimates it by $-\log Z \geq 0$ (Lieb). We also fix the Fawzi–Renner factor: with the squared-fidelity convention used throughout, the bound is $I(A : C|B) \geq -\log F$ (not $-2\log F$). All numerical claims of v1 (Petz slope table; prefactor–slope trade-off; crossover $w^* = 3$) have been independently reproduced from scratch; the verification script ships with this paper.

1 Beyond Gaussianity: extending the Clustering–Recovery bridge

The proof strategy used in the Gaussian setting [9] relies crucially on the quasi-free structure: all correlation data are encoded in the two-point function and the recovery map can be built from linear (symplectic) transformations. In interacting theories such a reduction is not available. This motivates the following program: replace Gaussian structure by *robust, model-independent* notions of clustering and quasi-locality, and derive recoverability (or small conditional mutual information) from these assumptions using genuinely non-Gaussian techniques.

1.1 Working framework and notions of clustering

We consider a lattice system with finite-dimensional local Hilbert spaces, so that finite-region restrictions are described by density matrices. Throughout this section we work with a family of finite-volume states $(\rho_\Lambda)_\Lambda$ (e.g. finite-volume Gibbs states for a finite-range interaction with a fixed choice of boundary conditions). All bounds are required to hold uniformly in Λ .

For each finite volume Λ , let $\text{dist}(\cdot, \cdot)$ denote the graph distance induced by Λ (with the chosen boundary conditions). Given regions $X, Y \subseteq \Lambda$,

$$\text{dist}(X, Y) := \min\{\text{dist}(x, y) : x \in X, y \in Y\}, \quad \text{dist}(x, A) := \min\{\text{dist}(x, y) : y \in A\}. \quad (1)$$

Definition 1 (Connected correlator and uniform exponential clustering). *Let ρ_Λ be a finite-volume state on Λ and let O_X, O_Y be observables supported on regions X and Y . The connected correlator is*

$$\text{Corr}_{\rho_\Lambda}(O_X, O_Y) := \text{Tr}(\rho_\Lambda O_X O_Y) - \text{Tr}(\rho_\Lambda O_X) \text{Tr}(\rho_\Lambda O_Y). \quad (2)$$

The family $(\rho_\Lambda)_\Lambda$ has uniform exponential clustering if there exist $c, \xi > 0$ such that for all volumes Λ , all disjoint $X, Y \subseteq \Lambda$, and all O_X, O_Y with $\|O_X\|_\infty, \|O_Y\|_\infty \leq 1$,

$$|\text{Corr}_{\rho_\Lambda}(O_X, O_Y)| \leq c e^{-\text{dist}(X, Y)/\xi}. \quad (3)$$

In the interacting case, Definition 1 is a *minimal, motivating* hypothesis, not one expected to be sufficient for CMI/recovery without refinement: for entropic statements one typically needs *uniform* bounds stable under conditioning on a buffer region and/or higher-order cumulant control (Section 1.5). Conjecture 1 should be read as conditional on such a strengthened clustering input.

1.2 Geometric setup: a collar buffer of width w

Let Λ be finite and $A \subset \Lambda$. For an integer $w \geq 1$ define the w -neighborhood and the induced tripartition

$$A^{(w)} := \{x \in \Lambda : \text{dist}(x, A) \leq w\}, \quad B := A^{(w)} \setminus A, \quad C := \Lambda \setminus A^{(w)}. \quad (4)$$

Then A, B, C are disjoint and B is a separating collar of width w . For open boundary conditions we restrict to an *admissible class* of regions with $\text{dist}(A, \partial\Lambda) \geq w$, so the collar avoids the boundary; on a torus no restriction is needed.

1.3 A non-Gaussian extension of Conjecture 5.1 of the companion

We take fidelity in the Uhlmann form $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$.

Conjecture 1 (General Clustering–Recovery bridge beyond Gaussianity). *Assume $(\rho_\Lambda)_\Lambda$ has uniform exponential clustering with correlation length ξ . For any finite volume Λ , admissible $A \subseteq \Lambda$, and collar width $w \geq 1$, with the tripartition of Eq. (4), there exist constants $c', \alpha > 0$ (depending only on microscopic locality data and ξ , not on Λ, A , or w) such that for each (Λ, A, w) there is a CPTP recovery channel $\mathcal{R}_{B \rightarrow BC}$ acting nontrivially only on B with*

$$1 - F(\rho_{ABC}, (\text{id}_A \otimes \mathcal{R}_{B \rightarrow BC})(\rho_{AB})) \leq c' e^{-\alpha w/\xi}. \quad (5)$$

Remark 1 (Locality of dynamics vs locality of recovery). *Locality of the interaction typically yields Lieb–Robinson bounds [2], but quasi-locality of recovery is not automatic; hence the optional operational strengthening below.*

Optional strengthening (operational quasi-locality of recovery). In the Heisenberg picture, with $\mathcal{R}_{B \rightarrow BC}^\dagger$ the adjoint channel on observables: there exist $c_{\text{ql}}, \alpha_{\text{ql}} > 0$ such that for any $C_0 \subseteq C$ with $\text{dist}(C_0, B) \geq r$ and any O_{C_0} with $\|O_{C_0}\|_\infty \leq 1$, where $\rho_{C_0} := \text{Tr}_{\Lambda \setminus C_0}(\rho_\Lambda)$,

$$\left\| \mathcal{R}_{B \rightarrow BC}^\dagger(O_{C_0}) - \text{Tr}(\rho_{C_0} O_{C_0}) \mathbf{1}_B \right\|_\infty \leq c_{\text{ql}} e^{-\alpha_{\text{ql}} r/\xi}. \quad (6)$$

Entropic conventions (finite-dimensional). All logarithms are natural. $S(\rho) := -\text{Tr}(\rho \log \rho)$, $D(\rho\|\sigma) := \text{Tr}(\rho(\log \rho - \log \sigma))$, with $D = +\infty$ if $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$; $D(\rho\|\sigma)$ is used below also for unnormalized positive σ , with the same formula. For a tripartite state,

$$I(A : C|B)_\rho := S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_B) - S(\rho_{ABC}). \quad (7)$$

1.3.1 CMI formulation via Fawzi–Renner

Applied to the finite-dimensional state ρ_{ABC} , the Fawzi–Renner inequality [1] implies existence of a channel $\mathcal{R}_{B \rightarrow BC}$ with

$$I(A : C|B)_\rho \geq -\log F(\rho_{ABC}, (\text{id}_A \otimes \mathcal{R}_{B \rightarrow BC})(\rho_{AB})), \quad (8)$$

where $F = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$ is the *squared* Uhlmann fidelity fixed above. Equivalently, in terms of the root fidelity $f := \|\sqrt{\rho}\sqrt{\sigma}\|_1 = \sqrt{F}$, the standard Fawzi–Renner form $I(A : C|B)_\rho \geq -2 \log f$ is recovered. Using $1 - F \leq -\log F$, this yields the recovery-error bound $1 - F \leq I(A : C|B)_\rho$ directly. A sufficient route to Conjecture 1 is therefore an exponential CMI estimate $I(A : C|B) \leq c'' e^{-\alpha w/\xi}$.

CMI as relative entropy to the Markov product (corrected in v2). Let

$$M_{ABC} := \exp(\log \rho_{AB} + \log \rho_{BC} - \log \rho_B), \quad Z := \text{Tr } M_{ABC}, \quad \sigma_{ABC}^{\text{MP}} := M_{ABC}/Z. \quad (9)$$

The *exact* identity is with the unnormalized Markov product:

$$I(A : C|B)_\rho = D(\rho_{ABC} \parallel M_{ABC}). \quad (10)$$

For the normalized state, since $\log \sigma^{\text{MP}} = \log M_{ABC} - (\log Z)\mathbf{1}$ and $\text{Tr } \rho = 1$,

$$D(\rho_{ABC} \parallel \sigma_{ABC}^{\text{MP}}) = D(\rho_{ABC} \parallel M_{ABC}) + \log Z = I(A : C|B)_\rho + \log Z, \quad (11)$$

so that

$$\boxed{I(A : C|B)_\rho = D(\rho_{ABC} \parallel \sigma_{ABC}^{\text{MP}}) - \log Z, \quad Z \leq 1.} \quad (12)$$

Here $Z \leq 1$ follows from Lieb’s triple-matrix inequality [8] (equality iff ρ_{ABC} is an exact quantum Markov chain, in which case $Z = 1$ and both formulas coincide). Since $\log Z \leq 0$, the normalized relative entropy *underestimates* the CMI by $-\log Z \geq 0$: $D(\rho \parallel \sigma^{\text{MP}}) \leq I(A : C|B)_\rho$. v1 stated Eq. (12) without the $-\log Z$ term, i.e. as an equality $I(A : C|B) = D(\rho \parallel \sigma^{\text{MP}})$, which therefore understates the CMI by $|\log Z|$ whenever the state is not exactly Markov. Both (10) and (12) are verified numerically in Appendix A.

1.4 Why Gaussian proofs do not port

In the Gaussian case the recovery map is determined by two-point data and the proof is matrix analysis. In interacting models none of the following hold in general: (1) Wick reduction of higher correlators; (2) linear/symplectic structure of an optimal recovery channel; (3) closed-form entropies/CMI in terms of finitely many correlators. The extension requires techniques that (a) connect correlator clustering to entropic statements and (b) construct or certify a quasi-local recovery map.

1.5 Technical tool-kit needed outside Gaussian structure

Candidate ingredients replacing Gaussian linear algebra: (1) quasi-locality bounds (Lieb–Robinson) [2]; (2) gapped ground states and exponential clustering [3]; (3) approximate factorization / buffer decoupling, $\|\rho_{AC} - \rho_A \otimes \rho_C\|_1 \leq c e^{-\text{dist}(A,C)/\xi}$; (4) from clustering of correlators to clustering of relative entropy / CMI; (5) cluster expansion / polymer methods (high temperature, weak coupling) [4]; (6) approximate quantum Markov chains and recovery (Petz / rotated Petz) [1, 5, 6],

$$\mathcal{R}_{B \rightarrow BC}^t(X) := \rho_{BC}^{\frac{1+it}{2}} \rho_B^{-\frac{1+it}{2}} X \rho_B^{-\frac{1-it}{2}} \rho_{BC}^{\frac{1-it}{2}}; \quad (13)$$

(7) approximate modular flow locality and Araki-type bounds; (8) cumulant control beyond two-point functions, $|\kappa_k(O_{X_1}, \dots, O_{X_k})| \leq c_k \exp(-\text{TreeDist}(X_1, \dots, X_k)/\xi)$.

2 Numerical experiments: TFIM Gibbs recovery

Why numerics here. A complete analytical proof of Conjecture 1 would require assembling several non-Gaussian ingredients (Section 1.5). Pending such a theory, we provide finite-size numerical evidence in the 1D TFIM Gibbs state and directly test the predicted exponential-in- w behavior of recovery errors for explicit candidate maps (Petz and averaged rotated-Petz).

2.1 Numerical methodology

Model and Gibbs state. Transverse-field Ising chain with open boundary conditions,

$$H = -J \sum_{i=1}^{N-1} X_i X_{i+1} - h \sum_{i=1}^N Z_i, \quad (14)$$

Gibbs state $\rho \propto e^{-\beta H}$ by exact diagonalization; $N = 10$, $J = 1$, $h \in \{1.05, 1.10, 1.20, 1.50\}$, $\beta \in \{0.3, 0.6, 1.0, 1.5, 2.0\}$.

Collar partition. $A = \{1, \dots, L_A\}$, $B = \{L_A+1, \dots, L_A+w\}$, $C = \{L_A+w+1, \dots, N\}$, with $L_A = 2$, $w \in \{1, \dots, 6\}$. The admissibility condition is *not* imposed in this finite 1D benchmark (for large w the collar meets the boundary); results are treated as controlled finite-size evidence for the qualitative w -dependence.

CMI and recovery. For each (h, β, w) we compute $\rho_{AB}, \rho_B, \rho_{BC}, \rho_{ABC}$ by partial trace and $I(A : C|B)$ from von Neumann entropies. The Petz map is

$$\mathcal{R}_{B \rightarrow BC}^{\text{Petz}}(X) = \rho_{BC}^{1/2} (\rho_B^{-1/2} X \rho_B^{-1/2} \otimes \mathbf{1}_C) \rho_{BC}^{1/2}, \quad (15)$$

with $\rho_B^{-1/2}$ the Moore–Penrose inverse on $\text{supp}(\rho_B)$, so that $\mathcal{R}^{\text{Petz}}$ is CPTP in exact arithmetic. The rotated-Petz method uses a discretized averaged map $\mathcal{R}^{\text{rot}} \approx \sum_k w_k \mathcal{R}^{t_k}$ with $w_k \geq 0$, $\sum_k w_k = 1$ (a convex combination of completely positive maps). The subsequent PSD-clipping and trace renormalization is numerical post-processing applied to $\tilde{\rho}_{ABC}$, not claimed to preserve exact channel properties at the map level.

Regularization. Eigenvalue flooring $\lambda_j \mapsto \max\{\lambda_j, \varepsilon_{\text{inv}}\}$ with $\varepsilon_{\text{inv}} = 10^{-12}$ before applying λ_j^α (including $\alpha = -1/2$); after each reconstruction, Hermitization, PSD clipping at $\varepsilon_{\text{psd}} = 10^{-14}$, and trace renormalization.

Rotated-Petz averaging (discretized). Truncated quadrature $t \in [-t_{\text{max}}, t_{\text{max}}]$, $t_{\text{max}} = 6$, $n_t = 61$ uniform nodes, trapezoidal weights proportional to $(\cosh(\pi t) + 1)^{-1}$, normalized on the truncated grid [6].

Error metrics and fits. Trace distance $T(w) := \frac{1}{2} \|\rho_{ABC} - \tilde{\rho}_{ABC}\|_1$, Hilbert–Schmidt norm, and $T(w)^2$ as a proxy motivated by Fuchs–van de Graaf $T^2 \leq 1 - F \leq 2T$ [7]. For each observable $y(w)$ we fit $\log y(w) \approx a - \mu w$ (least squares on positive data points) and report μ . By construction $\mu_{T^2} = 2\mu_T$; the column is retained for continuity with v1.

N	L_A	h	β	method	μ_{CMI}	μ_T	μ_{HS}	μ_{T^2}
10	2	1.05	0.30	petz	7.0354	5.3352	5.3002	10.6705
10	2	1.05	0.30	rotated	7.0354	4.6294	4.6326	9.2587
10	2	1.05	0.60	petz	4.4807	3.8992	3.8466	7.7983
10	2	1.05	0.60	rotated	4.4807	3.3117	3.2917	6.6233
10	2	1.05	1.00	petz	3.0063	2.7623	2.7326	5.5246
10	2	1.05	1.00	rotated	3.0063	2.3692	2.3489	4.7383
10	2	1.05	1.50	petz	2.0991	2.0078	1.9949	4.0156
10	2	1.05	1.50	rotated	2.0991	1.7075	1.6920	3.4150
10	2	1.05	2.00	petz	1.6014	1.5659	1.5608	3.1317
10	2	1.05	2.00	rotated	1.6014	1.3064	1.2975	2.6128
10	2	1.10	0.30	petz	4.9396	5.2877	5.2121	10.5754
10	2	1.10	0.30	rotated	4.9396	4.5992	4.6008	9.1984
10	2	1.10	0.60	petz	4.4085	3.8551	3.8054	7.7102
10	2	1.10	0.60	rotated	4.4085	3.2721	3.2534	6.5442
10	2	1.10	1.00	petz	2.9533	2.7282	2.7040	5.4565
10	2	1.10	1.00	rotated	2.9533	2.3376	2.3198	4.6752
10	2	1.10	1.50	petz	2.0622	1.9797	1.9743	3.9593
10	2	1.10	1.50	rotated	2.0622	1.6825	1.6716	3.3651
10	2	1.10	2.00	petz	1.5756	1.5386	1.5428	3.0773
10	2	1.10	2.00	rotated	1.5756	1.2868	1.2834	2.5736
10	2	1.20	0.30	petz	6.7935	5.2927	5.2593	10.5855
10	2	1.20	0.30	rotated	6.7935	4.5448	4.5389	9.0897
10	2	1.20	0.60	petz	4.2792	3.7692	3.7261	7.5383
10	2	1.20	0.60	rotated	4.2792	3.2002	3.1844	6.4005
10	2	1.20	1.00	petz	2.8631	2.6564	2.6414	5.3128
10	2	1.20	1.00	rotated	2.8631	2.2785	2.2655	4.5570
10	2	1.20	1.50	petz	2.0053	1.9110	1.9159	3.8221
10	2	1.20	1.50	rotated	2.0053	1.6315	1.6266	3.2630
10	2	1.20	2.00	petz	1.5420	1.4693	1.4834	2.9387
10	2	1.20	2.00	rotated	1.5420	1.2423	1.2446	2.4846
10	2	1.50	0.30	petz	6.3827	5.1851	5.1450	10.3701
10	2	1.50	0.30	rotated	6.3827	4.3791	4.3633	8.7581
10	2	1.50	0.60	petz	3.9869	3.5227	3.4867	7.0453
10	2	1.50	0.60	rotated	3.9869	3.0035	2.9915	6.0071
10	2	1.50	1.00	petz	2.6913	2.4388	2.4213	4.8775
10	2	1.50	1.00	rotated	2.6913	2.0939	2.0766	4.1878
10	2	1.50	1.50	petz	1.9329	1.7097	1.7041	3.4193
10	2	1.50	1.50	rotated	1.9329	1.4609	1.4492	2.9219
10	2	1.50	2.00	petz	1.5391	1.2830	1.2795	2.5659
10	2	1.50	2.00	rotated	1.5391	1.0952	1.0891	2.1904

Table 1: Exponential-fit slopes for $I(A : C|B)$, trace-distance error T , Hilbert–Schmidt error, and T^2 (v1 Colab values; independently reproduced, Appendix A).

2.2 Fitted exponential slopes

Table 1 reports the v1 (Colab) slopes; every Petz row and a representative subset of rotated rows have been reproduced from scratch by the independent verification suite (Appendix A).

2.3 Prefactor–slope trade-off and crossover widths

In the high-temperature regime we observe a consistent prefactor–slope trade-off: rotated-Petz yields smaller $T(w)$ at $w = 1$ while Petz exhibits a larger decay rate and overtakes at a modest crossover width w^* . For all eight disagreement cases ($h \in \{1.05, 1.10, 1.20, 1.50\}$, $\beta \in \{0.30, 0.60\}$) the crossover is $w^* = 3$ in every metric (T , HS, T^2); Figure 1 shows the regenerated curves.

Limitations. Finite-size numerics on a specific 1D model and collar geometry; no thermodynamic-limit uniformity claimed; no certified uncertainty bars for fitted slopes beyond the fitting procedure described. Fits at $\beta = 0.3$ include points near the 10^{-13} arithmetic floor: the T and HS slopes there are stable only to a few percent, and the μ_{CMI} entries at $\beta = 0.3$ are floor-dominated fitting noise and should be discarded (Appendix A).

A Independent verification

Distributed as `verification/0101/verify_0101.py` (NumPy only; no Colab dependence). It re-implements the entire pipeline from scratch (exact diagonalization, partial traces, Petz and discretized rotated-Petz with the stated regularization) and checks:

(a) Markov-product identity (the v2 correction). On random tripartite states it verifies $I(A : C|B) = D(\rho||M)$ exactly (machine precision) for the unnormalized M of Eq. (9), and $I(A : C|B) = D(\rho||\sigma^{\text{MP}}) - \log Z$ with $\log Z < 0$ strictly on non-Markov states (so $D(\rho||\sigma^{\text{MP}}) < I$) — confirming that v1’s normalized-state equality understated the CMI by exactly $|\log Z|$. The check $I - [D(\rho||\sigma^{\text{MP}}) - \log Z]$ vanishes to machine precision, while the wrong-sign combination $I - [D(\rho||\sigma^{\text{MP}}) + \log Z]$ leaves a residual $2|\log Z|$.

(b) Petz slope table. All twenty (h, β) Petz rows of Table 1 are reproduced. For $\beta \geq 0.6$ the agreement is to all four reported decimals in every column (e.g. $h=1.10$, $\beta=1.00$: $\mu_{\text{CMI}}=2.9533$, $\mu_T=2.7282$, $\mu_{\text{HS}}=2.7040$, $\mu_{T^2}=5.4565$). At $\beta = 0.3$ the T and HS slopes agree to 0.3–3% (e.g. $h=1.20$: $\mu_T = 5.4172$ vs v1’s 5.2927): there $T(w \geq 5) \lesssim 10^{-13}$ sits at the arithmetic floor and the fit is sensitive to regularization and BLAS details. The μ_{CMI} entries at $\beta = 0.3$ are *not* reproducible and should be discarded: at that temperature $I(A : C|B)$ reaches the machine floor ($|I| \sim 10^{-14}$, sign-indefinite) already at $w = 5$, so the positive-points-only fit retains different point sets in different arithmetics (v1 reports $\mu_{\text{CMI}} \in \{7.04, 4.94, 6.79, 6.38\}$ across h at fixed $\beta = 0.3$; the independent run gives $\{7.02, 6.97, 6.78, 5.18\}$ — both are fitting noise). This caveat, absent in v1, does not affect the physics: the exponential decay of $T(w)$ itself is unambiguous over six decades, and all $\beta \geq 0.6$ CMI slopes reproduce exactly.

(c) Rotated-Petz and crossover. The rotated-Petz rows are reproduced for all eight disagreement cases ($h \in \{1.05, 1.10, 1.20, 1.50\}$, $\beta \in \{0.30, 0.60\}$): exact to four decimals at $\beta = 0.6$, within 0.3–0.6% at $\beta = 0.3$. The prefactor–slope trade-off with crossover $w^* = 3$ (Table 2 of v1) is confirmed in all eight cases in the trace-distance metric (Figure 1 and `fig8_data.csv` are generated by the suite).

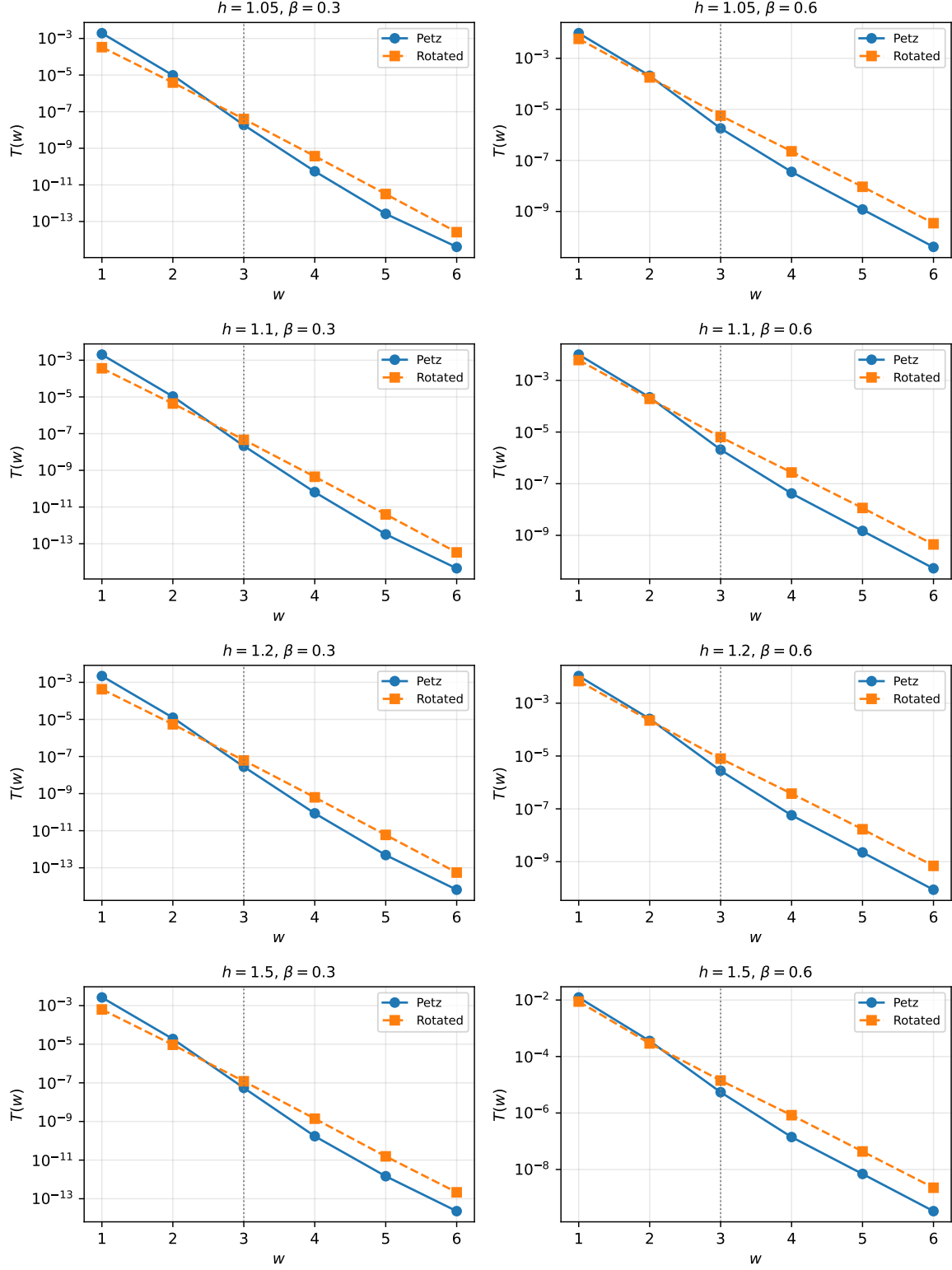


Figure 1: Trace-distance recovery error $T(w)$ vs buffer width w for the eight disagreement cases (regenerated by the verification suite): rotated-Petz is better at $w = 1$ (prefactor), Petz overtakes for $w \geq w^* = 3$ (slope).

B Changes relative to version 1

1. **Corrected identity (the only mathematical error found in v1):** v1’s “CMI as relative entropy” equation stated $I(A : C|B) = D(\rho||\sigma^{\text{MP}})$ with the *normalized* Markov product; the exact statement is Eq. (10) (unnormalized), or Eq. (12) with the $-\log Z$ correction and $Z \leq 1$ by Lieb’s triple-matrix inequality [8]. Verified numerically. Additionally, the Fawzi–Renner inequality is stated consistently with the squared-fidelity convention, $I(A : C|B) \geq -\log F$ (Eq. (8)); v1 carried the root-fidelity factor $-2\log F$ with F defined as squared fidelity, an inconsistent factor of two.
2. **Independent verification added:** full from-scratch re-implementation (Appendix A); v1’s figure artifacts (Colab PNG renderings) replaced by script-generated Figure 1 and CSV.
3. **Numerical-floor caveat added** for the $\beta = 0.3$ fits.
4. **Companion reference** updated to the v2 of the Gaussian clustering–recovery paper (2512.0060v2) and its repository; log-base convention made explicit; $\mu_{T^2} = 2\mu_T$ redundancy noted.

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