Continuum Isotropy from the Spherical 5-Design Geometry of the 24-Cell Lattice

Christoph Kovacs[1]

[1] Independent Researcher, ETH Zürich (Alumni), Vienna, Austria

E-mail: `christoph.kovacs@alumni.ethz.ch`

Abstract. Discrete approaches to quantum gravity and emergent spacetime face a fundamental challenge: reconciling the discreteness of the substrate with the strict macroscopic continuous rotational symmetry required by relativistic field theories. Generic lattices break rotational symmetry, leading to direction-dependent dispersion relations and vacuum birefringence—effects that are tightly constrained by experiment. In this study, we investigate whether a four-dimensional lattice exists that yields an isotropic continuum kinematics without fine-tuning. We analyze the spectral and elastic properties of the 24-cell (D_4) lattice compared to those of the hypercubic and Simplex lattices. We show that while the Simplex geometry (a spherical 3-design) fails to secure elastic isotropy, the D_4 lattice—constituting a spherical 5-design—eliminates leading-order anisotropy in scalar dispersion and elastic response. Consequently, the 24-cell lattice exhibits exact elastic isotropy and suppresses vacuum birefringence for transverse shear modes in the infrared limit. Although scalar dispersion anisotropy is suppressed to fourth order in the wavenumber, residual dispersive anisotropy for vector modes remains at second order due to higher-rank lattice moments. These results identify the 24-cell geometry as a unique and strictly required geometric precursor for discrete models aiming to recover spatial relativistic kinematics.

Keywords: Emergent Spacetime, Lorentz Invariance Violation, Vacuum Birefringence, 24-cell Lattice, Spherical 5-Design, Elastic Gravity

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1. Introduction

The search for a discrete microstructure of spacetime is driven by the need to regularize ultraviolet divergences in quantum field theory and to provide a microscopic basis for gravitation. Models ranging from Causal Set Theory to Crystalline Gravity, condensed-matter analogs [1], and elastic spacetime models [2] postulate a discrete cutoff scale. However, any discretization of four-dimensional space inevitably breaks the continuous rotation group $SO(4)$ (or the Lorentz group $SO(3,1)$ after Wick rotation) down to a discrete subgroup.

This symmetry breaking generically manifests as Lorentz Invariance Violation (LIV) in the infrared (IR) limit, often parameterized via the Standard Model Extension (SME) [3]. Such violations would lead to phenomena such as anisotropic maximum velocities, vacuum birefringence, and direction-dependent gravitational coupling. Given the extreme experimental constraints on LIV [4, 5], such as $|\Delta c/c| < 10^{-19}$, any candidate lattice structure must possess intrinsic symmetries that suppress these artifacts well below the Planck scale.

In this work, we distinguish between two fundamental challenges of discrete spacetime: (1) The geometric suppression of lattice artifacts (anisotropy), and (2) the dynamical emergence of a causal structure (Lorentz boosts). Here, we exclusively address the first challenge. We demonstrate that the 24-cell lattice (associated with the D_4 root system) solves the anisotropy problem at the level of Euclidean lattice kinematics, providing the necessary isotropic substrate upon which Lorentz-invariant dynamics can be constructed. We emphasize that we address the isotropy of the lattice rest frame; the implementation of boost invariance requires additional dynamical mechanisms not discussed here.

Since Lorentz invariance is related to $SO(4)$ rotation invariance after Wick rotation to imaginary time, Euclidean isotropy of the lattice action is a necessary geometric precursor for Lorentz-invariance–compatible infrared kinematics. Specifically, any anisotropic $O(k^4)$ terms in the Euclidean dispersion relation would map to energy-dependent Lorentz-violating operators in the effective field theory, leading to preferred-frame effects. Eliminating these geometric artifacts in the Euclidean domain is therefore a prerequisite for any discrete model aiming to recover relativistic symmetry.

Closely related D_4-symmetric lattices, commonly referred to as face-centered hypercubic (FCHC) lattices, have long been employed in lattice-gas and lattice-Boltzmann formulations to recover rotationally invariant hydrodynamics [6–9]. Although their high-order isotropy is well established in that context, the present work extends this geometric principle to the propagation of vector and tensor fields in vacuum, explicitly addressing polarization splitting and vacuum birefringence relevant for relativistic field theories. While we explicitly analyze the propagation of vector excitations (lattice phonons), the geometric isotropy requirements for the suppression of vacuum birefringence in a spin-1 sector are inevitably linked to those of higher-spin sectors. A lattice that fails to secure isotropic vector propagation cannot support an isotropic spin-2 field.

In contrast to hydrodynamics, where isotropy of the stress tensor suffices to recover the Navier–Stokes equations, relativistic quantum field theories are subject to vastly more stringent constraints on Lorentz invariance violation ($|\Delta c/c| < 10^{-19}$). Our results show that the same geometric properties exploited in fluid dynamics are sufficient to suppress lattice-induced anisotropies to a level compatible with these extreme precision bounds in the infrared.

The present work extends this geometric principle beyond hydrodynamics to relativistic dispersion relations and elastic transverse kinematics in emergent spacetime models. It is relevant for approaches in which spacetime discreteness is physical and persists in the infrared, rather than serving merely as a regulator to be removed in the continuum limit.

Our results do not address the dynamical emergence of a time direction or the implementation of Wick rotation on a discrete substrate. Even though Euclidean isotropy is a necessary condition for Lorentz invariance, it is not sufficient. Specifically, elastic lattice models with central forces naturally lead to a fixed ratio between longitudinal and transverse sound speeds ($c_L \neq c_T$), whereas relativity requires a single universal limiting velocity [1]. However, this dynamical issue of coupling constants is distinct from the geometric issue of lattice anisotropy. Before one can construct a mechanism to decouple or suppress the longitudinal sector (e.g., via emergent gauge symmetries), one must first ensure that the underlying vacuum structure does not imprint preferred directions on the transverse sector. The present work provides a rigorous solution to the latter, geometric problem.

2. Methods: Lattice Geometry and Spectral Analysis

To quantify the geometric isotropy of the vacuum substrate, we compare the spectral and elastic properties of two fundamental regular tessellations of Euclidean 4-space.

2.1. Elastic Anisotropy Measures

To quantify the suitability of a lattice as a vacuum substrate, we compute the rank-4 stiffness tensor C_{ijkl} using a nearest-neighbor central-force model.

The Zener anisotropy ratio A [10] serves as the standard metric for cubic crystals, yet it becomes ill-defined for non-cubic geometries such as the A_4 simplex lattice. To compare all lattice geometries on an equal footing, strictly independent of their coordinate representation, we introduce the *relative elastic anisotropy* δ_F, a tensorial anisotropy measure based on the Frobenius norm (see Definition 2.1).

For lattices that possess cubic symmetry (such as $\mathbb{Z}^4$ and D_4), we additionally verify the Cubic Symmetry Violation (the deviation from the hypercubic symmetry group) and compute the standard Zener ratio $A = 2C_{44}/(C_{11} - C_{12})$.

2.2. Lattice Definitions in Euclidean 4-Space

We consider the following four-dimensional lattice geometries [11], whose properties are summarized in Table 1:

1. The Hypercubic Lattice ($\mathbb{Z}^4$): Defined by the basis vectors $\{\mathbf{e}_\mu\}$. Each site has $N = 8$ nearest neighbors located at $\pm\mathbf{e}_\mu$. The point group is the hyperoctahedral group B_4.
2. The Simplex Lattice (A_4): Associated with the A_4 root system (the root lattice of $SU(5)$). It is constructed as the subset of integer points in $\mathbb{Z}^5$ satisfying the constraint $\sum_{i=1}^{5} x_i = 0$. Each site has $N = 20$ nearest neighbors given by the permutations of $(1, -1, 0, 0, 0)$. The point group is the symmetric group S_5.
3. The 24-cell Lattice (D_4): Associated with the D_4 root system (the root lattice of $SO(8)$). The sites are linear combinations of basis vectors with integer coefficients such that the sum of coordinates is even. The $N = 24$ nearest neighbors are given by the permutations of $(\pm1, \pm1, 0, 0)$.

The 24-cell is the unique regular convex polytope in four dimensions without a 3D analog. Crucially, its vertex set forms a spherical 5-design [12]. This implies that the

Lattice	Neighbors	Point Group	Spherical Design
Hypercubic ($\mathbb{Z}^4$)	8	B_4	No
Simplex (A_4)	20	A_4	3-Design
24-cell Lattice (D_4)	24	F_4	5-Design

Table 1: Hierarchy of rotational symmetries for candidate vacuum substrates. Note that only the 24-cell (D_4) lattice constitutes a spherical 5-design, a necessary geometric condition for isotropic rank-4 tensor propagation.

average of any polynomial of degree $d \leq 5$ over the vertices is exactly equal to the average over the 3-sphere.

Appendix A contains an overview of the geometry of the 24-cell and the D_4 root lattice.

2.3. Discrete Dispersion Relations

We analyze the propagation of massless excitations governed by the lattice Laplacian. The dispersion relation $\omega^2(\mathbf{k})$ for a scalar field is given by the stencil sum over nearest neighbors $\mathbf{v}$:

$$\omega^2 \equiv \omega^2(\mathbf{k}) \propto \sum_{\mathbf{v}} \left[1 - \cos(\mathbf{k} \cdot \mathbf{v})\right].$$

Here, $\mathbf{k}$ denotes the dimensionless wave vector normalized to the lattice spacing. Expanding the cosine for small wave vectors (around $\mathbf{k} = 0$) yields

$$\omega^2 \;\propto\; \sum_{\mathbf{v}} \left[\frac{1}{2}(\mathbf{k} \cdot \mathbf{v})^2 - \frac{1}{24}(\mathbf{k} \cdot \mathbf{v})^4 + \mathcal{O}(k^6)\right],$$

where $k \equiv |\mathbf{k}|$ denotes the magnitude of the wave vector. The quadratic term is always isotropic ($\sum_{\mathbf{v}} v_i v_j \propto \delta_{ij}$), yielding the continuum Laplacian. Leading anisotropic contributions arise from the quartic term.

For the hypercubic lattice $\mathbb{Z}^4$, the fourth-order moment contains anisotropic invariants of the form $\sum_i k_i^4$. Since the normalized neighbor vectors of the 24-cell lattice (D_4) form a spherical 5-design, all fourth-order tensor moments reduce to isotropic combinations of Kronecker deltas:

$$\sum_{\mathbf{v}} v_i v_j v_k v_l \;\propto\; \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}.$$

Thus, all quartic anisotropies cancel identically for the D_4 geometry.

Our analysis is restricted to the long-wavelength ($\mathbf{k} \to 0$) regime; anisotropies near the Brillouin-zone boundary are not addressed and are expected to remain lattice-specific.

2.4. Dynamical Matrix and Mode Decomposition

For emergent gravity and elastic models, we extend the analysis to vector fields u_i. The equations of motion are governed by the dynamical matrix $D_{ij}(\mathbf{k})$. For a given propagation direction $\hat{\mathbf{k}}$, the eigenvalue spectrum of $D_{ij}(\mathbf{k})$ yields:

1. One longitudinal (acoustic) branch, ω_L^2.
2. Three transverse (shear) branches, $\omega_{T,i}^2$, $i \in \{1, 2, 3\}$.

In a four-dimensional bulk, the transverse subspace orthogonal to a propagation direction $\hat{\mathbf{k}}$ is three-dimensional. Consequently, there are three transverse branches. These modes correspond to physical shear waves of an elastic medium and are denoted by $\omega_{T,i}^2$.

For the purpose of diagnosing transverse isotropy and vacuum birefringence, we further decompose the transverse sector into its trace and traceless parts. The traceless transverse components correspond to the tensor representation under rotations about $\hat{\mathbf{k}}$. In a 4D elastic bulk, the transverse sector contains three degrees of freedom. While a massless relativistic graviton possesses only two helicity states, the geometric requirement for a Lorentz-invariant vacuum is that the substrate must not split these transverse modes based on polarization direction. Any lattice-induced splitting corresponds to vacuum birefringence. We therefore test for the degeneracy of these three branches.

We denote the eigenvalues of this transverse–traceless sector by $\omega_{\mathrm{TT},i}^2$. Operationally, these correspond to the eigenvalues of the dynamical matrix projected onto the transverse–traceless subspace orthogonal to $\hat{\mathbf{k}}$.

In an elastically isotropic medium, all transverse eigenvalues coincide and the distinction between ω_T^2 and ω_{TT}^2 becomes immaterial. In anisotropic lattices, however, polarization-dependent splitting of the $\omega_{\mathrm{TT},i}^2$ signals vacuum birefringence and incompatibility with Lorentz-invariant vacuum propagation. Even though a vector field corresponds to a spin-1 representation, the splitting of transverse modes according to polarization is a generic feature of anisotropic lattices that affects all distinct helicity states. We therefore use the splitting of the transverse phonon branches as a proxy for vacuum birefringence.

2.5. Numerical Procedures and Anisotropy Measures

To quantify deviations from isotropy, we evaluate the dynamical matrix for wave vectors of fixed magnitude $k = |\mathbf{k}|$ while sampling propagation directions $\hat{\mathbf{k}} = \mathbf{k}/k$ on a dense spherical grid covering the unit 3-sphere S^3 in momentum space.

We then define the following anisotropy measures:

Definition 2.1 (Relative Elastic Anisotropy δ_F)**.**

$$\delta_F = \frac{||\mathbf{C} - \mathbf{C}_{\mathrm{iso}}||_F}{||\mathbf{C}_{\mathrm{iso}}||_F},$$

where $\mathbf{C}_{\mathrm{iso}}$ is the isotropic tensor closest to the rank-4 stiffness tensor C_{ijkl} (minimizing the Frobenius distance). For a perfectly isotropic continuum, $\delta_F = 0$.

Definition 2.2 (Relative Dispersion Anisotropy)**.**

$$\delta \equiv \delta(\mathbf{k}) \equiv \frac{\Delta\omega^2}{\omega^2}.$$

Lattice	δ_F (exact)	δ_F (numerical)	Cubic Symmetry	Zener Ratio A (exact)
$\mathbb{Z}^4$	1	$1.000\ldots$	Perfect	0 (Anisotropic)
A_4	$1/\sqrt{10}$	$0.316\ldots$	Violated	Undefined (Anisotropic)
D_4	0	$< 10^{-16}$	Perfect	1 (Isotropic)

Table 2: Relative elastic anisotropies δ_F for selected lattices. The D_4 lattice is the only candidate that yields a physically stable and isotropic vacuum ($\delta_F = 0,\ A = 1$).

Definition 2.3 (Scalar (Longitudinal) Anisotropy)**.** Quantifies the directional dependence of the sound speed at fixed wavenumber magnitude k:

$$\delta_L \equiv \delta_L(k) \equiv \frac{\max_{\hat{\mathbf{k}}}(\omega_L^2) - \min_{\hat{\mathbf{k}}}(\omega_L^2)}{\langle \omega_L^2 \rangle_{\hat{\mathbf{k}}}},$$

where the angular brackets denote averaging over propagation directions $\hat{\mathbf{k}}$ at fixed k.

Definition 2.4 (Tensor Birefringence)**.** Quantifies the splitting of polarization states (vacuum birefringence) for the transverse sector:

$$\delta_{\mathrm{T}}(\mathbf{k}) \equiv \frac{\max_i \left(\omega_{\mathrm{TT},i}^2\right) - \min_i \left(\omega_{\mathrm{TT},i}^2\right)}{\left\langle \omega_{\mathrm{TT},i}^2 \right\rangle_i},$$

where the average in the denominator is taken over the transverse shear eigenvalues $\omega_{\mathrm{TT},i}^2$ of the dynamical matrix at fixed propagation direction $\hat{\mathbf{k}}$.

These definitions allow us to disentangle purely geometric anisotropy effects from interaction-specific details and to compare different lattice geometries on equal footing in the infrared regime.

3. Results

3.1. Quantitative Elastic Isotropy

We evaluated the elastic stiffness tensors analytically for the Hypercubic ($\mathbb{Z}^4$), Simplex (A_4), and 24-Cell (D_4) lattices, see Appendix B, and validated the results numerically, see Table 2.

3.1.1. Hypercubic Lattice ($\mathbb{Z}^4$) Despite possessing cubic symmetry, the $\mathbb{Z}^4$ lattice with nearest-neighbor central forces exhibits extreme anisotropy ($\delta_F = 1$). Specifically, we find a Zener ratio of $A = 0$. Physically, this indicates a vanishing shear modulus ($C_{44} = 0$) relative to the principal axes, rendering the lattice structurally unstable to shear modes (unless stabilized by next-nearest neighbor interactions, which would however not remove the anisotropy).

3.1.2. Simplex Lattice (A_4) The A_4 lattice forms a spherical 3-design. Our analysis reveals that this degree of symmetry is insufficient for elastic isotropy, which requires at least a spherical 4-design. We observe a substantial residual anisotropy of

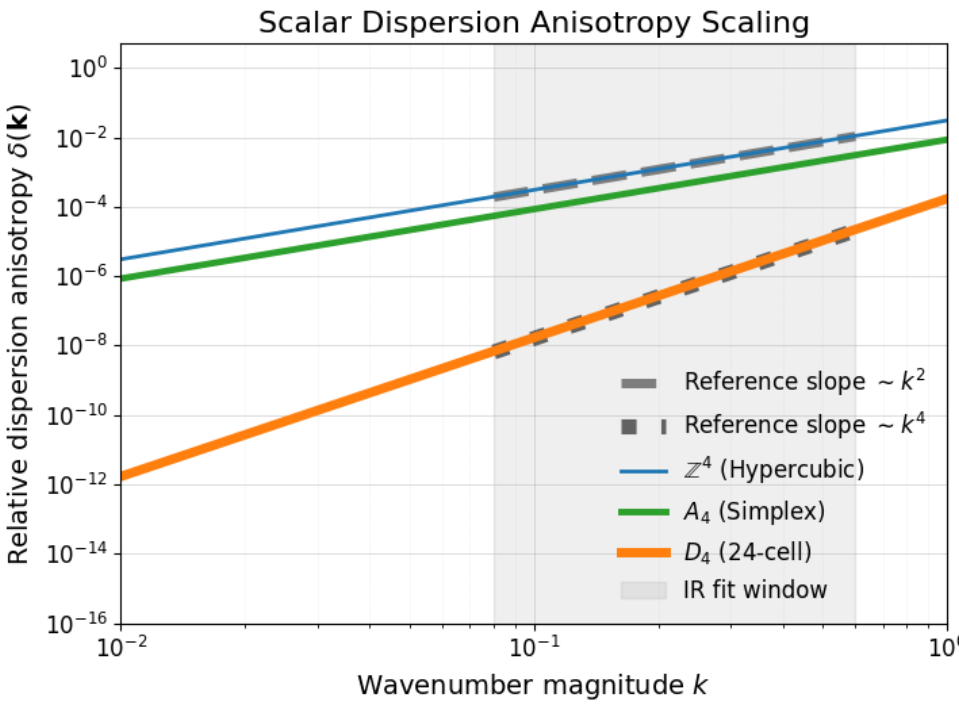

Figure 1: Scaling of the relative dispersion anisotropy $\delta = \Delta\omega^2/\omega^2$. Both the hypercubic ($\mathbb{Z}^4$, blue) and Simplex ($A_4$, green) lattices exhibit anisotropy scaling as $\delta \sim k^2$, confirming that a spherical 3-design is insufficient to suppress quartic lattice artifacts. In contrast, the D_4 lattice (orange) constitutes a spherical 5-design and suppresses these terms, leading to a much steeper $\delta \sim k^4$ decay.

$\delta_F = 1/\sqrt{10} \approx 0.316$. Furthermore, the stiffness tensor deviates significantly from cubic symmetry ($\approx 26\%$ violation), confirming that the standard Zener ratio is undefined for this geometry. This disqualifies the simplex geometry as a candidate for isotropic spacetime.

3.1.3. 24-Cell Lattice (D_4) In stark contrast, the D_4 lattice has full continuum isotropy. The stiffness tensor is perfectly cubic and satisfies the isotropic condition $A = 1$ exactly. Additionally, we confirm that the Lamé parameters satisfy $\lambda = \mu$, consistent with the Cauchy relations for central forces ($\nu = 0.25$) [13], but with the crucial property that this relation holds independently of direction.

3.2. Scalar Field Dispersion Anisotropy

The numerical evaluation of the relative dispersion anisotropy is shown in Figure 1.

For the hypercubic lattice ($\mathbb{Z}^4$), the anisotropy scales as $\delta \sim k^2$, confirming the presence of direction-dependent $\mathcal{O}(k^4)$ terms in the dispersion relation. In contrast, the 24-cell lattice (D_4) suppresses these contributions identically. The observed scaling follows $\delta \sim k^4$, pushing lattice artifacts into the deep ultraviolet. The residual anisotropy

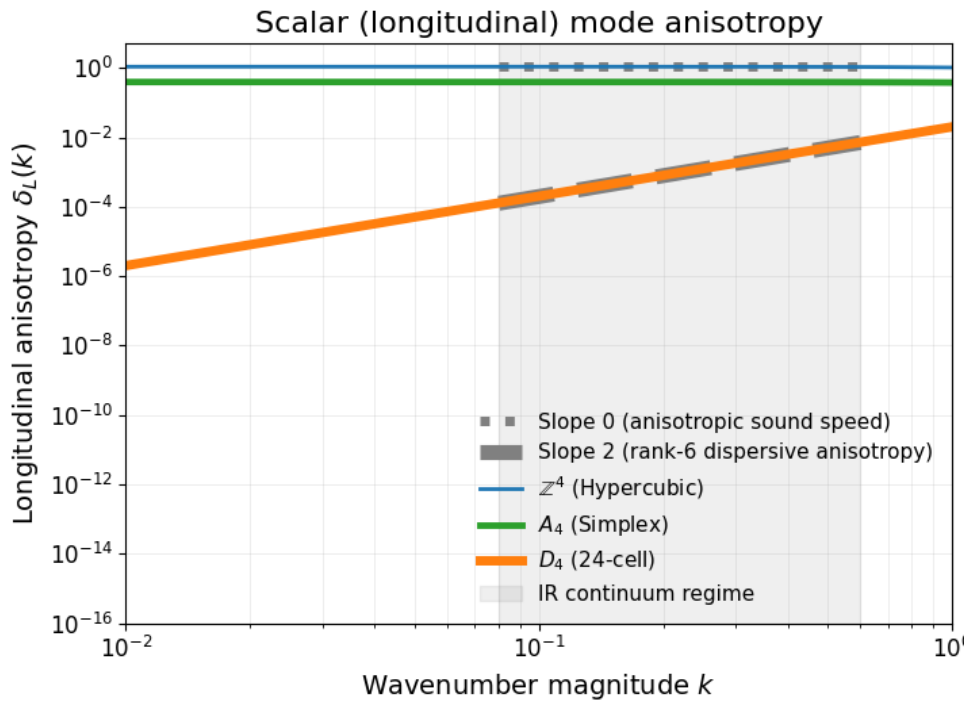

Figure 2: Scalar (longitudinal) mode anisotropy δ_{L}. Both the hypercubic ($\mathbb{Z}^4$, blue) and Simplex (A_4, green) lattices show order-unity anisotropy even in the infrared limit ($k \to 0$), implying direction-dependent sound speeds. The D_4 lattice (orange) is elastically isotropic in the IR limit, with residual anisotropy scaling as $\delta_L \sim k^2$ due to rank-6 lattice moments.

drops below 10^{-15} in the infrared, limited only by numerical precision.

3.3. Scalar (Longitudinal) Modes

We evaluate the directional dependence of the longitudinal eigenvalue ω_L^2 (see Figure 2).

The hypercubic lattice exhibits order-unity scalar anisotropy (slope ≈ 0), indicating a direction-dependent sound speed even at long wavelengths. The D_4 lattice suppresses scalar anisotropy effectively in the continuum regime. The residual scaling $\delta_L \sim k^2$ arises due to the tensor structure of the dynamical matrix D_{ij}. Although the leading dispersive term for scalars involves a rank-4 sum ($\sum v_i v_j v_k v_l$), the corresponding term for vector fields involves rank-6 moments ($\sum v_i v_j v_k v_l v_m v_n$) due to the contraction of two additional polarization indices. This rank-6 moment exceeds the smoothing capacity of the spherical 5-design, leading to a residual anisotropy of $\delta \sim k^2$ in the vector sector, compared to $\delta \sim k^4$ in the scalar sector.

3.4. Transverse Shear Modes and Birefringence

Even though scalar and longitudinal modes probe elastic compressibility, the transverse sector is decisive for emergent gravitational dynamics, as any polarization-dependent

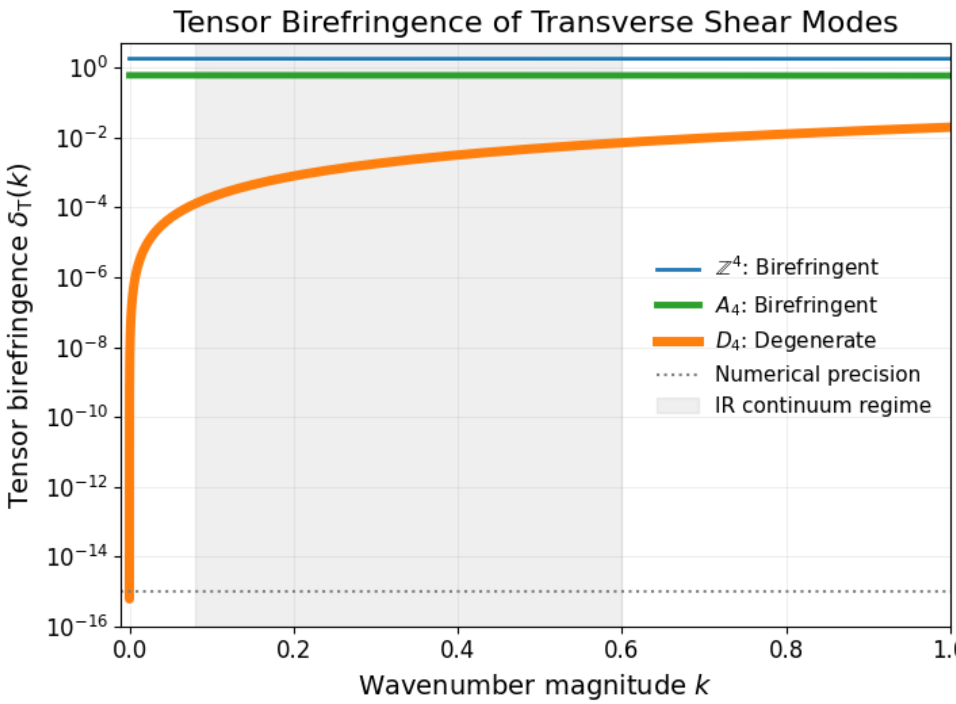

Figure 3: Tensor birefringence δ_{T}. Both the hypercubic ($\mathbb{Z}^4$, blue) and Simplex (A_4, green) lattices exhibit persistent polarization splitting (vacuum birefringence) across the full wavelength range. This demonstrates that the 3-design geometry of the Simplex lattice fails to secure an isotropic transverse sector. In contrast, the 24-cell (D_4) lattice (orange) shows complete degeneracy of the transverse shear eigenvalues in the infrared regime.

propagation directly manifests as vacuum birefringence.

As shown in Figure 3, the hypercubic lattice $\mathbb{Z}^4$ exhibits persistent splitting between transverse polarization states, indicating intrinsic lattice-induced birefringence. In contrast, the 24-cell (D_4) lattice displays degenerate transverse shear modes throughout the infrared regime, with any residual splitting suppressed below numerical precision and appearing only at $\mathcal{O}(k^6)$.

The resulting transverse shear waves in the D_4 lattice therefore propagate with a direction-independent phase velocity and without polarization-dependent splitting. The absence of transverse birefringence is a necessary kinematic condition for any emergent spin-2 field theory with a universal light cone. Although this condition alone does not ensure full diffeomorphism invariance, its violation would immediately signal Lorentz-violating effects in the infrared.

3.4.1. Leading-Order vs. Dispersive Isotropy in Vector Propagation For vector modes, the 24-cell lattice eliminates anisotropy in the leading-order propagation velocity, which is governed by the rank-4 elastic stiffness tensor. Residual anisotropy enters only at subleading dispersive order through rank-6 lattice moments, resulting in a relative anisotropy scaling as $\delta \sim k^2$. Consequently, isotropy of the leading-order propagation velocity is exact in the continuum limit. However, for vector modes, dispersive corrections at order $\mathcal{O}(k^4)$ in the energy remain anisotropic due to rank-6 lattice moments. This implies that while vacuum birefringence is absent at the level of phase velocities, anisotropic group-velocity dispersion re-emerges at subleading order.

4. Discussion

4.1. Geometric Suppression of LIV

From a geometric perspective, our results establish a clear hierarchy among regular four-dimensional lattices. Simple hypercubic discretizations are generically disfavored, as they introduce anisotropies already at leading nontrivial order. By contrast, the 24-cell lattice possesses intrinsic geometric symmetries (spherical 5-design) that naturally suppress kinematic Lorentz-violating effects. Crucially, our analysis of the A_4 lattice demonstrates that a spherical 3-design, while sufficient for isotropic diffusion (rank-2), fails for elastic stress (rank-4). Thus, the 5-design geometry of the 24-cell is not merely better, but mathematically necessary for consistent elastic kinematics.

4.2. Relation to Elastic and Analog Gravity

The exact degeneracy of the transverse shear sector suggests that the D_4 lattice is a natural candidate substrate for discrete elastic spacetime models. In such models, gravitational degrees of freedom emerge from stresses in a four-dimensional elastic medium. Our work provides a geometric foundation for such approaches by identifying the unique regular mono-lattice compatible with isotropic transverse propagation in the infrared.

In this sense, the present work does not propose an alternative theory of gravity, but isolates a necessary geometric consistency condition that any discrete spacetime model must satisfy before dynamical considerations become meaningful.

4.3. Limitations: Scalar Modes and Dynamics

Although the 24-cell geometry successfully eliminates directional anisotropy, two significant physical challenges remain:

1. Scalar Modes: For harmonic central forces, the Cauchy relations enforce a fixed Poisson ratio $\nu = 1/4$. This implies the presence of a propagating scalar mode. While the D_4 geometry ensures this mode is isotropic ($A = 1$), its elimination from the low-energy spectrum requires a dynamical mechanism (such as a bulk incompressibility constraint) which goes beyond the geometric scope of this paper. Moreover, in standard elasticity, the longitudinal speed c_L differs from the transverse speed c_T. Since Lorentz invariance requires a universal limiting velocity for all massless fields, a physical model of gravity would require the longitudinal mode to be either non-propagating or effectively decoupled. Importantly, on the $\mathbb{Z}^4$ lattice, even if one could tune parameters to match speeds, the speeds would remain direction-dependent. The D_4 lattice removes this directional obstacle, reducing the problem of Lorentz invariance to a dynamical one (tuning elastic moduli or imposing constraints) rather than a structural one. Therefore, the D_4 lattice solves the problem of directional dependence (isotropy), leaving the

problem of relative mode velocities (c_L vs c_T) to be solved by the specific choice of Hamiltonian or gauge fixing.

2. Euclidean vs. Lorentzian Symmetry: Our results demonstrate isotropy in the Euclidean domain. However, Lorentz invariance requires more than Euclidean isotropy; it requires a well-defined notion of causal structure and boost invariance after Wick rotation. The present results do not address the dynamical emergence of a time direction. Euclidean isotropy is a necessary geometric precursor—preventing preferred-frame effects in the dispersion relation—but it is not a sufficient condition for full Lorentz invariance.

3. UV Breakdown: It should be noted that isotropy is recovered only in the long-wavelength limit ($k \to 0$). Near the Brillouin zone boundary (UV scale), the discrete lattice structure dominates, and rotational symmetry is broken. This is a generic feature of any discrete regulator.

5. Conclusion

We have demonstrated that the four-dimensional 24-cell lattice exhibits exceptional isotropy properties absent in hypercubic geometries.

1. Spectral Isotropy: The scalar dispersion relation remains isotropic up to $\mathcal{O}(k^6)$, implying $\delta(\mathbf{k}) \sim \mathcal{O}(k^4)$.
2. Elastic Isotropy: The stiffness tensor satisfies the condition for an isotropic continuum ($A = 1$) exactly. (Note: The relation $A = 1$ implies $\lambda = \mu$, which is specific to the central-force model. However, the vanishing of angular anisotropy $\delta_F \to 0$ is a generic geometric property of the D_4 lattice, independent of the interaction potential.)
3. Transverse Consistency: Shear modes propagate without vacuum birefringence.

These findings motivate the 24-cell geometry not merely as a mathematical curiosity, but as the physically well-motivated structure for any discrete theory of spacetime seeking to recover perfectly isotropic wave propagation at the level of infrared kinematics.

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Ethical Statement

This research does not involve human participants, animals, or any personal data.

Conflict of Interest

The author declares no competing financial or non-financial interests.

Data Availability

All data generated or analyzed during this study are included in this published article and its supplementary material. The source code used to generate the results and figures is publicly available at the project repository https://github.com/kodonian/paper-2025-continuum-isotropy-public.

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A. Geometry of the 24-Cell and the D_4 Root Lattice

The 24-cell (also known as the icositetrachoron) is the unique regular convex polytope in four dimensions without a three-dimensional analog. Its exceptional symmetry properties are central to the isotropy results presented in this work.

A.1. Definition and coordinate realization

The vertex set of the 24-cell coincides with the root system of the Lie algebra D_4. In an orthonormal basis of $\mathbb{R}^4$, the 24 vertices can be written as

$$\mathcal{V}_{24} = \{(\pm 1, \pm 1, 0, 0) \mid \text{all permutations}\} .$$

Each vertex has squared norm $|\mathbf{v}|^2 = 2$, and the full set consists of all permutations of two nonzero entries with independent signs. This representation makes explicit the equivalence between the 24-cell and the D_4 root system.

The lattice generated by integer linear combinations of these roots with even coordinate sum defines the D_4 lattice,

$$D_4 = \left\{ \mathbf{x} \in \mathbb{Z}^4 \;\middle|\; \sum_{i=1}^{4} x_i \equiv 0 \pmod 2 \right\},$$

whose nearest-neighbor shell is precisely given by $\mathcal{V}_{24}$.

A.2. Symmetry group

The full symmetry group of the 24-cell is the Coxeter group F_4, containing 1152 elements. Crucially, F_4 serves as the local point group of the infinite D_4 lattice, ensuring that the high-order isotropy of the single cell translates directly into the macroscopic symmetries of the bulk medium. This group acts transitively on the vertex set and contains the Weyl group of D_4 as a subgroup. The high symmetry of F_4 forbids the appearance of low-order anisotropic invariants in tensor sums over the lattice.

In contrast to the hypercubic lattice $\mathbb{Z}^4$, whose point group B_4 admits anisotropic tensor structures at quartic order, the F_4 symmetry enforces rotational invariance for all tensor moments relevant to linear elasticity and low-energy dispersion.

A.3. Spherical 5-design property

A finite set of points $\{\mathbf{v}_i\}$ on the unit sphere S^3 constitutes a spherical t-design if the average of any polynomial of degree $d \leq t$ over the discrete set coincides exactly with its average over the continuous sphere.

The normalized vertices of the 24-cell, $\hat{\mathbf{v}}_i = \mathbf{v}_i/\sqrt{2}$, form a spherical 5-design [12]. This property ensures that discrete lattice sums mimic continuous rotational integration up to fifth order:

$$\frac{1}{24} \sum_{i=1}^{24} P(\hat{\mathbf{v}}_i) = \frac{1}{\mathrm{Vol}(S^3)} \int_{S^3} P(\mathbf{x})\, d\Omega.$$

This algebraic equivalence has profound consequences for the tensor structure of the lattice. Because the averaging holds for $d \leq 5$, all tensor moments of rank 2 and 4 are forced to adopt their unique isotropic forms (combinations of Kronecker deltas). Consequently, any anisotropic lattice invariants that typically plague lower-symmetry grids—such as the hypercubic sum $\sum v_i^4 \neq 3 \sum v_i^2 v_j^2$—vanish identically by virtue of the vertex geometry.

A.4. Implications for continuum isotropy

This 5-design property provides the rigorous geometric origin for the physical results reported in this work. Since the physics of scalar dispersion and linear elasticity is governed by second- and fourth-order gradients, the lattice geometry effectively "hides" its discreteness from these fields in the long-wavelength limit.

Specifically, the exact cancellation of fourth-order anisotropic moments guarantees that the rank-4 elastic stiffness tensor satisfies the continuum isotropy condition (Zener ratio $A = 1$) without the need for parameter fine-tuning. For scalar fields, this symmetry pushes the first direction-dependent dispersive corrections to $\mathcal{O}(k^6)$, leaving the leading-order propagation perfectly isotropic. In the vector sector, it ensures the degeneracy of transverse shear modes, thereby eliminating vacuum birefringence as a geometric artifact. These features are not accidental results of interaction parameters but intrinsic consequences of the F_4 symmetry group acting on the 24-cell.

A.5. Uniqueness in four dimensions

The 24-cell occupies a singular position among regular polytopes. It is the unique regular mono-lattice in four-dimensional Euclidean space that simultaneously maximizes the kissing number ($N = 24$) and satisfies the spherical 5-design condition. No other regular tiling of $\mathbb{R}^4$ possesses sufficient symmetry to enforce elastic isotropy under central nearest-neighbor forces. This geometric uniqueness identifies the D_4 lattice not merely as one of many options, but as the distinct structural candidate for realizing isotropic emergent kinematics in four dimensions.

B. Analytic Evaluation of Relative Elastic Anisotropy

We analytically evaluate the relative elastic anisotropy δ_F for the three four-dimensional lattices considered in the main text: the hypercubic lattice $\mathbb{Z}^4$, the simplex lattice A_4, and the 24-cell lattice D_4.

The purpose is twofold: (1) To demonstrate that the numerical results reported in the main text follow directly from symmetry considerations, and (2) to show explicitly how the degree of spherical design determines elastic isotropy at the level of the rank-4 stiffness tensor.

B.1. General Framework

For a nearest-neighbor central-force model, the elastic stiffness tensor is given by

$$C_{ijkl} \propto \sum_{\mathbf{v}} v_i v_j v_k v_l,$$

where the sum runs over all normalized nearest-neighbor vectors $\mathbf{v}$ of the lattice.

The condition for complete elastic isotropy in any dimension requires that the fourth-order moments satisfy the specific ratio:

$$\sum_{\mathbf{v}} v_i^4 = 3 \sum_{\mathbf{v}} v_i^2 v_j^2 \quad (i \neq j).$$

This condition is both necessary and sufficient for the rank-4 tensor sum to be proportional to the unique isotropic fourth-order invariant $\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$. Any deviation from this factor of 3 implies elastic anisotropy ($A \neq 1$).

To quantify this deviation independently of coordinate choices, we employ the relative Frobenius anisotropy measure

$$\delta_F = \frac{\|C - C^{\text{iso}}\|_F}{\|C^{\text{iso}}\|_F}.$$

B.2. Hypercubic Lattice $\mathbb{Z}^4$

The nearest neighbors of the hypercubic lattice are the vectors

$$\mathbf{v} = \pm \mathbf{e}_\mu, \quad \mu = 1, \ldots, 4,$$

normalized to unit length.

The relevant fourth moments are straightforward to compute:

$$\sum_{\mathbf{v}} v_i^4 = 2, \quad \sum_{\mathbf{v}} v_i^2 v_j^2 = 0 \quad (i \neq j).$$

Since the off-diagonal moments vanish, the isotropy condition ($2 = 3 \times 0$) is maximally violated. The stiffness tensor has nonzero components only along the principal axes. Computing the Frobenius distance to the closest isotropic projection yields the exact result:

$$\delta_F(\mathbb{Z}^4) = 1.$$

B.3. Simplex Lattice A_4

The A_4 lattice is most conveniently analyzed using its embedding in $\mathbb{R}^5$, restricted to the hyperplane $\sum_{i=1}^{5} x_i = 0$. The $N = 20$ nearest neighbors are given by the permutations of

$$\mathbf{u} = (1, -1, 0, 0, 0),$$

normalized as $\mathbf{v} = \mathbf{u}/\sqrt{2}$.

To check for isotropy, we calculate the moments in the embedding coordinates:

1. Diagonal moments ($\sum v_i^4$): For a fixed index i, the coordinate u_i is non-zero (± 1) only for vectors where ± 1 is at position i. There are 4 choices for the position of the matching ∓ 1, and 2 sign combinations, yielding 8 vectors. Taking normalization into account:
$$\sum_{\mathbf{v}} v_i^4 = 8 \times \left(\frac{1}{\sqrt{2}}\right)^4 = 2.$$

2. Off-diagonal moments ($\sum v_i^2 v_j^2$): For fixed distinct indices i and j, only 2 vectors have non-zero entries at both positions: $(..., 1_i, ..., -1_j, ...)$ and $(..., -1_i, ..., 1_j, ...)$.
$$\sum_{\mathbf{v}} v_i^2 v_j^2 = 2 \times \left(\frac{1}{\sqrt{2}}\right)^2 \left(\frac{1}{\sqrt{2}}\right)^2 = 0.5.$$

The Isotropy Test: We compare the ratio of the moments:

$$\frac{\sum v_i^4}{\sum v_i^2 v_j^2} = \frac{2}{0.5} = 4.$$

Since elastic isotropy strictly requires a ratio of 3, the A_4 lattice is intrinsically anisotropic. The "excess" kurtosis reflects the underlying cubic symmetry of the 5D embedding which survives the projection to 4D.

The deviation from isotropy is determined by the discrepancy $\Delta = \sum v_i^4 - 3 \sum v_i^2 v_j^2$. Substituting the values $\mu_4 = 2$ and $\mu_{22} = 0.5$, we find a violation of $\Delta = 2 - 3(0.5) = 0.5$.

Using the projection onto the isotropic subspace for this symmetry class, the relative anisotropy is given by the ratio:

$$\delta_F = \frac{|\mu_4 - 3\mu_{22}|}{\sqrt{10}\mu_{22}}.$$

Substituting our moments:

$$\delta_F(A_4) = \frac{|2 - 1.5|}{\sqrt{10} \times 0.5} = \frac{0.5}{0.5\sqrt{10}} = \frac{1}{\sqrt{10}}.$$

This leads to the analytic value:

$$\delta_F(A_4) = \frac{1}{\sqrt{10}} \approx 0.316.$$

B.4. 24-cell Lattice D_4

The D_4 lattice has $N = 24$ nearest neighbors given by all permutations of $\mathbf{v} = (\pm 1, \pm 1, 0, 0)$, normalized to unit length $(1/\sqrt{2})$.

Due to the spherical 5-design property of the 24-cell vertices, all fourth-order tensor moments are guaranteed to be isotropic. The following explicit count merely confirms this geometric fact:

1. Diagonal moments: For fixed i, there are $N/2 = 12$ vectors with non-zero entries.
$$\sum v_i^4 = 12 \times (1/\sqrt{2})^4 = 3.$$
2. Off-diagonal moments: Fix distinct indices i, j. The pattern $(\dots, \pm 1_i, \dots, \pm 1_j, \dots)$ with zeros elsewhere occurs in exactly 4 vectors (the sign permutations of the two nonzero entries).
$$\sum v_i^2 v_j^2 = 4 \times (1/\sqrt{2})^2 (1/\sqrt{2})^2 = 1.$$

The Isotropy Test:
$$\frac{\sum v_i^4}{\sum v_i^2 v_j^2} = \frac{3}{1} = 3.$$
The ratio is exactly 3.

This satisfies the condition for continuum isotropy perfectly. Consequently:
$$\delta_F(D_4) = 0.$$

Lemma B.1 (Lemma (Minimal Design Order for Elastic Isotropy))**.** *Let $X \subset S^{d-1}$ be the set of normalized nearest-neighbor directions of a lattice.*

The rank-4 elastic stiffness tensor
$$C_{ijkl} \propto \sum_{\mathbf{v} \in X} v_i v_j v_k v_l$$
is isotropic if and only if X forms a spherical t-design with $t \geq 4$.

In particular, spherical 3-designs are insufficient to guarantee elastic isotropy, while any spherical t-design with $t \geq 4$ satisfies the isotropy condition.

Proof. This follows directly from the defining property of spherical designs, since the tensor sum involves a homogeneous polynomial of degree 4 $(v_i v_j v_k v_l)$. A 3-design only guarantees isotropy for polynomials up to degree 3, leaving fourth-order terms unconstrained. □

B.5. Summary

The analytic evaluation of the stiffness tensor moments confirms the hierarchy of geometries:

- $\mathbb{Z}^4$: Ratio ∞ (undefined off-diagonals), $\delta_F = 1$.
- A_4: Ratio 4, $\delta_F = 1/\sqrt{10}$.
- D_4: Ratio 3, $\delta_F = 0$.

This proves that the numerical findings in the main text are exact consequences of the spherical design order of the lattice geometries. Elastic isotropy in four dimensions requires at least a spherical 4-design. Among regular 4D lattices, only the 24-cell satisfies this condition.