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Finite-Dimensional Davies Interface Lemmas and TFIM Witness Tests for the Heisenberg Cut as a Resource Boundary

Lluis Eriksson

Version 2 - supersession and correction note

Purpose of this replacement

This replacement preserves public version 1 and adds an explicit correction and supersession record. The general Davies-channel Bohr decomposition, suppression proof route, constants, envelope interfaces and inline listing must not be used from version 1 without the corrections in ai.viXra:2601.0023v2.

The exact omega=0 statements and witness mechanism of version 1 stand unchanged. Proposition 4.4 survives with a repaired proof. The historical paper is not erased; the successor controls the corrected general decomposition, suppression hypotheses and constants.

Citation rule

For the corrected general Bohr decomposition, proof of Proposition 4.4, suppression constants and TFIM implementation, cite ai.viXra:2601.0023v2. The exact omega=0 statements may still be cited from version 1 together with this notice.

Status boundary

SUPERSEDED does not mean that every sentence of version 1 is false. It means that load-bearing technical statements were corrected in the named successor and version 1 is no longer the authoritative mathematical source for them.

The exact claim map and provenance record continue on notice page 2. The historical public manuscript then follows unchanged.

CORRECTION AND CLAIM MAP

Results retained from the historical paper

- The exact omega=0 identity and all omega=0 statements of version 1 stand unchanged.

- The witness mechanism and the conclusion of Proposition 4.4 survive; the successor repairs the proof using the exact general decomposition.

- The finite TFIM exact-diagonalization witness protocol remains historical numerical evidence under the corrected implementation conventions.

Claims superseded or narrowed

- Lemma 4.3's printed plain-commutator Bohr decomposition is false in general; this does not invalidate its omega=0 reduction.

- The proofs of Lemmas 4.12-4.13 use a false KMS submultiplicativity bound; their constants, inheritance route and claim of avoiding an inverse-smallest-eigenvalue factor are superseded.

- The inline appendix/listing and any numerical conclusions depending on the superseded normalization are not authoritative.

What the successor adds

- An exact general Bohr decomposition that reduces to the unchanged omega=0 identity and is pairwise nonnegative.

- A repaired proof of Proposition 4.4 plus corrected c_sigma-dependent suppression constants, hypotheses and pinning identity.

- A verification suite for the decomposition, counterexample, corrected bounds and TFIM witness.

Evidence classification

The omega=0 identity and surviving witness conclusion are exact finite-dimensional statements. The successor verifies the corrected general decomposition and finite TFIM tests; no general lower envelope follows without its assumptions.

Provenance

Notice pages 1-2 are new. The following 22 pages reproduce public version 1 with identical rendered content. Frozen copies of both public v1 and the public successor v2 are included in the audited package.

Finite-Dimensional Davies Interface Lemmas and TFIM Witness Tests for the Heisenberg Cut as a Resource Boundary

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December 2025

Status of claims (read first)

- **Imported result.** The maintenance inequality $P_{\mathrm{extra}}(\rho) \geq k_{\mathrm{B}}T\,\dot{C}_{\mathrm{loss}}(\rho)$ is imported from the published preprint [2].
- **Proved here (finite-dimensional).** The exact $\omega = 0$ Dirichlet identity (Lemma 1), the Bohr-block Dirichlet decomposition (Lemma 2), and the witness lower bound (Proposition 1).
- **Sufficient conditional bound.** The envelope suppression lemma (Lemma 3) is a finite-dimensional sufficient statement under explicit hypotheses (IR exclusion + quasi-local spectral tails).
- **Numerics.** TFIM results are exact diagonalization and stability-checked finite-size diagnostics (here $N \leq 12$); no thermodynamic-limit uniformity is claimed.

Abstract

We present a finite-dimensional technical core for interpreting the quantum–classical boundary as a control-resource limitation: a state is operationally "classical" (relative to a chosen conditional expectation) whenever sustaining coherence is infeasible under a finite available power budget.

Using battery-assisted thermal operations at temperature T and energy pinching Δ, we quantify coherence by $C(\rho) = S(\rho\|\Delta[\rho])$ and define its instantaneous loss rate $\dot{C}_{\mathrm{loss}}(\rho)$ under uncontrolled Markovian dynamics. An imported maintenance inequality yields the operational bound $P_{\mathrm{extra}}(\rho) \geq k_{\mathrm{B}}T\,\dot{C}_{\mathrm{loss}}(\rho)$.

We isolate the dynamical hinge required for geometric scaling claims: the relation between geometric separation and effective decoherence rates. In Davies generators we give two interface lemmas: a negative lemma showing that an active $\omega = 0$ channel yields a witness-based lower bound via a KMS commutator with $S(0)$, and a positive sufficient lemma yielding envelope suppression under explicit infrared exclusion plus a quasi-local spectral tail hypothesis.

Finally, we provide TFIM exact-diagonalization witness protocols computing ratios $R(\epsilon)$ and optimized local witnesses $R_{\mathrm{opt}}(\epsilon)$, including scaling tests with $\epsilon = \epsilon(N)$ and temperature sweeps over β. We also report direct state-side interface micro-tests on weak-coherence families, confirming plateau behavior of $\dot{C}_{\mathrm{loss}}/C$ and controlled $\omega = 0$ scaling in $\gamma(0)$ and β, with stability checks in the Euler step size and PSD clipping threshold.

Contents

1 Scope and non-claims

This work does not propose modified quantum dynamics, collapse mechanisms, derivations of the Born rule, or ontological resolutions of the measurement problem. The claims are operational: they concern minimal control resources required to maintain coherence against uncontrolled open-system dynamics under an explicit thermodynamic bookkeeping model.

Remark 1 (Figures)**.** *This manuscript expects several figure files (e.g.* `scaling_Ropt_N_over_4.png`*,* `interface_ratio_vs_C.png`*) to be present in 'figures/' (or the working directory). If they are missing, compilation will fail.*

Table 1: Program Map: logical structure of the coherence-maintenance series.

ID	Role	Key Contribution
	Foundational Preprints (Published)	
P1	Geometric Foundation	AQFT static clustering bounds and Petz recovery.
P2	Thermodynamic Law	Maintenance inequality: $P_{\text{extra}}(\rho) \geq k_{\mathrm{B}}T\,\dot{C}_{\text{loss}}(\rho)$.
P3	Synthesis/Outlook	Operational cut framing and interface discussion.
	Core Technical Paper (This Work)	
P7	**Finite-Dimensional Core**	**Davies interface lemmas, TFIM witnesses, and reproducible micro-tests.**
	Companion Notes (to be posted separately)	
P4	Hypothesis Framing	RIP: weak vs. strong forms, plausibility routes, and falsification criteria.
P6	Stress Tests	Operatorial $\kappa(\epsilon)$ tests and $\omega = 0$-floor mechanisms.
P5	Logical Closure	Separation of proved results vs. conditional interfaces vs. open dynamical gaps.

2 Finite-dimensional operational core

2.1 Thermal state, energy pinching, and coherence

Let the system Hamiltonian be

$$H_S = \sum_n E_n \Pi_n,$$

and define at temperature T the thermal state

$$\gamma_S := \frac{e^{-\beta H_S}}{\mathrm{Tr}(e^{-\beta H_S})}, \qquad \beta = \frac{1}{k_{\mathrm{B}}T}.$$

Define energy pinching

$$\Delta[\rho] := \sum_n \Pi_n \rho \Pi_n,$$

and relative-entropy coherence (nats)

$$C(\rho) := S(\rho \| \Delta[\rho]), \qquad S(\rho\|\sigma) := \mathrm{Tr}[\rho(\log\rho - \log\sigma)]\,.$$

2.2 Uncontrolled Markovian dynamics and coherence-loss rate

Let $\rho_t = e^{t\mathcal{L}}(\rho)$ be a Markovian semigroup with a thermal stationary state $\mathcal{L}(\gamma_S) = 0$. Define the instantaneous coherence-loss rate

$$\dot{C}_{\text{loss}}(\rho) := -\left.\frac{d}{dt} C(\rho_t)\right|_{t=0}.$$

2.3 Control model and extra power

We assume the control primitive of battery-assisted thermal operations at temperature T, and define incremental (extra) power P_{extra} as an infimum over pairs of strategies maintaining ρ and maintaining $\Delta[\rho]$, respectively.

2.4 Core maintenance inequality (imported)

Theorem 1 (Extra-power coherence maintenance bound (imported))**.** *Assume strategies exist that maintain ρ and that maintain $\Delta[\rho]$ under battery-assisted thermal operations at temperature T. Then*

$$P_{\text{extra}}(\rho) \geq k_{\text{B}} T \, \dot{C}_{\text{loss}}(\rho).$$

Corollary 1 (No-go from a uniform loss floor)**.** *Fix $C_0 > 0$ and a nonempty target family $\mathcal{F}_{C_0} \subset \{\rho : C(\rho) \geq C_0\}$. If there exists*

$$g(C_0) := \inf_{\rho \in \mathcal{F}_{C_0}} \dot{C}_{\text{loss}}(\rho) > 0,$$

then any controller maintaining any $\rho \in \mathcal{F}_{C_0}$ requires $P_{\text{extra}}(\rho) \geq k_{\text{B}} T \, g(C_0)$. If $P_{\max} < k_{\text{B}} T \, g(C_0)$, sustained maintenance is operationally impossible on $\mathcal{F}_{C_0}$.

3 Davies generators, KMS Dirichlet form, and an effective envelope

3.1 Bohr components

Consider a Davies generator derived in the weak-coupling–secular limit with a single system coupling operator $S = S^\dagger$. Define Bohr components

$$S(\omega) := \sum_{E_m - E_n = \omega} \Pi_m S \Pi_n, \qquad S = \sum_{\omega} S(\omega), \qquad S(\omega)^\dagger = S(-\omega).$$

Bath rates satisfy the KMS relation

$$\gamma(-\omega) = e^{-\beta\omega} \gamma(\omega).$$

3.2 KMS inner product and Dirichlet form

Define the KMS (GNS) inner product

$$\langle A, B \rangle_\sigma := \mathrm{Tr}\Big(\sigma^{1/2} A^\dagger \sigma^{1/2} B\Big), \qquad \|A\|_{2,\sigma}^2 := \langle A, A \rangle_\sigma,$$

with $\sigma = \gamma_S$. For the Heisenberg-picture generator $\mathcal{L}^\dagger$, define the Dirichlet form

$$\mathcal{E}_\sigma(O) := -\mathrm{Re}\, \langle O, \mathcal{L}^\dagger(O) \rangle_\sigma.$$

3.3 Separation and variational envelope

Fix a coupling region (e.g. a site j_0) and let $\mathcal{A}_\epsilon$ denote operators supported at distance at least ϵ from that region. Define the operatorial envelope

$$\kappa(\epsilon) := \sup_{O\in\mathcal{A}_\epsilon,\ O\neq 0} \frac{\mathcal{E}_\sigma(O)}{\|O\|_{2,\sigma}^2}.$$

3.4 A concrete tail map on spin chains (Pauli-string conditional expectation)

In lattice spin systems we use a canonical "tail" map built from the Pauli-string expansion. Let Λ be the set of lattice sites and let $\mathcal{A}(\Omega)$ denote the local operator algebra on region $\Omega \subset \Lambda$.

Fix a coupling region Ω_0 (e.g. $\Omega_0 = \{j_0\}$). For $\epsilon \geq 1$ define the far region

$$\Omega_\epsilon := \{j \in \Lambda :\ d(j, \Omega_0) \geq \epsilon\}.$$

Let $\mathcal{A}_{<\epsilon} := \mathcal{A}(\Lambda \setminus \Omega_\epsilon)$ be the algebra supported within distance $< \epsilon$ of the coupling region.

Define the *Pauli-string conditional expectation* $\mathbb{E}_{<\epsilon} : \mathcal{A}(\Lambda) \to \mathcal{A}_{<\epsilon}$ as the orthogonal projection (with respect to the normalized Hilbert–Schmidt inner product) onto the span of Pauli strings supported in $\Lambda \setminus \Omega_\epsilon$. Equivalently, writing any operator as a Pauli expansion $X = \sum_P x_P P$ with Pauli strings P,

$$\mathbb{E}_{<\epsilon}[X] := \sum_{P:\ \mathrm{supp}(P)\subseteq\Lambda\setminus\Omega_\epsilon} x_P\, P.$$

We then define the tail at separation ϵ by

$$\mathrm{tail}_\epsilon(X) := X - \mathbb{E}_{<\epsilon}[X].$$

Remark 2. *We define $\mathbb{E}_{<\epsilon}$ as a normalized Hilbert–Schmidt projection for concreteness and computational convenience. Since the system is finite-dimensional, $\|\cdot\|_{2,\sigma}$ and the Hilbert–Schmidt norm are equivalent; thus any exponential tail bound in one induces an exponential tail bound in the other up to multiplicative constants depending on σ.*

These constants may depend on (N, β) in finite-size tests. Alternatively, one may define $\mathbb{E}_{<\epsilon}$ as the orthogonal projection with respect to $\langle\cdot,\cdot\rangle_\sigma$ to align norms more tightly.

4 Interface lemmas: rate floors and conditional envelope suppression

4.1 Negative lemma: the $\omega = 0$ channel yields a witness lower bound

Define the $\omega = 0$ Bohr component

$$S(0) := \sum_n \Pi_n S \Pi_n.$$

Assume $\gamma(0) > 0$ (e.g. after infrared regularization in an Ohmic model).

Lemma 1 (Exact $\omega = 0$ Dirichlet identity)**.** *If the $\omega = 0$ Davies channel is*

$$\mathcal{L}_0^\dagger(O) = \gamma(0)\left(S(0)OS(0) - \frac{1}{2}\{S(0)^2, O\}\right),$$

then for all O,

$$\mathcal{E}_\sigma^{(0)}(O) := -\mathrm{Re}\,\langle O, \mathcal{L}_0^\dagger(O)\rangle_\sigma = \frac{\gamma(0)}{2}\,\|[S(0), O]\|_{2,\sigma}^2.$$

Proof. Let $\sigma = \gamma_S$ and set $S_0 := S(0)$. Since S_0 is diagonal in the energy eigenbasis, we have $[S_0, H_S] = 0$ and hence $[S_0, \sigma] = 0$, i.e. $\sigma^{1/2} S_0 = S_0 \sigma^{1/2}$.

Write

$$\mathcal{L}_0^\dagger(O) = \gamma(0) \left(S_0 O S_0 - \frac{1}{2}\{S_0^2, O\} \right).$$

Using $\langle A, B\rangle_\sigma = \mathrm{Tr}(\sigma^{1/2} A^\dagger \sigma^{1/2} B)$ and $[S_0, \sigma^{1/2}] = 0$, one expands both $\langle O, \mathcal{L}_0^\dagger(O)\rangle_\sigma$ and $\|[S_0, O]\|_{2,\sigma}^2$ and matches terms, obtaining

$$-\mathrm{Re}\,\langle O, \mathcal{L}_0^\dagger(O)\rangle_\sigma = \frac{\gamma(0)}{2} \, \|[S_0, O]\|_{2,\sigma}^2.$$

□

Proposition 1 (Fixed-separation witness lower bound (sufficient))**.** *Fix $\epsilon_0 \geq 1$. If there exists $c_{\epsilon_0} > 0$ and an operator $O_{\epsilon_0} \in \mathcal{A}_{\epsilon_0}$ such that*

$$\frac{\|[S(0), O_{\epsilon_0}]\|_{2,\sigma}^2}{\|O_{\epsilon_0}\|_{2,\sigma}^2} \geq c_{\epsilon_0},$$

then

$$\kappa(\epsilon_0) \geq \frac{\gamma(0)}{2} \, c_{\epsilon_0}.$$

4.2 Bohr-block decomposition of the KMS Dirichlet form

Lemma 2 (Bohr-block decomposition of the KMS Dirichlet form (single-channel Davies))**.** *Assume a single Hermitian coupling operator $S = S^\dagger$ and a Davies generator in the Heisenberg picture of the form*

$$\mathcal{L}^\dagger(O) = \sum_\omega \gamma(\omega) \left(S(\omega)^\dagger O S(\omega) - \frac{1}{2}\{S(\omega)^\dagger S(\omega), O\} \right), \qquad S(\omega)^\dagger = S(-\omega),$$

with rates satisfying the KMS relation $\gamma(-\omega) = e^{-\beta\omega}\gamma(\omega)$ and reference state $\sigma = \gamma_S$. Let

$$\langle A, B\rangle_\sigma := \mathrm{Tr}\left(\sigma^{1/2} A^\dagger \sigma^{1/2} B\right), \qquad \mathcal{E}_\sigma(O) := -\mathrm{Re}\,\langle O, \mathcal{L}^\dagger(O)\rangle_\sigma.$$

Then

$$\mathcal{E}_\sigma(O) = \frac{1}{2} \sum_\omega \gamma(\omega) \, \|[S(\omega), O]\|_{2,\sigma}^2.$$

Proof. Fix ω and set $L := S(\omega)$ and $\gamma := \gamma(\omega)$. Define

$$\mathcal{L}_\omega^\dagger(O) := \gamma \left(L^\dagger O L - \frac{1}{2}\{L^\dagger L, O\} \right).$$

Since $\sigma = \gamma_S \propto e^{-\beta H_S}$ and L has Bohr frequency ω, the intertwining relations hold:

$$\sigma^{1/2} L = e^{-\beta\omega/2} L \sigma^{1/2}, \qquad \sigma^{1/2} L^\dagger = e^{+\beta\omega/2} L^\dagger \sigma^{1/2}.$$

Using these relations and cyclicity of trace, one rewrites $-\mathrm{Re}\,\langle O, \mathcal{L}_\omega^\dagger(O)\rangle_\sigma$ as $\frac{\gamma}{2}\|[L, O]\|_{2,\sigma}^2$. Summing over ω gives the claim. □

4.3 Positive lemma: envelope suppression under IR exclusion and quasi-local spectral tails

Definition 1 (Infrared exclusion)**.** *We say the coupling satisfies IR exclusion at scale $\omega_* > 0$ if $S(\omega) = 0$ for all $|\omega| < \omega_*$ (or alternatively $\gamma(\omega) = 0$ on that band).*

Lemma 3 (Envelope suppression under IR exclusion and quasi-local spectral tails (sufficient))**.** *Assume IR exclusion and suppose there exist $A > 0$, $\xi > 0$, and $p \geq 0$ such that for all $|\omega| \geq \omega_*$,*

$$\|\mathrm{tail}_\epsilon(S(\omega))\|_{2,\sigma} \leq A(1+\epsilon)^p e^{-\epsilon/\xi},$$

and $\sup_{|\omega|\geq\omega_*} \gamma(\omega) < \infty$. Then there exists $C > 0$ such that

$$\kappa(\epsilon) \leq C(1+\epsilon)^{2p} e^{-2\epsilon/\xi}.$$

Proof (sketch). By Lemma 2,

$$\mathcal{E}_\sigma(O) = \frac{1}{2}\sum_\omega \gamma(\omega)\, \|[S(\omega), O]\|_{2,\sigma}^2.$$

By IR exclusion, only $|\omega| \geq \omega_*$ contribute. For $O \in \mathcal{A}_\epsilon$, write

$$S(\omega) = \mathbb{E}_{<\epsilon}[S(\omega)] + \mathrm{tail}_\epsilon(S(\omega)).$$

Since $\mathbb{E}_{<\epsilon}[S(\omega)]$ is supported within distance $< \epsilon$ while O is supported at distance at least ϵ,

$$[\mathbb{E}_{<\epsilon}[S(\omega)], O] = 0, \qquad \Rightarrow \qquad [S(\omega), O] = [\mathrm{tail}_\epsilon(S(\omega)), O].$$

Then

$$\|[A, O]\|_{2,\sigma} \leq \|AO\|_{2,\sigma} + \|OA\|_{2,\sigma} \leq 2\|A\|_\infty \, \|O\|_{2,\sigma},$$

and hence

$$\frac{\|[S(\omega), O]\|_{2,\sigma}^2}{\|O\|_{2,\sigma}^2} \leq 4\, \|\mathrm{tail}_\epsilon(S(\omega))\|_\infty^2.$$

Using $\|X\|_\infty \leq \|X\|_2 \leq \lambda_{\min}(\sigma)^{-1/2}\|X\|_{2,\sigma}$ yields

$$\|\mathrm{tail}_\epsilon(S(\omega))\|_\infty^2 \leq \frac{1}{\lambda_{\min}(\sigma)}\, \|\mathrm{tail}_\epsilon(S(\omega))\|_{2,\sigma}^2.$$

Combining the bounded-rate assumption with the tail hypothesis gives the stated envelope for $\kappa(\epsilon)$ after taking the supremum over $O \in \mathcal{A}_\epsilon$. □

Remark 3 (Finite-size scaling of the prefactor)**.** *The constant C contains a factor $\lambda_{\min}(\sigma)^{-1}$ arising from norm comparisons and may grow with (N, β) since $\sigma = \gamma_S \propto e^{-\beta H_S}$. Accordingly, Lemma 3 is stated as a sufficient finite-dimensional bound at fixed (N, β) and is not claimed to be uniform in any thermodynamic limit.*

5 TFIM witness computations (exact diagonalization)

5.1 Model

We consider the transverse-field Ising model on N spins with open boundary conditions:

$$H_S = -J \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x - h \sum_{i=1}^{N} \sigma_i^z, \qquad h > J.$$

We choose a coupling site $j_0 = \lfloor N/2 \rfloor$ and take $S = \sigma_{j_0}^z$. We consider thermal states $\sigma = \gamma_S$ at inverse temperatures $\beta \in \{0.5, 1, 2\}$.

5.2 Witness ratios

For a distant site $k = j_0 \pm \epsilon$ define

$$R(\epsilon; O) := \frac{\|[S(0), O]\|_{2,\sigma}^2}{\|O\|_{2,\sigma}^2},$$

and define the optimized local witness ratio

$$R_{\text{opt}}(\epsilon) := \max_{O \in \text{span}\{\sigma_k^x, \sigma_k^y, \sigma_k^z\}} R(\epsilon; O).$$

By Lemma 1,

$$\kappa(\epsilon) \geq \frac{\gamma(0)}{2} R_{\text{opt}}(\epsilon).$$

5.3 Fixed ϵ results (at $\beta = 1$)

For $J = 1$, $h = 1.5$, $\beta = 1$:

N	$R_{\text{opt}}(1)$	$R_{\text{opt}}(2)$	$R_{\text{opt}}(3)$
8	0.134648	0.0885503	0.0684289
10	0.0999406	0.0696450	0.0550748
12	0.0793451	0.0574518	0.0464596

Table 2: Optimized local KMS commutator witness $R_{\text{opt}}(\epsilon)$ for TFIM with $S = \sigma_{j_0}^z$, center coupling, $\beta = 1$, $h = 1.5$, $J = 1$.

5.4 Scaling tests with $\epsilon = \epsilon(N)$ (at $\beta = 1$)

We computed $R_{\text{opt}}(\epsilon(N))$ for four rules:

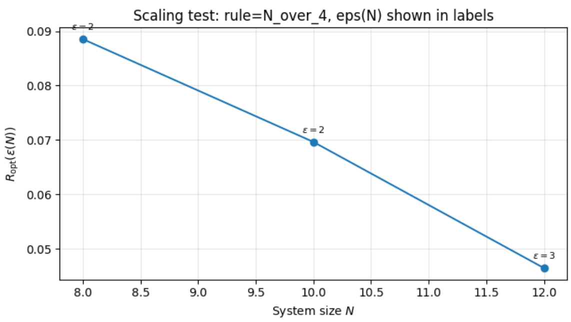
$$\epsilon(N) \in \left\{ \left\lfloor \frac{N}{4} \right\rfloor, \ \left\lfloor \frac{N}{3} \right\rfloor, \ \epsilon_{\max}(N), \ 3 \right\}, \qquad \epsilon_{\max}(N) = \min(j_0, N - 1 - j_0).$$

5.5 Temperature sweep tests over β

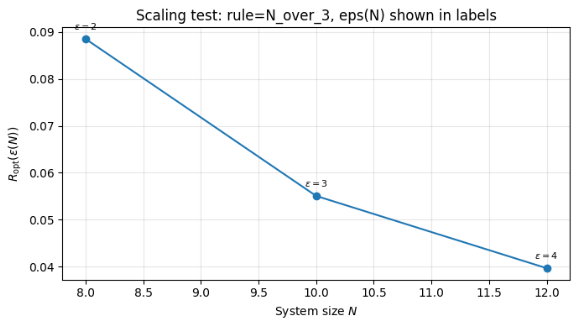
We swept $\beta \in \{0.5, 1, 2\}$ for the same $\epsilon(N)$ rules and $N \in \{8, 10, 12\}$.

Rule	$N = 8$	$N = 10$	$N = 12$
$\lfloor N/4 \rfloor$	0.0885503 ($\epsilon = 2$)	0.0696450 ($\epsilon = 2$)	0.0464596 ($\epsilon = 3$)
$\lfloor N/3 \rfloor$	0.0885503 ($\epsilon = 2$)	0.0550748 ($\epsilon = 3$)	0.0396591 ($\epsilon = 4$)
$\epsilon_{\max}(N)$	0.0684289 ($\epsilon = 3$)	0.0461050 ($\epsilon = 4$)	0.0330550 ($\epsilon = 5$)
3 (fixed)	0.0684289 ($\epsilon = 3$)	0.0550748 ($\epsilon = 3$)	0.0464596 ($\epsilon = 3$)

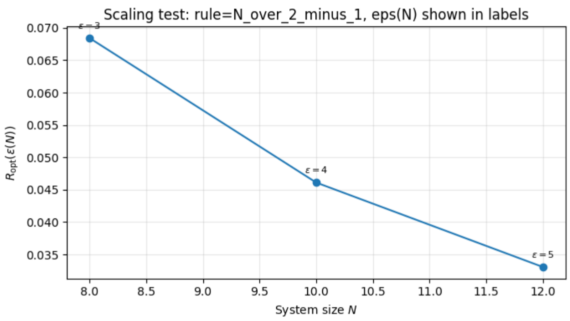
Table 3: Scaling test values $R_{\text{opt}}(\epsilon(N))$ for TFIM with $S = \sigma^z_{j_0}$, $\beta = 1$, $h = 1.5$, $J = 1$.



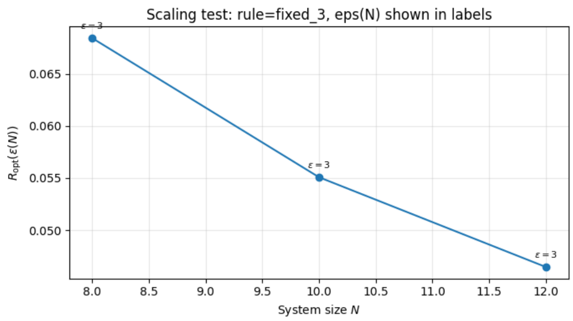
(a) $\epsilon = \lfloor N/4 \rfloor$



(b) $\epsilon = \lfloor N/3 \rfloor$

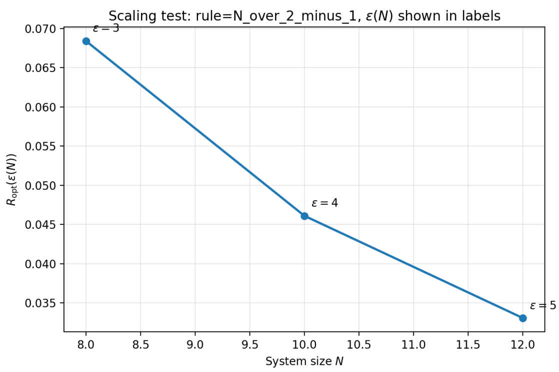


(c) $\epsilon = \epsilon_{\max}(N)$

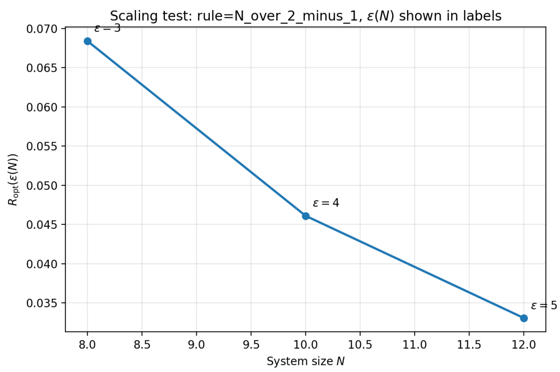


(d) $\epsilon = 3$

Figure 1: Scaling plots of $R_{\text{opt}}(\epsilon(N))$ under four $\epsilon(N)$ rules (at $\beta = 1$).

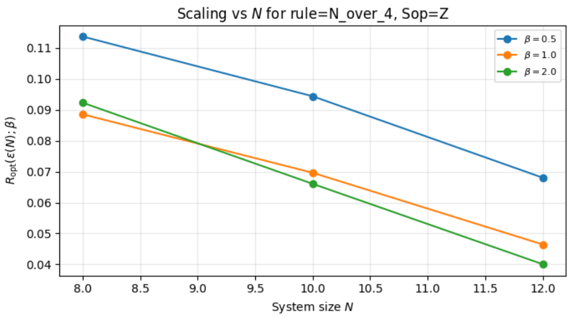


(a) Scaling test: rule=fixed_3, $\epsilon(N)$ shown in labels.

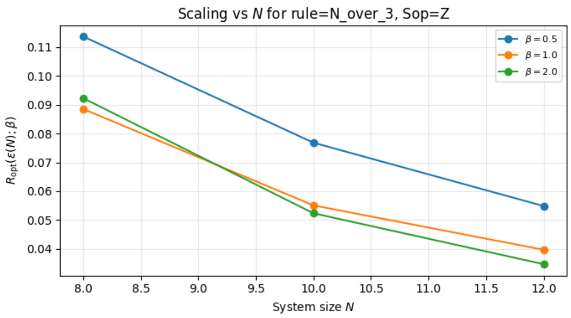


(b) Scaling test: rule=$\epsilon_{\max}(N)$, $\epsilon(N)$ shown in labels.

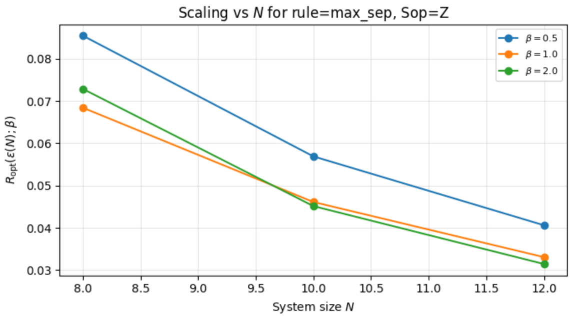
Figure 2: Scaling tests where $\epsilon(N)$ is printed on each data point.



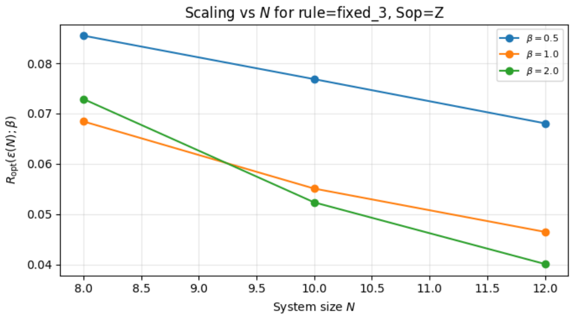
(a) $\epsilon = \lfloor N/4 \rfloor$



(b) $\epsilon = \lfloor N/3 \rfloor$



(c) $\epsilon = \epsilon_{\max}(N)$



(d) $\epsilon = 3$

Figure 3: Optimized witness $R_{\mathrm{opt}}(\epsilon(N); \beta)$ versus N for multiple β values.

6 State-side interface micro-tests: weak-coherence families and $\omega = 0$ scaling

The witness computations above are observable-side tests based on the exact $\omega = 0$ Dirichlet identity. To directly probe the state-side quantity appearing in the maintenance inequality, we also test the interface numerically on explicit weak-coherence families.

Weak-coherence family. Fix an energy eigenbasis $\{n\}$ of H_S, thermal weights $p_n \propto e^{-\beta E_n}$, and a pair (m, n) with $m \neq n$. Define in the energy basis

$$\rho_E(\alpha) := \mathrm{diag}(p) + \alpha\sqrt{p_m p_n}\,(|m\rangle\langle n| + |n\rangle\langle m|)\,,$$

and set $\rho(\alpha) = V \rho_E(\alpha) V^\dagger$ where V diagonalizes H_S. For small α, this is positive and satisfies $\Delta[\rho(\alpha)] = \mathrm{diag}(p)$ by construction. We evaluate $C(\rho) = S(\rho \| \Delta[\rho])$ and approximate

$$\dot{C}_{\mathrm{loss}}(\rho) \approx -\frac{C(\rho_t) - C(\rho)}{t}, \qquad \rho_t \approx \Pi(\rho + t\,\mathcal{L}(\rho))\,,$$

where Π denotes numerical projection to a valid density matrix (Hermitian, PSD with eigenvalue clipping, trace one). We report the empirical ratio

$$r(\rho) := \frac{\dot{C}_{\mathrm{loss}}(\rho)}{C(\rho)}.$$

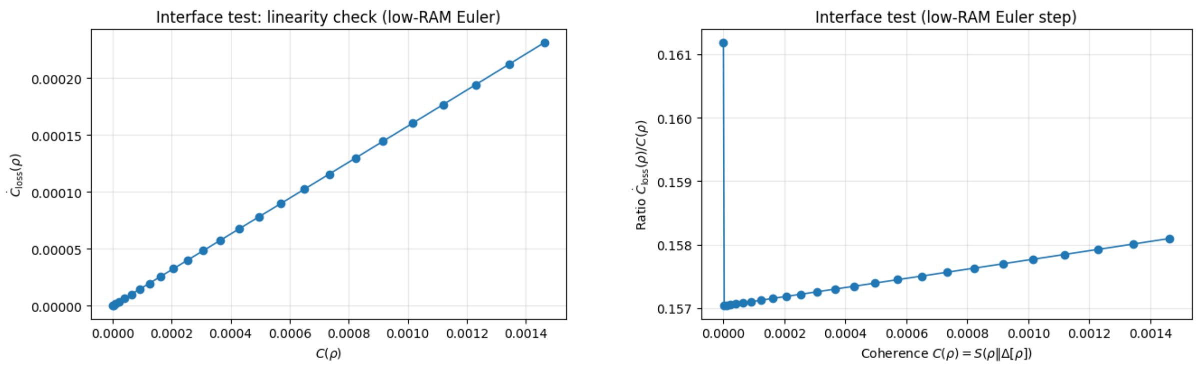
Observed plateau (truncated Davies blocks). For $N = 6$, $J = 1$, $h = 1.5$, $\beta = 1$, coupling $S = \sigma^z_{j_0}$ at the center site, and a transition-selected pair optimized for a nonzero Bohr block, we observed that $r(\rho)$ is approximately constant across several decades of $C(\rho)$ (excluding ultra-small C where PSD projection dominates), with a plateau around

$$r(\rho) \approx 0.157\text{–}0.158$$

for the $\{\pm\omega\}$ truncation, and a slightly larger plateau

$$r(\rho) \approx 0.160\text{–}0.161$$

when an additional $\omega = 0$ channel with $\gamma(0) = 0.2$ is included.



(a) Linearity check: $\dot{C}_{\rm loss}(\rho)$ versus $C(\rho)$. (b) Ratio $r(\rho) = \dot{C}_{\rm loss}(\rho)/C(\rho)$ versus $C(\rho)$.

Figure 4: State-side interface micro-test (Euler step with PSD projection). Ultra-small $C(\rho)$ points can be affected by eigenvalue clipping; the plateau region is stable.

$\omega = 0$-only regime: linear scaling in $\gamma(0)$. To isolate the commutator-squared mechanism of the negative lemma, we ran an $\omega = 0$-only truncation and chose (m, n) to maximize $|s_m - s_n|\sqrt{p_m p_n}$ where $s_k := kSk$ in the energy basis. With $\beta = 1$ and numerical prescription $\gamma(0) = \gamma_0/\beta$, the measured plateau ratio $r(\alpha = 0.15)$ scales linearly with $\gamma(0)$:

γ_0	$\gamma(0) = \gamma_0/\beta$	$r(\alpha = 0.15)$	$r/\gamma(0)$
0.05	0.05	0.0101957	0.2039
0.10	0.10	0.0203914	0.2039
0.40	0.40	0.0815656	0.2039

Table 4: TFIM $\omega = 0$-only micro-test ($N = 6$, $J = 1$, $h = 1.5$, $\beta = 1$). The empirical ratio $r = \dot{C}_{\rm loss}/C$ scales linearly in $\gamma(0) = \gamma_0/\beta$.

Fixed-pair temperature scaling: $r(\beta) \propto 1/\beta$ at fixed γ_0. To avoid conflating temperature dependence with changes in the coherence sector, we fixed $(m, n) = (0, 1)$ and kept $\gamma_0 = 0.10$ fixed (so $\gamma(0) = \gamma_0/\beta$). The measured plateau satisfies $r(\beta) \approx k/\beta$:

β	$\gamma(0) = \gamma_0/\beta$ ($\gamma_0 = 0.10$)	$r(\alpha = 0.15)$	$\beta\, r(\alpha = 0.15)$
0.5	0.20	0.004317824675	0.002158912338
1.0	0.10	0.002158560224	0.002158560224
2.0	0.05	0.001078731632	0.002157463263
4.0	0.025	0.000538771757	0.002155087026

Table 5: Fixed-pair $\omega = 0$-only temperature sweep ($N = 6$, $J = 1$, $h = 1.5$, $(m, n) = (0, 1)$, $\gamma_0 = 0.10$). The near-constancy of βr confirms $r(\beta) \propto 1/\beta$ at fixed γ_0.

7 State–observable interface (explicit hypothesis)

The maintenance inequality bounds power in terms of the state functional $\dot{C}_{\rm loss}(\rho)$, whereas $\kappa(\epsilon)$ is defined via a supremum over observables in $\mathcal{A}_\epsilon$. To avoid a quantifier mismatch, we state the necessary bridge as an explicit hypothesis.

Definition 2 (Weak-coherence regime on a family). *A family of states $\mathcal{F}_\epsilon$ lies in the weak-coherence regime if there exists $\delta_\epsilon > 0$ such that $C(\rho) \le \delta_\epsilon$ for all $\rho \in \mathcal{F}_\epsilon$.*

Definition 3 (Population non-extremality on a family). *A family $\mathcal{F}_\epsilon$ is population-nonextremal if there exists $p_{\min} > 0$ such that for all $\rho \in \mathcal{F}_\epsilon$,*

$$\Delta[\rho] \;\ge\; p_{\min}\mathbf{1}.$$

Remark 4. *This assumption rules out diagonals arbitrarily close to the boundary of the simplex, ensuring uniform local expansions of relative entropy around $\Delta[\rho]$ and preventing degenerate small-denominator effects in weak-coherence estimates.*

Definition 4 (Effective coherence sector). *Fix a set Ω of Bohr frequencies. We say ρ has coherence supported in Ω if all its off-diagonal matrix elements in the energy basis lie in frequency blocks $\omega \in \Omega$.*

Definition 5 (Sector rate lower bound). *For a Davies generator, let $\Gamma(\omega)$ denote the Kossakowski rate matrix in the Bohr block ω. In the single-channel case, $\Gamma(\omega)$ is a scalar and we identify $\lambda_{\min}(\Gamma(\omega)) = \Gamma(\omega)$. Define*

$$\underline{\Gamma}_\Omega := \min_{\omega\in\Omega} \lambda_{\min}(\Gamma(\omega)).$$

Hypothesis 1 (State–observable interface (lower proportionality on a family)). *Fix ϵ and a target family $\mathcal{F}_\epsilon$. Assume:*

1. *$\mathcal{F}_\epsilon$ lies in a weak-coherence regime;*
2. *$\mathcal{F}_\epsilon$ is population-nonextremal;*
3. *all $\rho \in \mathcal{F}_\epsilon$ have coherence supported in a fixed sector set Ω;*
4. *the Davies generator is validated in the regime of interest.*

Then there exists $\eta_\epsilon \to 0$ as the coherence scale $\delta_\epsilon \to 0$ such that for all $\rho \in \mathcal{F}_\epsilon$,

$$\dot{C}_{\rm loss}(\rho) \;\ge\; (1-\eta_\epsilon)\,\underline{\Gamma}_\Omega\, C(\rho).$$

Remark 5. *This interface is falsifiable: one can numerically test whether $\dot{C}_{\rm loss}(\rho)/C(\rho)$ is bounded below on the specified family.*

8 Operational consequence (conditional, well-typed)

Combining Theorem 1 with Hypothesis 1 yields, for $\rho \in \mathcal{F}_\epsilon$,

$$P_{\text{extra}}(\rho) \;\geq\; k_{\mathrm{B}}T\,\dot{C}_{\text{loss}}(\rho) \;\geq\; k_{\mathrm{B}}T\,(1-\eta_\epsilon)\,\underline{\Gamma}_\Omega\,C(\rho).$$

In particular, if $C(\rho) \geq C_0$ on $\mathcal{F}_\epsilon$, then

$$P_{\text{extra}}(\rho) \;\geq\; k_{\mathrm{B}}T\,(1-\eta_\epsilon)\,\underline{\Gamma}_\Omega\,C_0.$$

9 Conclusion

We separated a proven finite-dimensional thermodynamic maintenance bound from conditional dynamical interface claims. In Davies dynamics, an exact $\omega = 0$ identity yields operational witness-based lower bounds. In TFIM witness computations, $R_{\text{opt}}(\epsilon)$ is robust and non-negligible at fixed separations, while decreasing as ϵ scales with N over the tested sizes.

We emphasize that the TFIM numerics ($N \leq 12$) are presented as finite-size witness diagnostics, not as definitive thermodynamic-limit scaling evidence.

A Numerical stability checks: time step and PSD clipping

We estimate $\dot{C}_{\text{loss}}(\rho)$ via a forward Euler step $\rho_t \approx \Pi(\rho + t\mathcal{L}(\rho))$ with PSD projection Π (eigenvalue clipping) and a finite-difference quotient.

dt	r (last sample)
$1e-4$	0.1613363453
$5e-5$	0.1613377388
$1e-5$	0.1613385899

Table 6: Sensitivity of the plateau ratio $r = \dot{C}_{\text{loss}}/C$ to the Euler step size dt (interface micro-test).

`psd_clip`	r (last sample)
$1e-12$	0.1613377388
$1e-14$	0.1613377388
$1e-16$	0.1613377388

Table 7: Sensitivity of the plateau ratio $r = \dot{C}_{\text{loss}}/C$ to the PSD projection threshold (interface micro-test).

B Imported maintenance inequality: precise pointer map

The main maintenance inequality used in this manuscript is

$$P_{\text{extra}}(\rho;\mathcal{L},T) \;\geq\; k_{\mathrm{B}}T\,\dot{C}_{\text{loss}}(\rho), \qquad \dot{C}_{\text{loss}}(\rho) := -\frac{d}{dt}S(\rho_t\|\Delta[\rho_t])\bigg|_{t=0}, \quad \rho_t = e^{t\mathcal{L}}(\rho).$$

A complete finite-dimensional proof is given in the published preprint [2].

C Reproducibility: Colab-friendly scripts (included inline)

This appendix includes scripts used to generate the state-side interface micro-tests (including figures) and the $\omega = 0$ fixed-pair β sweep.

C.1 Low-RAM interface micro-test script (Davies blocks; optional $\omega = 0$; pair selection)

```
# file: test_interface_hypothesis_tfim_lowram.py
import argparse
import numpy as np
import scipy.linalg as la
import matplotlib.pyplot as plt


def parse_args():
    ap = argparse.ArgumentParser()
    ap.add_argument("--N", type=int, default=6)
    ap.add_argument("--J", type=float, default=1.0)
    ap.add_argument("--h", type=float, default=1.5)
    ap.add_argument("--beta", type=float, default=1.0)

    ap.add_argument("--Sop", type=str, default="Z", choices=["X", "Y", "Z"])
    ap.add_argument("--center", action="store_true")
    ap.add_argument("--j0", type=int, default=None)

    ap.add_argument("--dt", type=float, default=5e-5)
    ap.add_argument("--nsamples", type=int, default=25)
    ap.add_argument("--alpha_max", type=float, default=0.15)

    ap.add_argument("--bin_tol", type=float, default=1e-10)
    ap.add_argument("--gamma0", type=float, default=0.2)
    ap.add_argument("--s", type=float, default=1.0)

    ap.add_argument("--use_omega0", action="store_true")
    ap.add_argument("--only_omega0", action="store_true")
    ap.add_argument("--only_pm_omega", action="store_true")

    ap.add_argument("--pair_mode", type=str, default="pmomega", choices=["pmomega", "
        omega0"])
    ap.add_argument("--psd_clip", type=float, default=1e-14)
    ap.add_argument("--seed", type=int, default=0)

    ap.add_argument("--save_prefix", type=str, default="interface")
    ap.add_argument("--no_plots", action="store_true")

    args, unknown = ap.parse_known_args()
    if unknown:
        print(f"Ignoring unknown args: {unknown}")

    if args.only_omega0 and args.only_pm_omega:
        raise ValueError("Choose at most one of --only_omega0 or --only_pm_omega.")
    if args.only_omega0 and (not args.use_omega0):
        raise ValueError("--only_omega0 requires --use_omega0.")

    if not args.center and args.j0 is None:
        args.center = True
```

```
    return args


def paulis():
    I = np.array([[1, 0], [0, 1]], dtype=complex)
    X = np.array([[0, 1], [1, 0]], dtype=complex)
    Y = np.array([[0, -1j], [1j, 0]], dtype=complex)
    Z = np.array([[1, 0], [0, -1]], dtype=complex)
    return I, X, Y, Z


def kron_all(ops):
    out = ops[0]
    for op in ops[1:]:
        out = np.kron(out, op)
    return out


def local_pauli(N, site, which):
    I, X, Y, Z = paulis()
    P = {"I": I, "X": X, "Y": Y, "Z": Z}[which.upper()]
    ops = [I] * N
    ops[site] = P
    return kron_all(ops)


def two_site_xx(N, i):
    I, X, Y, Z = paulis()
    ops = [I] * N
    ops[i] = X
    ops[i + 1] = X
    return kron_all(ops)


def tfim_hamiltonian(N, J, h):
    dim = 2**N
    H = np.zeros((dim, dim), dtype=complex)
    for i in range(N - 1):
        H += -J * two_site_xx(N, i)
    for i in range(N):
        H += -h * local_pauli(N, i, "Z")
    return (H + H.conj().T) / 2


def thermal_probs(E, beta):
    Emin = np.min(E.real)
    w = np.exp(-beta * (E.real - Emin))
    return w / np.sum(w)


def gamma_model(omega, beta, gamma0=0.2, s=1.0):
    if omega > 0:
        denom = 1.0 - np.exp(-beta * omega)
        denom = denom if denom > 1e-15 else 1e-15
        return float(gamma0 * (omega**s) / denom)
    if omega < 0:
        op = abs(omega)
        return float(np.exp(-beta * op) * gamma_model(op, beta, gamma0=gamma0, s=s))
```

```
    return 0.0


def project_to_density(rho, eps=1e-14):
    rho = 0.5 * (rho + rho.conj().T)
    evals, evecs = la.eigh(rho)
    evals = np.maximum(evals.real, eps)
    rho_psd = (evecs * evals) @ evecs.conj().T
    rho_psd = 0.5 * (rho_psd + rho_psd.conj().T)
    rho_psd = rho_psd / np.trace(rho_psd)
    return rho_psd


def rel_entropy(rho, sigma):
    rho = 0.5 * (rho + rho.conj().T)
    sigma = 0.5 * (sigma + sigma.conj().T)
    log_rho = la.logm(rho)
    log_sig = la.logm(sigma)
    return float(np.trace(rho @ (log_rho - log_sig)).real)


def lindblad_L_of_rho(rho, L_ops, rates):
    out = np.zeros_like(rho)
    for Lk, rk in zip(L_ops, rates):
        if rk == 0:
            continue
        LdL = Lk.conj().T @ Lk
        out += rk * (Lk @ rho @ Lk.conj().T - 0.5 * (LdL @ rho + rho @ LdL))
    return out


def build_Somega_bin(S_E, E, omega_target, bin_tol):
    d = S_E.shape[0]
    Sbin = np.zeros((d, d), dtype=complex)
    for m in range(d):
        for n in range(d):
            omega = float((E[m] - E[n]).real)
            if abs(omega - omega_target) <= bin_tol:
                Sbin[m, n] = S_E[m, n]
    return Sbin


def choose_pair_pmomega(S_E, p, E, bin_tol):
    d = S_E.shape[0]
    best = (-1.0, None)
    for m in range(d):
        for n in range(d):
            if m == n:
                continue
            omega = float((E[m] - E[n]).real)
            if omega <= bin_tol:
                continue
            score = float(abs(S_E[m, n]) * np.sqrt(p[m] * p[n]))
            if score > best[0]:
                best = (score, (m, n, omega))
    if best[1] is None:
        raise RuntimeError("Could not find a positive-frequency pair.")
    return best[1]
```

```python
def choose_pair_omega0(S_E, p, E, bin_tol):
    d = S_E.shape[0]
    sdiag = np.diag(S_E).real
    best = (-1.0, None)
    for m in range(d):
        for n in range(d):
            if m == n:
                continue
            omega = float((E[m] - E[n]).real)
            if abs(omega) <= bin_tol:
                continue
            score = float(abs(sdiag[m] - sdiag[n]) * np.sqrt(p[m] * p[n]))
            if score > best[0]:
                best = (score, (m, n, abs(omega)))
    if best[1] is None:
        raise RuntimeError("Could not find a pair for omega0 mode.")
    return best[1]


def make_rho_E(p, m, n, alpha):
    rho_E = np.diag(p.astype(complex))
    amp = alpha * np.sqrt(p[m] * p[n])
    rho_E[m, n] += amp
    rho_E[n, m] += amp
    return rho_E


def main():
    args = parse_args()
    np.random.seed(args.seed)

    N, J, h, beta = args.N, args.J, args.h, args.beta
    j0 = args.j0 if args.j0 is not None else (N // 2 if args.center else 0)

    H = tfim_hamiltonian(N, J, h)
    E, V = la.eigh(H)
    Vdag = V.conj().T
    p = thermal_probs(E, beta)

    S = local_pauli(N, j0, args.Sop)
    S_E = Vdag @ S @ V

    if args.pair_mode == "pmomega":
        m, n, omega = choose_pair_pmomega(S_E, p, E, args.bin_tol)
        print("Pair mode: pmomega")
    else:
        m, n, omega = choose_pair_omega0(S_E, p, E, args.bin_tol)
        print("Pair mode: omega0")

    print(f"Selected pair (m,n)=({m},{n}), omega≈{omega:.12g}")

    Somega_E = build_Somega_bin(S_E, E, omega, args.bin_tol)
    Sminus_E = Somega_E.conj().T
    Somega = V @ Somega_E @ Vdag
    Sminus = V @ Sminus_E @ Vdag
```

```python
    g_plus = gamma_model(omega, beta, gamma0=args.gamma0, s=args.s)
    g_minus = gamma_model(-omega, beta, gamma0=args.gamma0, s=args.s)

    omega0_ops, omega0_rates = [], []
    pm_ops, pm_rates = [Somega, Sminus], [g_plus, g_minus]

    if args.use_omega0:
        S0_E = np.diag(np.diag(S_E))
        S0 = V @ S0_E @ Vdag
        gamma0_val = float(args.gamma0 / beta)
        omega0_ops, omega0_rates = [S0], [gamma0_val]
        print(f"Included omega=0 channel with gamma(0)=gamma0/beta={gamma0_val:.6g}")

    if args.only_omega0:
        L_ops, rates = omega0_ops, omega0_rates
        print("Mode: ONLY omega=0")
    elif args.only_pm_omega:
        L_ops, rates = pm_ops, pm_rates
        print("Mode: ONLY ±omega")
    else:
        L_ops, rates = omega0_ops + pm_ops, omega0_rates + pm_rates
        print("Mode: omega=0 (if enabled) PLUS ±omega")

    alphas = np.linspace(1e-4, args.alpha_max, args.nsamples)
    dt = args.dt

    Cs, dCs, ratios = [], [], []
    for alpha in alphas:
        rho_E = make_rho_E(p, m, n, alpha)
        rho = project_to_density(V @ rho_E @ Vdag, eps=args.psd_clip)

        rho_E_chk = Vdag @ rho @ V
        Delta_rho = project_to_density(V @ np.diag(np.diag(rho_E_chk)) @ Vdag, eps=args
            .psd_clip)
        C = rel_entropy(rho, Delta_rho)

        Lrho = lindblad_L_of_rho(rho, L_ops, rates)
        rho_t = project_to_density(rho + dt * Lrho, eps=args.psd_clip)

        rho_t_E = Vdag @ rho_t @ V
        Delta_rho_t = project_to_density(V @ np.diag(np.diag(rho_t_E)) @ Vdag, eps=args
            .psd_clip)
        C_t = rel_entropy(rho_t, Delta_rho_t)

        dotC_loss = -(C_t - C) / dt

        Cs.append(C)
        dCs.append(dotC_loss)
        ratios.append(dotC_loss / C if C > 0 else np.nan)

    Cs = np.array(Cs, dtype=float)
    dCs = np.array(dCs, dtype=float)
    ratios = np.array(ratios, dtype=float)

    alpha_last = float(alphas[-1])
    r_last = float(ratios[-1])
    C_last = float(Cs[-1])
```

```
    print(f"RESULT alpha_last={alpha_last:.12g} C_last={C_last:.12g} r_last={r_last
        :.12g}")

    if not args.no_plots:
        prefix = args.save_prefix

        plt.figure(figsize=(6.5, 4.2))
        plt.plot(Cs, ratios, marker="o", linewidth=1.2)
        plt.grid(True, alpha=0.3)
        plt.xlabel("Coherence $C(\\rho)=S(\\rho\\|\\Delta[\\rho])$")
        plt.ylabel("Ratio $\\dot C_{\\mathrm{loss}}(\\rho)/C(\\rho)$")
        plt.title("Interface test (low-RAM Euler step)")
        plt.tight_layout()
        plt.savefig(f"{prefix}_ratio_vs_C.png", dpi=200)

        plt.figure(figsize=(6.5, 4.2))
        plt.plot(Cs, dCs, marker="o", linewidth=1.2)
        plt.grid(True, alpha=0.3)
        plt.xlabel("$C(\\rho)$")
        plt.ylabel("$\\dot C_{\\mathrm{loss}}(\\rho)$")
        plt.title("Interface test: linearity check (low-RAM Euler)")
        plt.tight_layout()
        plt.savefig(f"{prefix}_dotC_vs_C.png", dpi=200)

        print(f"Saved -> {prefix}_ratio_vs_C.png")
        print(f"Saved -> {prefix}_dotC_vs_C.png")


if __name__ == "__main__":
    main()
```

C.2 $\omega = 0$-only fixed-pair sweep script (minimal, for the β table)

```
# file: tfim_omega0_fixedpair.py
import argparse
import numpy as np
import scipy.linalg as la


def parse_args():
    ap = argparse.ArgumentParser()
    ap.add_argument("--N", type=int, default=6)
    ap.add_argument("--J", type=float, default=1.0)
    ap.add_argument("--h", type=float, default=1.5)
    ap.add_argument("--beta", type=float, default=1.0)
    ap.add_argument("--Sop", type=str, default="Z", choices=["X", "Y", "Z"])
    ap.add_argument("--center", action="store_true")
    ap.add_argument("--j0", type=int, default=None)

    ap.add_argument("--m", type=int, required=True)
    ap.add_argument("--n", type=int, required=True)

    ap.add_argument("--gamma0", type=float, default=0.10, help="Implements gamma(0)=
        gamma0/beta")
    ap.add_argument("--dt", type=float, default=5e-5)
    ap.add_argument("--alpha_max", type=float, default=0.15)
    ap.add_argument("--nsamples", type=int, default=25)
```

```
    ap.add_argument("--psd_clip", type=float, default=1e-14)
    ap.add_argument("--seed", type=int, default=0)
    return ap.parse_args()


def paulis():
    I = np.array([[1, 0], [0, 1]], dtype=complex)
    X = np.array([[0, 1], [1, 0]], dtype=complex)
    Y = np.array([[0, -1j], [1j, 0]], dtype=complex)
    Z = np.array([[1, 0], [0, -1]], dtype=complex)
    return I, X, Y, Z


def kron_all(ops):
    out = ops[0]
    for op in ops[1:]:
        out = np.kron(out, op)
    return out


def local_pauli(N, site, which):
    I, X, Y, Z = paulis()
    P = {"X": X, "Y": Y, "Z": Z}[which.upper()]
    ops = [I] * N
    ops[site] = P
    return kron_all(ops)


def two_site_xx(N, i):
    I, X, Y, Z = paulis()
    ops = [I] * N
    ops[i] = X
    ops[i + 1] = X
    return kron_all(ops)


def tfim_hamiltonian(N, J, h):
    dim = 2**N
    H = np.zeros((dim, dim), dtype=complex)
    for i in range(N - 1):
        H += -J * two_site_xx(N, i)
    for i in range(N):
        H += -h * local_pauli(N, i, "Z")
    return (H + H.conj().T) / 2


def thermal_probs(E, beta):
    Emin = np.min(E.real)
    w = np.exp(-beta * (E.real - Emin))
    return w / np.sum(w)


def project_to_density(rho, eps=1e-14):
    rho = 0.5 * (rho + rho.conj().T)
    evals, evecs = la.eigh(rho)
    evals = np.maximum(evals.real, eps)
    rho_psd = (evecs * evals) @ evecs.conj().T
    rho_psd = 0.5 * (rho_psd + rho_psd.conj().T)
```

```
    rho_psd = rho_psd / np.trace(rho_psd)
    return rho_psd


def rel_entropy(rho, sigma):
    rho = 0.5 * (rho + rho.conj().T)
    sigma = 0.5 * (sigma + sigma.conj().T)
    return float(np.trace(rho @ (la.logm(rho) - la.logm(sigma))).real)


def lindblad_omega0_L(rho, S0, gamma0_val):
    S0sq = S0 @ S0
    return gamma0_val * (S0 @ rho @ S0 - 0.5 * (S0sq @ rho + rho @ S0sq))


def main():
    args = parse_args()
    np.random.seed(args.seed)

    N, J, h, beta = args.N, args.J, args.h, args.beta
    j0 = args.j0 if args.j0 is not None else (N // 2 if args.center else 0)

    H = tfim_hamiltonian(N, J, h)
    E, V = la.eigh(H)
    Vdag = V.conj().T
    p = thermal_probs(E, beta)

    S = local_pauli(N, j0, args.Sop)
    S_E = Vdag @ S @ V

    S0_E = np.diag(np.diag(S_E))
    S0 = V @ S0_E @ Vdag

    m, n = args.m, args.n
    omega = abs(float((E[m] - E[n]).real))
    smsn = abs((S_E[m, m] - S_E[n, n]).real)
    gamma0_val = float(args.gamma0 / beta)

    print(f"Fixed pair (m,n)=({m},{n}), omega≈{omega:.12g}")
    print(f"beta={beta}, gamma0={args.gamma0}, gamma(0)=gamma0/beta={gamma0_val}")
    print(f"p_m={p[m]:.6g}, p_n={p[n]:.6g}, |s_m-s_n|={smsn:.6g}")

    alphas = np.linspace(1e-4, args.alpha_max, args.nsamples)
    dt = args.dt

    diag_E = np.diag(p.astype(complex))

    print("\nalpha C(rho) dotC_loss dotC_loss/C")
    last_ratio = None
    for alpha in alphas:
        rho_E = diag_E.copy()
        amp = alpha * np.sqrt(p[m] * p[n])
        rho_E[m, n] += amp
        rho_E[n, m] += amp

        rho = project_to_density(V @ rho_E @ Vdag, eps=args.psd_clip)

        rho_E_chk = Vdag @ rho @ V
```

```python
        Delta_rho_E = np.diag(np.diag(rho_E_chk))
        Delta_rho = project_to_density(V @ Delta_rho_E @ Vdag, eps=args.psd_clip)

        C = rel_entropy(rho, Delta_rho)

        rho_t = project_to_density(
            rho + dt * lindblad_omega0_L(rho, S0, gamma0_val),
            eps=args.psd_clip
        )

        rho_t_E = Vdag @ rho_t @ V
        Delta_rho_t_E = np.diag(np.diag(rho_t_E))
        Delta_rho_t = project_to_density(V @ Delta_rho_t_E @ Vdag, eps=args.psd_clip)

        C_t = rel_entropy(rho_t, Delta_rho_t)
        dotC_loss = -(C_t - C) / dt
        ratio = dotC_loss / C if C > 0 else float("nan")
        last_ratio = ratio
        print(f"{alpha: .6f} {C: .6e} {dotC_loss: .6e} {ratio: .6g}")

    print(f"\nRESULT r(alpha={args.alpha_max}) = {last_ratio}")


if __name__ == "__main__":
    main()
```

References

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