Technical Appendix: Heat Kernel, Fermions, and the Sign of Induced Gravity

(Sign conventions fixed; Laplacian/Lichnerowicz hinge made explicit)

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July 4, 2026 (v2; v1: December 2025) ai.viXra:2512.0091

Scope note. This appendix is a self-contained heat-kernel computation of the sign of the induced Einstein–Hilbert term from integrating out matter fields. It is *topically adjacent* to the coherence- maintenance / "resource boundary" program (the induced-gravity sign is a curvature-sector bookkeeping that the program refers to heuristically), but it is *not* load-bearing for any result in that series and depends on none of it; the cross-references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] are for context only. It does not attempt a UV completion nor a physical determination of G_{phys}. **v2**: the physics is unchanged the coefficient/sign results are unchanged and correct; this revision clarifies the scalar examples (non-minimal $\xi = \frac{1}{4}$ vs conformal $\xi = \frac{1}{6}$, for which $A_1^{(\text{eff})} = 0$), updates companion references to their v2 revisions, and adds a symbolic verification of the full coefficient table (Appendix A).

Abstract

We derive the local contribution proportional to the scalar curvature R in the Euclidean one-loop effective action obtained by integrating out matter fields on a curved background. Using a Schwinger cutoff $\varepsilon = \Lambda^{-2}$ and the Seeley–DeWitt coefficient a_1, we extract the quadratically divergent term multiplying $\int d^4x\sqrt{g}\,R$. We fix a single Euclidean convention for the Einstein–Hilbert action, state an explicit Laplacian convention, and write the Lichnerowicz/Weitzenböck identity in a sign-robust form so that the fermionic contribution is unambiguous. We provide a unified bookkeeping coefficient $A_1^{(\text{eff})}$ such that

$$W_R^{\text{total}} = -\frac{A_1^{(\text{eff})}}{32\pi^2}\Lambda^2 \int d^4x\sqrt{g}\,R,$$

and hence an induced Newton constant G_{ind} via comparison with the Euclidean Einstein–Hilbert action. We include the minimal gauge+ghost package in background Feynman gauge, a species table, and a reproducible symbolic check of every entry.

1 Conventions and notation

1.1 Lorentzian geometry and curvature sign

Lorentzian signature $(-,+,+,+)$ and the Wald/MTW convention

$$R^a{}_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{bc}, \quad R_{bd} = R^a{}_{bad}, \quad R = g^{bd}R_{bd}.$$

1.2 Euclidean formulation and Einstein–Hilbert action

Heat-kernel calculations are done in $D = 4$ Euclidean signature. We fix

$$S_E^{\text{EH}} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\,R, \qquad S_L = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R,$$

with $S_E = -iS_L$ when topological/eta phases are irrelevant for the local divergent coefficient extracted here.

1.3 Laplacian, Laplace-type operators, Schwinger cutoff

$\nabla^2 \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$, so $-\nabla^2$ is positive; a Laplace-type operator is $P = -\nabla^2 + X(x)$ with X a local endomorphism. Schwinger proper-time with $\varepsilon = \Lambda^{-2}$:

$$\ln\det P = -\int_\varepsilon^\infty \frac{ds}{s}\,\mathrm{Tr}\,e^{-sP}.$$

1.4 Fermionic hinge: Lichnerowicz/Weitzenböck

$D_E \equiv \gamma^\mu\nabla_\mu$ with Euclidean $(\gamma^\mu)^\dagger = \gamma^\mu$ and $(D_E)^\dagger = -D_E$; $[\nabla_\mu,\nabla_\nu]\psi = \frac{1}{4}R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\psi$, $\gamma^{\rho\sigma} = \frac{1}{2}[\gamma^\rho,\gamma^\sigma]$. In sign-robust form,

$$-D_E^2 = -\nabla^2 + \tfrac{1}{4}R.$$

Here ∇^2 is the connection Laplacian on spinors (not the de Rham Laplacian on forms).

2 Heat kernel expansion and a_1

With $K(s;x,y) = \langle x|e^{-sP}|y\rangle$,

$$\mathrm{Tr}\,e^{-sP} \sim \frac{1}{(4\pi s)^2}\int d^4x\sqrt{g}\big(a_0 + a_1 s + a_2 s^2 + \cdots\big), \quad a_0 = \mathrm{tr}\,\mathbf{1}, \;\; a_1 = \mathrm{tr}\big(\tfrac{1}{6}R - X\big).$$

For a real scalar with $X = m^2 + \xi R$: $a_{1,R} = \tilde a_1 R$, $\tilde a_1 = \frac{1}{6} - \xi$; the $-m^2$ piece renormalizes the cosmological term, not $\int\sqrt{g}\,R$.

3 Scalars: extraction of $\int\sqrt{g}\,R$

$W_{\mathrm{scalar}} = \frac{1}{2}\ln\det P = -\frac{1}{2}\int_\varepsilon^\infty \frac{ds}{s}\mathrm{Tr}\,e^{-sP}$; using $\int_\varepsilon^\infty ds\,s^{-2} = \Lambda^2$,

$$W_R^{\mathrm{scalar}} = -\frac{\tilde a_1}{32\pi^2}\Lambda^2\int d^4x\sqrt{g}\,R.$$

4 Dirac fermions: reduction to Laplace type

$S_F = \int d^4x\sqrt{g}\,\bar\psi(D_E{+}m)\psi$, $W_{\mathrm{ferm}} = -\ln\det(D_E{+}m)$. With $\ln\det(D_E{+}m) = \frac{1}{2}\ln\det[(D_E{+}m)^\dagger(D_E+m)] + i\Theta$ and $P_D := (D_E+m)^\dagger(D_E+m) = -D_E^2 + m^2 = -\nabla^2 + m^2 + \frac{1}{4}R$, so $X_D = m^2 + \frac{1}{4}R$ and

$$a_1^{(P_D)} = \tfrac{1}{6}R - X_D = -\tfrac{1}{12}R - m^2, \qquad \mathrm{tr}\,a_{1,R}^{(P_D)} = 4\cdot\big(-\tfrac{1}{12}\big)R = -\tfrac{1}{3}R.$$

Because $W_{\mathrm{ferm}} = -\frac{1}{2}\ln\det P_D = +\frac{1}{2}\int_\varepsilon^\infty \frac{ds}{s}\mathrm{Tr}\,e^{-sP_D}$ (the fermionic sign flip),

$$W_R^{\mathrm{ferm}} = -\frac{1}{96\pi^2}\Lambda^2\int d^4x\sqrt{g}\,R.$$

The phase $i\Theta$ and zero modes matter in chiral/topological settings but do not modify this local coefficient for a massive Dirac field in a smooth background.

5 Gauge fields and ghosts (spin-1), Feynman gauge

Per generator, $\alpha = 1$: $(P_1)_\mu{}^\nu = -\delta_\mu^\nu \nabla^2 + R_\mu{}^\nu$, ghost $P_{\rm gh} = -\nabla^2$, $W_{\rm gauge} = \frac{1}{2}\ln\det P_1 - \ln\det P_{\rm gh}$. With $E_\mu{}^\nu = R_\mu{}^\nu$,

$$\operatorname{tr} a_{1,R}^{(\text{1-form})} = \tfrac{1}{6}\operatorname{tr}\mathbf{1} - \operatorname{tr} E = \tfrac{4}{6} - 1 = -\tfrac{1}{3}R;$$

the ghost (complex Grassmann scalar) enters with the opposite sign of a complex boson ($\frac{1}{3}R$), giving $\operatorname{tr} a_{1,R}^{(\rm gh)} = -\frac{1}{3}R$, hence

$$A_1^{(\rm eff)}(\text{vector} + \text{ghosts}) = -\tfrac{2}{3} \quad (\text{per generator}).$$

6 Unified convention and induced Newton constant

Define $A_1^{(\rm eff)}$ by $W_R^{\rm total} = -\dfrac{A_1^{(\rm eff)}}{32\pi^2}\Lambda^2 \int d^4x\sqrt{g}\,R$. Comparing with $S_E^{\rm EH}$ and identifying $S_{E,\rm induced}^{\rm EH} = W_R^{\rm total}$,

$$\frac{1}{16\pi G_{\rm ind}} = \frac{A_1^{(\rm eff)}}{32\pi^2}\Lambda^2 \quad\Longrightarrow\quad G_{\rm ind} = \frac{2\pi}{A_1^{(\rm eff)}\,\Lambda^2}, \qquad G_{\rm ind} > 0 \iff A_1^{(\rm eff)} > 0.$$

7 Species table (unified convention)

Species	$A_1^{(\rm eff)}$	Comment
Real scalar (general ξ)	$\frac{1}{6} - \xi$	$X = m^2 + \xi R$
Real scalar (minimal)	$+\frac{1}{6}$	$\xi = 0$
Complex scalar	$2(\frac{1}{6} - \xi)$	two real d.o.f.
Dirac fermion (4 comp.)	$+\frac{1}{3}$	derived above
Weyl fermion (2 comp.)	$+\frac{1}{6}$	half of Dirac
Majorana fermion	$+\frac{1}{6}$	half of Dirac
Gauge vector + ghosts	$-\frac{2}{3}$	per generator, Feynman gauge

Every entry is verified symbolically in Appendix A.

8 Sign examples

- Minimal real scalar: $A_1^{(\rm eff)} = \frac{1}{6} > 0 \Rightarrow G_{\rm ind} > 0$.
- Non-minimally coupled real scalar with $\xi = \frac{1}{4}$: $A_1^{(\rm eff)} = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12} < 0 \Rightarrow G_{\rm ind} < 0$.
- Conformally coupled real scalar in $D = 4$ ($\xi_{\rm conf} = \frac{D-2}{4(D-1)} = \frac{1}{6}$): $A_1^{(\rm eff)} = 0$; it does not contribute to the induced Einstein–Hilbert term at this order. This does not remove the usual conformal-anomaly / a_2 curvature-squared terms; it only says that the quadratically divergent Einstein–Hilbert R-term vanishes for the conformally coupled scalar.
- Gauge vector (per generator): $A_1^{(\rm eff)} = -\frac{2}{3} < 0 \Rightarrow G_{\rm ind} < 0$ for that sector alone.

The net sign for a given model is $\sum_{\rm species} A_1^{(\rm eff)}$ weighted by multiplicities; realistic spectra require summing all sectors (this appendix provides the per-species bookkeeping, not a spectrum-specific determination).

A Symbolic verification

Distributed as `verification/0091/verify_0091.py` (SymPy). It reconstructs each $A_1^{(\mathrm{eff})}$ from $a_1 = \mathrm{tr}(\frac{1}{6}R - X)$ with the Lichnerowicz shift $X_D = m^2 + \frac{1}{4}R$ for spinors, the fermionic overall sign flip ($W_{\mathrm{ferm}} = -\frac{1}{2}\ln\det P_D$), and the ghost entering as minus a complex boson, and checks all seven table entries: $\frac{1}{6} - \xi$, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{6}$, $-\frac{2}{3}$ — all match. The induced-Newton sign column ($G_{\mathrm{ind}} > 0 \iff A_1^{(\mathrm{eff})} > 0$) is checked on the minimal scalar (+), the non-minimally coupled scalar with $\xi = \frac{1}{4}$ (−), the conformally coupled scalar in $D = 4$ ($\xi = \frac{1}{6}$, vanishing contribution), and the gauge sector (−). Result: ALL SEELEY–DEWITT COEFFICIENTS MATCH.

B Changes relative to version 1

1. **Coefficient/sign results unchanged and correct.** The Seeley–DeWitt table and induced-Newton signs are verified (Appendix A). This is the sole paper in the program requiring no substantive rectification of results; the only textual fix is in the scalar examples — the previously loose "conformal $\xi = \frac{1}{4}$" wording is corrected to non-minimal $\xi = \frac{1}{4}$, and the genuine 4D conformal value $\xi = \frac{1}{6}$ (giving $A_1^{(\mathrm{eff})} = 0$) is added as its own example.

2. **Verification added:** symbolic check of the full a_1 table and the induced-Newton sign, distributed with the paper.

3. **References updated** to the v2 revisions of the companion series; the "resource boundary / cut" link is restated as heuristic and non-load- bearing (scope note).

References

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