

Heterotic-Orbifold Embedding of the e-step Yukawa Ladder

(MSSM language; phenomenological target textures)

Jarosław Kaczorowski

ORCID: 0009-0000-7461-1179

This note provides a compact bridge between (i) heterotic orbifold model-building mechanisms for Yukawa hierarchies and (ii) the e-step / N-style numerical ladder relations for quark and lepton masses. The key idea is to interpret the e-step ladder as a concrete target for MSSM Yukawa textures, with a single step parameter $\varepsilon \equiv e^{-3}$ arising naturally from orbifold selection rules, higher-dimensional operators, and/or worldsheet (geometric) suppressions.

Scope.

We aim for a minimal, model-independent embedding: we do not claim a complete UV derivation of all SM parameters from a single compactification. Instead, we show how standard orbifold mechanisms can generate exactly the kind of discrete, exponent-controlled hierarchies that the e-step ladder postulates.

1. MSSM mass-Yukawa dictionary and $\tan\beta$ dependence

In the MSSM, two Higgs doublets H_u and H_d generate masses via $v_u \equiv \langle H_u^0 \rangle = v \sin\beta$ and $v_d \equiv \langle H_d^0 \rangle = v \cos\beta$, with $v \approx 246$ GeV. At a fixed renormalisation scale μ (and scheme), running masses relate to Yukawas as:

$$\begin{aligned} m_u(\mu) &= (v \sin\beta / \sqrt{2}) \cdot y_u(\mu), & m_d(\mu) &= (v \cos\beta / \sqrt{2}) \cdot y_d(\mu), \\ m_\ell(\mu) &= (v \cos\beta / \sqrt{2}) \cdot y_\ell(\mu) \quad (\text{charged leptons couple to } H_d). \end{aligned}$$

Down-type ratios cancel $\tan\beta$, while up-versus-down relations introduce factors of $\cot\beta$. This bookkeeping is essential when translating cross-sector numerical mass relations into Yukawa statements.

2. The e-step ladder as a target for Yukawa textures

The e-step framework fixes a strange-quark node using a lepton-anchored identity and builds the remaining quark masses via discrete steps. In compact form:

$$\begin{aligned} s &\equiv (m_e + m_\mu + m_\tau) / e^3, \\ d &= s \cdot e^{-3}, \quad u = 9 \cdot s \cdot e^{-6}, \\ c &= (2/3) \cdot s \cdot e^{+3}, \quad b = (1/9) \cdot s \cdot e^{+6}, \\ t &= b \cdot (2 e^3 + 1). \end{aligned}$$

Define the single step parameter $\varepsilon \equiv e^{-3} \approx 0.0498$. Then the ladder implies specific Yukawa texture constraints.

For same-Higgs down-type relations:

$$\begin{aligned} y_d / y_s &= m_d / m_s = \varepsilon, \\ y_b / y_s &= m_b / m_s = (1/9) \cdot \varepsilon^{-2}. \end{aligned}$$

For cross-Higgs relations (up vs down), MSSM introduces $\cot\beta$:

$$\begin{aligned} y_u / y_s &= (m_u / m_s) \cdot \cot\beta = 9 \cdot \varepsilon^2 \cdot \cot\beta, \\ y_c / y_s &= (m_c / m_s) \cdot \cot\beta = (2/3) \cdot \varepsilon^{-1} \cdot \cot\beta, \\ y_t / y_b &= (m_t / m_b) \cdot \cot\beta = (2 e^3 + 1) \cdot \cot\beta. \end{aligned}$$

The numerical prefactors (9, 2/3, 1/9, ...) are naturally interpreted as $O(1)$ coefficients from mixing, normalisation (Kähler metrics), or group-theoretic factors.

Scale/scheme caveat. Light-quark masses (u,d,s) are conventionally quoted as running \overline{MS} quantities, whereas charged leptons are usually treated as pole

masses. Percent-level mismatches are therefore expected unless all relations are evaluated consistently at a common μ and scheme.

3. Why heterotic orbifolds naturally generate stepwise hierarchies

In heterotic orbifold compactifications (e.g. T^6/Z_{6-II}), the 4D superpotential contains MSSM Yukawa terms:

$$W \supset (y_u)_{ij} Q_i U_j^c H_u + (y_d)_{ij} Q_i D_j^c H_d + (y_e)_{ij} L_i E_j^c H_d.$$

Orbifold selection rules (gauge invariance plus space-group constraints and related discrete symmetries) often forbid many renormalisable couplings. When a renormalisable entry is forbidden, the leading contribution arises from higher-dimensional operators involving singlets S_a that acquire VEVs:

$$W \supset (C_{ij} / M^{n_{ij}}) \cdot (Q_i U_j^c H_u) \cdot \prod_a S_a^{n_{ij}^{(a)}} \\ \Rightarrow (y_u)_{ij}^{\text{eff}} \sim C_{ij} \cdot \prod_a (\langle S_a \rangle / M)^{n_{ij}^{(a)}}, \text{ with integer exponents } n_{ij}^{(a)} \geq 0.$$

For twisted states localised at different fixed points, further geometric suppressions can appear (schematically):

$$(y)_{ij} \sim C_{ij} \cdot \exp(-A_{ij}) \cdot \prod_a (\langle S_a \rangle / M)^{n_{ij}^{(a)}}.$$

Thus, heterotic orbifolds provide exactly the kind of integer-controlled exponent structure needed for an e-step target: discrete charge data and localisation control the powers n_{ij} and/or actions A_{ij} , while coefficients C_{ij} remain $O(1)$.

4. Minimal identification of the e-step spurion

A minimal embedding identifies a single dominant suppression parameter with the e-step step:

$$\varepsilon \equiv e^{-3} \leftrightarrow \langle S \rangle / M \quad (\text{Froggatt-Nielsen-like}), \\ \text{and/or} \\ \varepsilon \leftrightarrow \exp(-A_0) \quad (\text{worldsheet / geometric suppression}).$$

Under this identification, the ladder becomes a concrete target for the eigenvalues (or dominant entries) of the MSSM Yukawa matrices, with $\tan\beta$ controlling the standard MSSM conversion between up- and down-type sectors.

Appendix A. Compact notation

We use $\varepsilon \equiv e^{-3}$ and interpret the e-step relations as statements about Yukawa hierarchies at a chosen scale μ . All $O(1)$ prefactors (including 9, 2/3, 1/9) are collected into coefficients C_{ij} that may originate from field normalisations, mixing, or group-theoretic Clebsch factors in a given UV completion.

Appendix B. Toy FN charge assignment (illustrative)

This appendix provides a minimal Froggatt-Nielsen (FN) toy model that reproduces the e-step-style exponent pattern at the level of diagonal Yukawa suppressions. It is not meant to be unique nor a full UV-complete construction; it simply demonstrates that a single $U(1)_X$ symmetry plus one flavon can generate the required integer exponents.

B.1 Setup

Assume an additional abelian flavour symmetry $U(1)_X$ and one flavon superfield S with charge $X(S)=-1$. Take $X(H_u)=X(H_d)=0$ for simplicity. Effective Yukawa operators then scale as

$$\begin{aligned}(y_u)_{ij} &\sim C_{ij}^u \cdot \varepsilon^{X(Q_i)+X(U_j^c)}, \\(y_d)_{ij} &\sim C_{ij}^d \cdot \varepsilon^{X(Q_i)+X(D_j^c)}, \\(y_e)_{ij} &\sim C_{ij}^e \cdot \varepsilon^{X(L_i)+X(E_j^c)},\end{aligned}$$

where $\varepsilon \equiv \langle S \rangle / M \approx e^{-3}$.

Integer exponents follow from integer charges; in orbifold compactifications, such integer data can arise from discrete symmetries, selection rules, and localisation assignments.

B.2 Example charge choice matching the e-step diagonal exponents

Choose diagonal suppression exponents (in units of ε) as follows:

$$\begin{aligned}\text{Up-type: } &n_t=0, \quad n_c=1, \quad n_u=4. \\ \text{Down-type: } &n_b=0, \quad n_s=2, \quad n_d=3.\end{aligned}$$

One simple charge assignment that realises these diagonal exponents is:

Field	X charge
Q1, Q2, Q3	2, 1, 0

$U1^c, U2^c, U3^c$	2, 0, 0
$D1^c, D2^c, D3^c$	1, 1, 0
H_u, H_d	0, 0
S	-1

With these charges, the diagonal exponents are:

$$n_u = X(Q1) + X(U1^c) = 2 + 2 = 4, \quad n_c = X(Q2) + X(U2^c) = 1 + 0 = 1, \quad n_t = X(Q3) + X(U3^c) = 0 + 0 = 0.$$

$$n_d = X(Q1) + X(D1^c) = 2 + 1 = 3, \quad n_s = X(Q2) + X(D2^c) = 1 + 1 = 2, \quad n_b = X(Q3) + X(D3^c) = 0 + 0 = 0.$$

Therefore y_t and y_b can be renormalisable (unsuppressed), while y_s, y_d, y_c, y_u arise with controlled powers of ϵ . The e-step prefactors (9, 2/3, 1/9) are then absorbed into the $O(1)$ coefficients C_{ij} , which may encode mixing, normalisation, or additional geometric effects.

B.3 Remarks on mixing and texture building

Off-diagonal entries follow from the same charge sums $X(Q_i) + X(U_j^c)$ and $X(Q_i) + X(D_j^c)$. This naturally yields hierarchical mixing. A realistic CKM pattern typically requires (i) allowing selected off-diagonal entries, (ii) including multiple singlets S_a or modular/geometric effects, and (iii) taking into account Kähler metric normalisation. The present toy assignment merely demonstrates that an integer-exponent ladder compatible with $\epsilon \approx e^{-3}$ is straightforward to realise in an FN-like effective description.

Numerical Check: e-step Ladder Masses vs. PDG 2025

This page adds a compact numerical cross-check of the e-step ladder defined earlier in the note. Lepton masses are treated as inputs; quark masses are then computed from the ladder and compared to Particle Data Group (PDG) 2025 summary values.

Particle	Predicted	PDG 2025	Δ (Pred-Exp)	$\Delta\%$
e (input)	0.510999 MeV	0.510999 MeV	+0.000000 MeV	+0.000%
μ (input)	105.658376 MeV	105.658376 MeV	+0.000000 MeV	+0.000%
τ (input)	1 776.930000 MeV	1 776.930000 MeV	+0.000000 MeV	+0.000%
u	2.091536 MeV	2.160000 MeV	-0.068464 MeV	-3.170%
d	4.667737 MeV	4.700000 MeV	-0.032263 MeV	-0.686%
s	93.753997 MeV	93.500000 MeV	+0.253997 MeV	+0.272%
c	1.255400 GeV	1.273000 GeV	-0.017600 GeV	-1.383%
b	4.202562 GeV	4.183000 GeV	+0.019562 GeV	+0.468%
t	173.024009 GeV	172.560000 GeV	+0.464009 GeV	+0.269%

Conventions and caveats. The PDG values for u, d, s are MSbar masses at $\mu = 2$ GeV; c and b are MSbar running masses at $\mu = m$; the quoted t value is the direct-measurement mass extracted from event kinematics. Light-quark masses are scheme/scale dependent; the percent differences are intended only as a quick numerical sanity check within these conventions.

Ladder inputs used here. $m_e = 0.510998950690$ MeV (CODATA 2022 in your framework), $m_\mu = 105.6583755$ MeV (PDG 2025), $m_\tau = 1776.93$ MeV (PDG 2025), $e = 2.718281828$.

Derived knot and quark masses. $s = (m_e + m_\mu + m_\tau)/e^3 = 93.753997$ MeV; $d = s \cdot e^{-3} = 4.667737$ MeV; $u = 9 \cdot s \cdot e^{-6} = 2.091536$ MeV; $c = (2/3) \cdot s \cdot e^3 = 1.255400$ GeV; $b = (1/9) \cdot s \cdot e^6 = 4.202562$ GeV; $t = b \cdot (2 \cdot e^3 + 1) = 173.024009$ GeV.