Mersenne Block Dynamics: A Framework for the Collatz Conjecture

Stephen R. Campbell
Simon Fraser University
sencael@sfu.ca

December 18, 2025

Abstract

We introduce *Mersenne block dynamics*, a bit-level structural decomposition of the accelerated Collatz (Syracuse) map on odd integers based on the 2-adic valuation $n(x) := \nu_2(x+1)$. This yields the canonical decomposition $x = P(x)\,2^{n(x)} + (2^{n(x)} - 1)$, where $n(x)$ is the length of the *Mersenne tail*. Using the associated odd factor $a(x)$, we partition each Syracuse orbit into deterministic *Mersenne blocks*. Inside a block, the odd values increase strictly while the tail length decreases by one at each step, producing a rigid "wedge" pattern in the binary expansion. The exit from a block is governed by the exponent $r(x) := \nu_2(3^{n(x)}a(x) - 1)$ and induces a coarse-grained *block map* $B(x) := S^{n(x)}(x)$. We derive explicit transition identities for B and exact step-count bookkeeping across the time scales $C \to T \to S \to B$. Using the exact natural-density law for (n(x),r(x)), and assuming an orbit-mixing hypothesis for typical B-orbits, we model successive block parameters (n(xk) and r(xk)) as independent and identically distributed (i.i.d.) geometric(1/2) random variables. This intrinsic statistical model predicts an expected logarithmic drift of $2\log_2(3/2) - 2 \approx -0.83$ bits per block step, recovering the classical probabilistic heuristic within a structural coordinate system.

2020 Mathematics Subject Classification: Primary 11B83; Secondary 37P99.
Keywords: Collatz conjecture, Syracuse map, Mersenne blocks, 2-adic valuations, probabilistic models.

Contents

Introduction

The *Collatz conjecture* (the $3x+1$ problem) asks whether repeated iteration of

$$C(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even,} \end{cases} \qquad n \in \mathbb{N},$$

eventually reaches 1 for every starting value $n \geq 1$. Despite its elementary definition, the conjecture remains open and is widely regarded as extraordinarily difficult [8]. It has been verified computationally to very large bounds (see, e.g., [13, 10] and references therein), and no nontrivial cycles are known; recent work of Hercher rules out all odd m-cycles up to length 91 [5]. On the analytic side, results of Terras and Tao provide rigorous evidence of "typical" descent (in suitable senses) without resolving convergence for *all* starting values [16, 15].

A standard reduction (formalized by Möller [12]) compresses Collatz dynamics to odd integers by folding each odd step together with all subsequent divisions by 2. This yields the *accelerated Collatz* (or *Syracuse*) map S on odd integers,

$$S(x) = \frac{3x+1}{2^{\nu_2(3x+1)}},$$

so that $S(x)$ is the next odd value after x in the original Collatz trajectory. Between C and S, one also encounters the intermediate Terras acceleration [16],

$$T(n) = \begin{cases} n/2 & n \text{ even,} \\ (3n+1)/2 & n \text{ odd,} \end{cases}$$

giving a natural ladder of time scales $C \to T \to S$ from "step-by-step" to "odd-to-odd" dynamics.

Mersenne block dynamics as a canonical coarse-graining. This paper introduces a further acceleration of the Syracuse dynamics obtained by grouping the orbit into *variable-length blocks* determined directly by the binary expansion of the current odd value. For an odd integer x, define

$$n(x) := \nu_2(x+1) \geq 1,$$

so $n(x)$ is the length of the trailing run of 1's in the binary expansion of x (the *Mersenne tail*). Equivalently, writing $x+1 = 2^{n(x)}a(x)$ with $a(x)$ odd yields the canonical decomposition

$$x = (a(x)-1)\,2^{n(x)} + \left(2^{n(x)}-1\right), \qquad a(x) := \frac{x+1}{2^{n(x)}} \text{ odd.}$$

This tail decomposition partitions each Syracuse orbit into contiguous *Mersenne blocks*. Starting from a *block start* x, the block consists of the $n(x)$ odd values

$$x,\ S(x),\ S^2(x),\ \ldots,\ S^{n(x)-1}(x),$$

whose trailing-1 run decreases deterministically in length from $n(x)$ down to 1, producing a rigid right-angled triangular "wedge" pattern in the trailing bits (Figure 2). Inside a block the dynamics is fully explicit (Theorem 3.1) and strictly increasing (Proposition 3.4); in particular, the first $n(x)-1$ Syracuse steps in a nondegenerate block are forced *stair steps* in the valuation sense $s(u) := \nu_2(3u+1) = 1$ (Lemma 3.7). Thus the only nontrivial 2-adic division in a block is concentrated in its final *exit* step.

The exit is measured by the exponent

$$r(x) := \nu_2\big(3^{n(x)}a(x) - 1\big),$$

and it induces a coarse-grained *block map*

$$B(x) := S^{n(x)}(x),$$

which jumps from one block start to the next. On this scale, the Collatz conjecture is equivalent to the pure block statement that every odd $x \geq 1$ eventually satisfies $B^k(x) = 1$ (Theorem 2.6).

Main results snapshot. Deterministically, the Mersenne Block Dynamics Framework (MBDF) yields an explicit closed form for the Syracuse iterates inside each block and an exact start-to-start transition identity for the induced block map B (Theorems 3.1 and 4.4), together with exact ratio/time bookkeeping and an explicit correction term for the block ratio $\frac{B(x)}{x}$ (Corollaries 4.12 and 4.5). We also characterize the arithmetic image and inverse structure of B: one always has $3 \nmid B(x)$ (Proposition 4.15), and B is surjective onto the admissible odd set $\{y : y \text{ odd}, 3 \nmid y\}$, in fact with infinitely many preimages at every fixed block length n (Theorem 4.17). Finally, the intrinsic joint law of $(n(x), r(x))$ is *exactly* independent geometric(1/2) in natural density (Proposition 5.2); assuming orbit-mixing along typical B-orbits (Heuristic 5.3), this intrinsic law recovers the classical negative expected logarithmic drift $2\log_2(3/2) - 2 \approx -0.83007$ per block step in block coordinates (Proposition 5.6).

What the framework delivers (and what it does not). We do *not* prove the Collatz conjecture. The purpose of the MBDF is to provide a structural coordinate system in which (i) the deterministic part of the dynamics is isolated and solvable in closed form, and (ii) the remaining arithmetic difficulty is localized to the behavior of the exit exponent $r(x)$ along block orbits. Concretely, this manuscript provides:

(i) **Explicit intra-block mechanics and a rigid bit-level wedge:** closed forms for $S^j(x)$ across a block and a deterministic decrement of the tail length at each step (Theorem 3.1, Corollary 3.2).

(ii) **Exact transition identities and time bookkeeping across scales:** a direct start-to-start block transition formula (Theorem 4.4) together with exact step-count relations across $C \to T \to S \to B$ (Corollary 4.12); in particular, one block step corresponds to exactly n Syracuse steps, $n+r$ Terras steps, and $2n+r$ Collatz steps, with natural-density expectations $E[n] = 2$, $E[n+r] = 4$, $E[2n+r] = 6$ (Proposition 2.12).

(iii) **A clean dominant factor plus an exact correction term:** an exact ratio identity for $B(x)/x$ which isolates the dominant heuristic factor $3^{n(x)}/2^{n(x)+r(x)}$ and controls the remaining correction (Corollary 4.5). This makes precise what is dropped when one models block ratios by $3^n/2^{n+r}$.

(iv) **Residue-class control of exits and a block-level inverse picture:** a congruence characterization of $r(x)$ (Lemma 4.13) and the resulting geometric residue-class law for r at fixed n (Corollary 4.14), together with a modular description of the image of B and explicit infinite families of preimages. In particular, $B(x)$ is never divisible by 3 (Proposition 4.15), and B is surjective onto the admissible odd set $\{y : \ y \text{ odd}, 3 \nmid y\}$; moreover, this surjectivity holds for every fixed block length n (Theorem 4.17).

(v) **Exact intrinsic statistics and an explicit single heuristic input:** the joint law of $(n(x), r(x))$ is *exactly* independent geometric(1/2) in natural density (Proposition 5.2). To pass from these static residue-class laws to orbit-level predictions along a deterministic B-orbit, we isolate a single non-rigorous assumption—an orbit-mixing hypothesis for typical block orbits (Heuristic 5.3). Under this hypothesis the classical expected logarithmic drift is recovered in block coordinates (Proposition 5.6).

The upshot is a sharp separation: *static* congruence structure and intrinsic distributions are proved exactly, while *dynamical* independence/mixing along orbits is explicitly identified as the remaining heuristic gap. Orbit-level diagnostics and reproducibility details are provided in the computational appendix.

Relation to classical encodings. Many standard approaches encode accelerated Collatz dynamics by recording parity vectors or, equivalently, the valuation sequence $s_j := \nu_2(3x_j + 1)$ along a Syracuse orbit $x_{j+1} = S(x_j)$; see, e.g., [8, 17, 9]. MBDF can be viewed as a coarse-graining of this encoding: if x is a block start, then $s(S^j(x)) = 1$ for $0 \le j \le n(x) - 2$ (Lemma 3.7), while the unique non-stair valuation completing the block satisfies $s(L(x)) = 1 + r(x)$ (Lemma 4.10). Thus the pair $(n(x), r(x))$ packages long forced runs of $s = 1$ into a single block-length parameter and isolates all nontrivial 2-adic division to the exit exponent. This viewpoint is also compatible with 2-adic formulations (e.g. the Bernstein–Lagarias conjugacy map) in which valuations and digit patterns govern time scales [1]. Finally, from the inverse-iteration (preimage tree/graph) perspective [9, 17], MBDF supplies a residue-class-friendly organization of backward branches for the coarse-grained map B via explicit preimage families (Theorem 4.17).

Organization. Section 1 introduces the tail decomposition and defines Mersenne blocks and wedges. Section 2 defines the block map B and proves the equivalence between block and Syracuse formulations of Collatz, together with time-scale comparisons. Section 3 develops the rigid intra-block dynamics, including a matrix formulation. Section 4 derives explicit exit and transition formulas for B in terms of $(n(x), r(x))$ and records a sufficient condition for a net block contraction. Section 5 proves the intrinsic residue-class statistics and derives the expected drift under the orbit-mixing hypothesis. We conclude with a summary, outlook, and concrete prospects for further work.

1 An Overview of Mersenne Block Dynamics

To understand the long-term behavior of the Collatz map, we introduce a structural framework called *Mersenne Block Dynamics*. This framework shifts the perspective from the step-by-step iteration of individual integers (be they both odd and even, or just odd) to the dynamics of Mersenne blocks, the deterministic activities within them, transitions between them, and their global orbits. In this overview, we establish the fundamental definitions of the Collatz, Terras, and Syracuse maps, and the canonical decomposition of odd integers that underpins the entire framework.

1.1 The Collatz, Terras, and Syracuse maps

For a nonzero integer m, let $\nu_2(m)$ denote the 2-adic valuation, i.e. the largest $k \geq 0$ with $2^k \mid m$.

Definition 1.1 (Collatz, Terras, and Syracuse maps). The *Collatz map* $C : \mathbb{N} \to \mathbb{N}$ is

$$C(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

The *Terras map* $T : \mathbb{N} \to \mathbb{N}$ is

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Equivalently, $T(n) = C(n)$ for even n and $T(n) = C^2(n)$ for odd n.

The *accelerated Collatz* or *Syracuse* map S acts on odd integers by

$$S(x) := \frac{3x+1}{2^{\nu_2(3x+1)}}, \qquad x \text{ odd,}$$

so that $S(x)$ is always odd.

Each step of S corresponds to one application of $x \mapsto 3x+1$ followed by *all* possible divisions by 2, i.e. one full Collatz "odd-to-odd" segment.

Remark 1.2 (Equivalent formulations and time scales). It is standard (e.g., [8]) that the classical Collatz conjecture

$$\forall n \geq 1,\ \exists i \geq 0 :\ C^i(n) = 1$$

is equivalent to either of the accelerated formulations

$$\forall n \geq 1,\ \exists t \geq 0 :\ T^t(n) = 1 \qquad \text{and} \qquad \forall x \geq 1 \text{ odd},\ \exists j \geq 0 :\ S^j(x) = 1.$$

In this manuscript we work primarily on the odd integers with S (and with the further coarse-grained block map B defined in §2). The maps C and T will be used mainly for time-scale comparisons and for connecting with the existing literature.

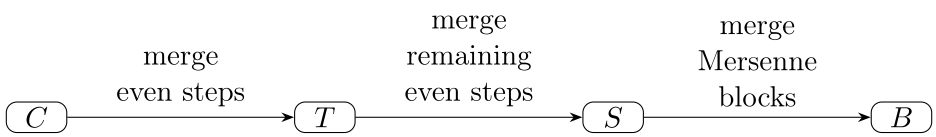

Figure 1: A hierarchy of time scales. The Collatz map C is the finest scale; the Terras map T folds each odd step together with one division by 2; the Syracuse map S folds each odd step together with all divisions by 2 up to the next odd integer; and the block map B (§2) folds whole Mersenne blocks into a single step.

1.2 Notation and conventions

- **Iteration indices.** We use i for C^i (Collatz time), t for T^t (Terras time), j for S^j (Syracuse time), and k for B^k (Mersenne Block time).
- **Domain.** Unless explicitly stated otherwise, symbols such as x denote positive *odd* integers. The functions $n(x)$, $a(x)$, $P(x)$, $L(x)$, and $r(x)$ are only defined for odd x.
- **Blocks vs. exits.** A *Mersenne block* rooted at x will mean the wedge segment $x, S(x), \ldots, S^{n(x)-1}(x)$. Its last value $L(x) := S^{n(x)-1}(x)$ is the *block peak*. The next odd iterate $B(x) := S^{n(x)}(x)$ begins the next block, and the step $L(x) \mapsto B(x)$ is the *block exit*.

1.3 Bit strings, Mersenne tails, and even prefix segments

We now isolate the maximal trailing run of ones in the binary expansion of an odd integer.

Lemma 1.3 (Canonical Mersenne-tail decomposition)**.** *Let $x \geq 1$ be odd and define*

$$n(x) := \nu_2(x+1) \ (\geq 1).$$

Then there is a unique even integer $P(x) \geq 0$ such that

$$x = P(x)\, 2^{n(x)} + \big(2^{n(x)} - 1\big). \tag{1}$$

Equivalently, $x + 1 = 2^{n(x)} a(x)$ with

$$a(x) := \frac{x+1}{2^{n(x)}} \quad odd, \qquad P(x) = a(x) - 1 \quad even.$$

Proof. Since x is odd, $x+1$ is even, so we can write

$$x + 1 = 2^n a,$$

with $n = \nu_2(x+1) \geq 1$ and a odd. Then

$$x = 2^n a - 1 = (a-1)2^n + 2^n - 1,$$

so setting $P(x) := a-1$ and $n(x) := n$ gives (1) with $P(x)$ even.

For uniqueness, suppose

$$x = P2^n + (2^n - 1) = P'2^{n'} + (2^{n'} - 1)$$

with P, P' even and $n, n' \geq 1$. Then

$$x + 1 = 2^n(P+1) = 2^{n'}(P'+1),$$

and $P+1$, $P'+1$ are odd. Hence $n = n'$ and $P+1 = P'+1$, so $P = P'$. □

Definition 1.4 (Mersenne tail, even prefix segment, and odd factor). For odd $x \geq 1$ we define:

- the *Mersenne-tail length*
$$n(x) := \nu_2(x+1) \ (\geq 1);$$
- the *odd factor*
$$a(x) := \frac{x+1}{2^{n(x)}} \quad \text{(odd)};$$
- the *even prefix segment*
$$P(x) := a(x) - 1 \quad \text{(even)}.$$

Equivalently, $n(x)$ is the length of the trailing run of 1's in the binary expansion of x.

Remark 1.5 (Binary picture and Mersenne tail). Write $(y)_2$ for the binary expansion of a nonnegative integer y as a bit string. Since $P(x)$ is even, $(P(x))_2$ ends in a 0-bit. The decomposition (1) says that in base 2 we can write x as

$$x = \underbrace{(P(x))_2}_{\text{ends with a 0-bit}} \ \underbrace{11\cdots1}_{n(x) \text{ ones}}$$

The trailing block of $n(x)$ ones is the *Mersenne tail* of x. The special case $P(x) = 0$ gives the pure Mersenne numbers $2^{n(x)} - 1$, whose entire binary expansion is the tail.

For example, $x = 27$ has $(27)_{10} = (11011)_2$ with a trailing run of two 1's, so $n(27) = \nu_2(28) = 2$ and $(P(27))_2 = (110)_2$.

1.4 Definition of Mersenne blocks and wedges

Definition 1.6 (Mersenne block and Mersenne wedge). Let $x \geq 1$ be odd and set $n := n(x) = \nu_2(x+1) \geq 1$.

- The finite Syracuse segment
$$x,\ S(x),\ S^2(x), \ldots, S^{n-1}(x)$$
is called the *Mersenne block* rooted at x. It contains n odd terms and $n-1$ Syracuse steps. We refer to x as the *block start*.

- The last value in the block,
$$L(x) := S^{n-1}(x),$$
is called the *block peak*. The next odd iterate
$$B(x) := S^n(x)$$
Equivalently, because $n := n(x)$, the start-to-start block transition consists of $n(x)$ Syracuse steps: $B(x) = S^{n(x)}(x)$.

- In binary, the iterates in the block have Mersenne-tail lengths
$$n,\ n-1,\ \dots,\ 1.$$
If we stack their bit strings vertically (with $S^0(x) = x$ on the top row and $S^j(x)$ on row j below it, for $0 \le j \le n-1$), the ones in the tails form a stair-step right-angled triangular pattern. We call this triangle of ones the *Mersenne wedge rooted at x* (See Corollary 3.2).

Remark 1.7 (Degenerate and nondegenerate blocks). If $n(x) = 1$, then the Mersenne block at x consists only of the single value x; there is no visible wedge. For a Mersenne tail of length $n(x) \ge 2$ we obtain a genuine wedge of height $n(x)$ in the trailing bits. For example, if $x = 31$ then $n(x) = \nu_2(32) = 5$ and the wedge has the schematic form

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{array}$$

The full Syracuse iteration of the block, including even prefixes $P(S^{n(x)}(x))$, is as follows:

$$\begin{array}{cccccccc} & & & 1 & 1 & 1 & 1 & 1 \\ & & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Remark 1.8 (Bit-string vs block dynamics). We will consistently distinguish:

- *bit-string dynamics*: how the bits of a single integer x evolve under one step $x \mapsto S(x)$;
- *block dynamics*: how finite segments of the Syracuse orbit, grouped into Mersenne blocks, evolve as we apply a coarse-grained block map B.

Mersenne tails and the triple $(n(x), a(x), P(x))$ are bit-string objects. Mersenne blocks, Mersenne wedges, and the block map B (defined in Section 2 below) live at the block level.

j	$S^j(27)$	k	$B^k(27)$	$n(x)$	Mersenne Block
0	27	0	27	$n = 2$	11011
1	41				101001
2	31	1	31	$n = 5$	11111
3	47				101111
4	71				1000111
5	107				1101011
6	161				10100001
7	121	2	121	$n = 1$	1111001
8	91	3	91	$n = 2$	1011011
9	137				10001001
10	103	4	103	$n = 3$	1100111
11	155				10011011
12	233				11101001
13	175	5	175	$n = 4$	10101111
14	263				100000111
15	395				110001011
16	593				1001010001
17	445	6	445	$n = 1$	110111101
18	167	7	167	$n = 3$	10100111
19	251				11111011
20	377				101111001
21	283	8	283	$n = 2$	100011011
22	425				110101001
23	319	9	319	$n = 6$	100111111
24	479				111011111
25	719				1011001111
26	1079				10000110111
27	1619				11001010011
28	2429				100101111101
29	911	10	911	$n = 4$	1110001111
30	1367				10101010111
31	2051				100000000011
32	3077				110000000101
33	577	11	577	$n = 1$	1001000001
34	433	12	433	$n = 1$	110110001
35	325	13	325	$n = 1$	101000101
36	61	14	61	$n = 1$	111101
37	23	15	23	$n = 3$	10111
38	35				100011
39	53				110101
40	5	16	5	$n = 1$	101
41	1	17	1	$n = 1$	1

Figure 2: Evolution of the Syracuse orbit for $x_0 = 27$ under Mersenne Block Dynamics. The left columns track the standard Syracuse steps $S^j(27)$, while the center columns identify the coarse-grained Block starts $B^k(27)$ and their tail lengths $n(x)$. The right panel visualizes the binary expansion, where red shaded regions indicate nondegenerate Mersenne wedges ($n \geq 2$) showing the deterministic decay of trailing ones.

Remark 1.9 (A tripartite classification). The Mersenne Block Dynamics Framework (MBDF) naturally separates into three overlapping viewpoints:

- *Inter-block dynamics*: the coarse-grained orbit of block starts under the Mersenne block map B.
- *Intra-block dynamics*: the deterministic evolution of bit strings *inside* a single block, i.e. the wedge segment $x, S(x), \ldots, S^{n(x)-1}(x)$ and its associated Mersenne wedge.
- *Trans-block dynamics*: the bit-level bookkeeping needed to pass from one block to the next. This can be viewed either
  - coarsely, as the start-to-start transition $x \mapsto B(x)$, or
  - finely, as the peak-to-next-start transition $L(x) \mapsto B(x)$ (one Syracuse step).

  In this paper, trans-block dynamics refers primarily to the arithmetic of the block exit exponent $r(x)$ and its effect on $x \mapsto B(x)$.

We will discuss each of these three areas of the MBDF in greater detail.

2 Mersenne Inter-Block Dynamics

Having established the structural units, we now define the *Mersenne block map* B. This map operates on the largest scale, stitching together the blocks to form the global orbit.

2.1 The Mersenne block map

Definition 2.1 (Mersenne block map B). For odd $x \geq 1$, let $n(x)$ be as in Lemma 1.3. The *Mersenne block map* B is

$$B(x) := S^{n(x)}(x).$$

Thus $B(x)$ is the next odd value reached after traversing the Mersenne block rooted at x and taking its exit step: the start of the next block.

Proposition 2.2 (Unfolding the Syracuse orbit from B)**.** *Let $x_0 \geq 1$ be odd, and define the block starts*

$$x_k := B^k(x_0) \qquad (k \geq 0),$$

with corresponding block lengths $n_k := n(x_k)$. Set $N_0 := 0$ and

$$N_{k+1} := N_k + n_k \qquad (k \geq 0),$$

so that $N_k = \sum_{r=0}^{k-1} n(x_r)$ for $k \geq 1$. Then for every $k \geq 0$ and every $0 \leq j \leq n_k - 1$,

$$S^{N_k+j}(x_0) = S^j(x_k).$$

Consequently, the odd-only Syracuse orbit of x_0 is obtained by concatenating the successive Mersenne blocks

$$x_k,\ S(x_k),\ \ldots,\ S^{n(x_k)-1}(x_k) \qquad (k = 0, 1, 2, \ldots),$$

stitched together by the block map B.

Proof. We first prove by induction on k that $S^{N_k}(x_0) = x_k$ for all $k \geq 0$. For $k = 0$ this is immediate since $N_0 = 0$ and $x_0 = B^0(x_0)$. Assume $S^{N_k}(x_0) = x_k$. Then using $N_{k+1} = N_k + n_k$ and $n_k = n(x_k)$,

$$S^{N_{k+1}}(x_0) = S^{N_k+n_k}(x_0) = S^{n_k}\big(S^{N_k}(x_0)\big) = S^{n_k}(x_k) = B(x_k) = x_{k+1}.$$

Now for $0 \leq j \leq n_k - 1$,

$$S^{N_k+j}(x_0) = S^j\big(S^{N_k}(x_0)\big) = S^j(x_k),$$

which gives the claimed block-by-block concatenation. □

2.2 Collatz in Mersenne block form

Conjecture 2.3 (Collatz in block form)**.** *For every odd integer $x \geq 1$ there exists $k \geq 0$ such that*

$$B^k(x) = 1.$$

Equivalently, every B-orbit on odd integers eventually hits the fixed point 1.

Conjecture 2.4 (Collatz in Syracuse form)**.** *For every odd integer $x \geq 1$ there exists $j \geq 0$ such that*

$$S^j(x) = 1.$$

Remark 2.5. $B(1) = 1$ since $S(1) = 1$ and $n(1) = \nu_2(2) = 1$.

Theorem 2.6 (Equivalence of block and Syracuse formulations)**.** *Conjectures 2.3 and 2.4 are equivalent.*

Proof. Assume Conjecture 2.3. Let x_0 be odd, and suppose $B^k(x_0) = 1$ for some k. By definition, each application of B corresponds to $n(\cdot)$ applications of S, so there is some $j \geq 0$ with $S^j(x_0) = B^k(x_0) = 1$. This gives Conjecture 2.4.

Conversely, assume Conjecture 2.4. Let x_0 be odd, and let j be the least integer with $S^j(x_0) = 1$. The odd values along the Syracuse orbit of x_0 are exactly the block starts and the interior points of the Mersenne blocks. The block starts are precisely the iterates $B^k(x_0)$, and 1 itself is a block start (since $B(1) = 1$). Hence $S^j(x_0)$ coincides with some $B^k(x_0)$, and so $B^k(x_0) = 1$. This is Conjecture 2.3. □

Remark 2.7. If $S^j(x_0) = 1$ occurs at a block start, then $B^k(x_0) = 1$ for that block index; if it occurs at a block peak, then the next block start is $S^{j+1}(x_0) = S(1) = 1$, so some iterate of B still hits 1.

Together with the standard equivalence between the Syracuse and classical Collatz conjectures, this shows that the Collatz problem can be stated entirely in terms of the block dynamics of B.

2.3 Total stopping times and time scales

Definition 2.8 (Total stopping times and Terras stopping time). Let $x_0 \geq 1$ be odd. Define:

- The *total Collatz stopping time* $\tau_C(x_0)$ as the least $i \geq 0$ with

$$C^i(x_0) = 1.$$

- The *total Terras stopping time* $\tau_T(x_0)$ as the least $t \geq 0$ with

$$T^t(x_0) = 1.$$

- The *total Syracuse stopping time* $\tau_S(x_0)$ as the least $j \geq 0$ with

$$S^j(x_0) = 1.$$

- The *total block stopping time* $\tau_B(x_0)$ as the least $k \geq 0$ with

$$B^k(x_0) = 1.$$

If no such i, t, j, k exist, the corresponding total stopping time is taken to be $+\infty$.

In addition, following Terras [16] we define the (Terras) *stopping time*

$$\sigma_T(x_0) := \min\{t \geq 1 : \ T^t(x_0) < x_0\},$$

the first time the T-orbit falls below its starting value.

Lemma 2.9 (Finite-prefix time bookkeeping along the Syracuse skeleton). *Let $x_0 \geq 1$ be odd and fix an integer $J \geq 1$. Define the Syracuse prefix*

$$x_\ell := S^\ell(x_0) \qquad (0 \leq \ell \leq J),$$

and set

$$s_\ell := \nu_2(3x_\ell + 1) \qquad (0 \leq \ell \leq J-1).$$

Define the cumulative Terras and Collatz step counts

$$t_J := \sum_{\ell=0}^{J-1} s_\ell, \qquad c_J := \sum_{\ell=0}^{J-1}(1 + s_\ell) = t_J + J.$$

Then

$$T^{t_J}(x_0) = x_J \qquad \text{and} \qquad C^{c_J}(x_0) = x_J.$$

Proof. Fix $0 \le \ell \le J-1$. By definition of s_ℓ and S we have

$$3x_\ell + 1 = 2^{s_\ell} x_{\ell+1}.$$

Terras time. Starting at the odd value x_ℓ, one Terras odd-step gives

$$T(x_\ell) = \frac{3x_\ell + 1}{2} = 2^{s_\ell - 1} x_{\ell+1},$$

and then $s_\ell - 1$ further Terras steps divide by 2 until reaching $x_{\ell+1}$. Hence

$$T^{s_\ell}(x_\ell) = x_{\ell+1}.$$

Concatenating these segments for $\ell = 0, 1, \ldots, J-1$ yields

$$T^{t_J}(x_0) = x_J.$$

Collatz time. Starting at the odd value x_ℓ, one Collatz odd-step gives $3x_\ell + 1$, and then s_ℓ successive Collatz even-steps divide by 2 until reaching $x_{\ell+1}$. Hence

$$C^{1+s_\ell}(x_\ell) = x_{\ell+1}.$$

Concatenating these segments for $\ell = 0, 1, \ldots, J-1$ yields

$$C^{c_J}(x_0) = x_J.$$

□

Corollary 2.10 (Stopping-time identities under finiteness)**.** *Let $x_0 \ge 1$ be odd and assume $\tau_S(x_0) < \infty$. Let $x_\ell := S^\ell(x_0)$ for $0 \le \ell \le \tau_S(x_0)$ and set $s_\ell := \nu_2(3x_\ell + 1)$ for $0 \le \ell \le \tau_S(x_0) - 1$. Then*

$$\tau_T(x_0) = \sum_{\ell=0}^{\tau_S(x_0)-1} s_\ell, \qquad \tau_C(x_0) = \sum_{\ell=0}^{\tau_S(x_0)-1} (1 + s_\ell) = \tau_T(x_0) + \tau_S(x_0).$$

In particular, $\tau_C(x_0) \ge \tau_T(x_0) \ge \tau_S(x_0)$ and $\tau_C(x_0) \ge 2\tau_S(x_0)$. If $x_0 > 1$, then $\tau_C(x_0) > \tau_T(x_0) > \tau_S(x_0) \ge 1$.

Proof. Apply Lemma 2.9 with $J = \tau_S(x_0)$, so that $x_J = 1$. This shows that $T^{t_J}(x_0) = 1$ and $C^{c_J}(x_0) = 1$, where $t_J = \sum_{\ell=0}^{J-1} s_\ell$ and $c_J = \sum_{\ell=0}^{J-1}(1+s_\ell)$. Thus $\tau_T(x_0) \le t_J$ and $\tau_C(x_0) \le c_J$.

To see these are equalities, note that for each $\ell < J$ the Terras segment from x_ℓ to $x_{\ell+1}$ has exactly s_ℓ steps and the intermediate values are $2^{s_\ell - 1}x_{\ell+1}, 2^{s_\ell - 2}x_{\ell+1}, \ldots, 2x_{\ell+1}, x_{\ell+1}$. Hence this segment hits 1 if and only if $x_{\ell+1} = 1$, and then only at the final step. Since $x_J = 1$ is the *first* occurrence of 1 along the Syracuse orbit by definition of $J = \tau_S(x_0)$, the first occurrence of 1 along the Terras orbit occurs at time t_J. Therefore $\tau_T(x_0) = t_J$. The same argument applies to the Collatz segments, giving $\tau_C(x_0) = c_J$.

The inequalities follow from $s_\ell \ge 1$ for all ℓ. If $x_0 > 1$, then $J = \tau_S(x_0) \ge 1$ and the final Syracuse step to reach 1 has $s_{J-1} \ge 2$, yielding $\tau_T(x_0) > \tau_S(x_0)$, and hence also $\tau_C(x_0) > \tau_T(x_0)$. □

Remark 2.11 (Time scales: C vs T vs S vs B). Conceptually, the four maps operate at different time resolutions:

$$C\,(\text{finest});\ T\,(\text{merge one }/2\text{ after odd step});\ S\,(\text{odd-to-odd});\ B\,(\text{block-to-block}).$$

Lemma 2.9 shows that each level compresses several steps of the previous one: for any fixed Syracuse prefix endpoint x_J, the Collatz and Terras step counts to reach x_J are larger than the Syracuse count J, while the block time count is smaller (depending on the block decomposition).

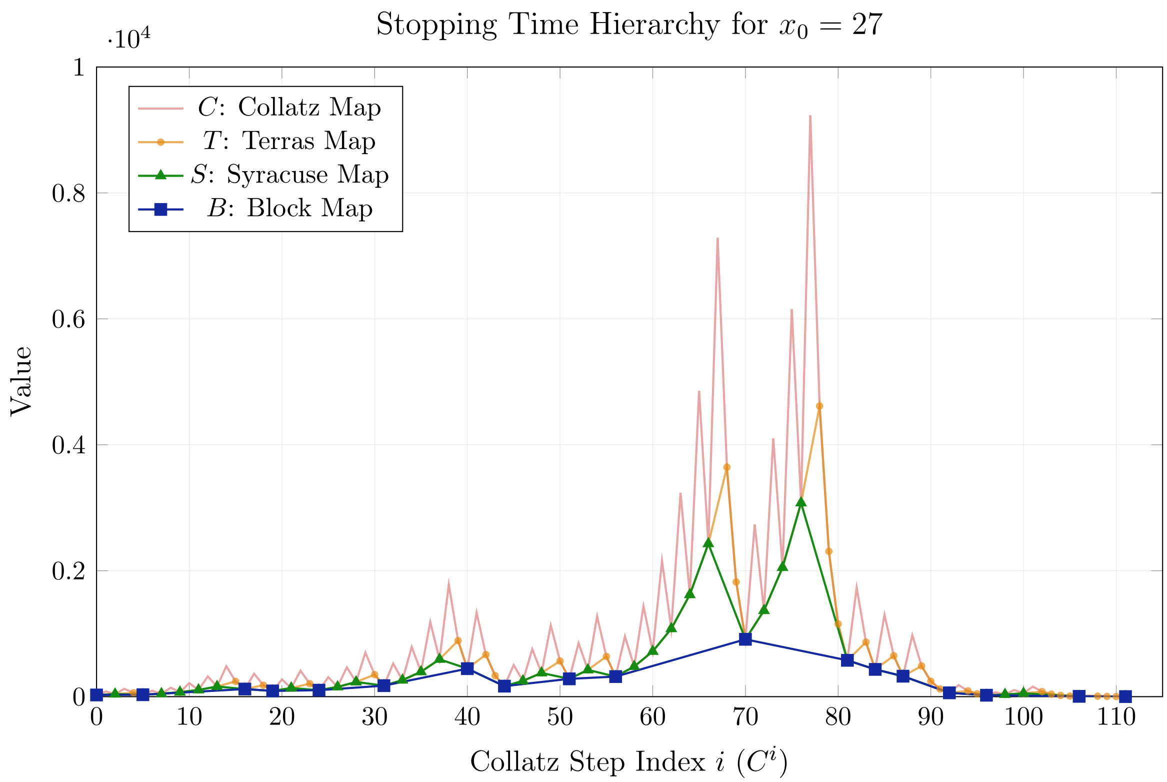

Figure 3: Evolution of the orbit for $x_0 = 27$ across four hierarchy levels. The fundamental Collatz trajectory $C^i(27)$ (red line) visits every integer. The accelerated maps show increasing levels of time compression: the Terras orbit $T^t(27)$ (orange circles) merges single parity steps; the Syracuse orbit $S^j(27)$ (green triangles) samples only odd integers; and the Mersenne Block orbit $B^k(27)$ (blue squares) achieves maximum compression by recording only the start of each Mersenne block. Note how the block map B skips the "intra-block" stair steps entirely, isolating the structural transitions.

Proposition 2.12 (Expected time compression per block step)**.** *Let x be odd, and set $n := n(x)$ and $r := r(x)$. The block transition $x \mapsto B(x)$ consists of exactly n Syracuse steps, $n + r$ Terras steps, and $2n + r$ Collatz steps. Moreover, with respect to natural density on odd integers,*

$$\mathbb{E}[n] = 2, \qquad \mathbb{E}[n + r] = 4, \qquad \mathbb{E}[2n + r] = 6.$$

Map f	$\tau_f(27)$	$\max\{f^t(27) : 0 \le t \le \tau_f(27)\}$	$\tau_C(27)/\tau_f(27)$
C	111	9232	1
T	70	4616	≈ 1.59
S	41	3077	≈ 2.71
B	17	911	≈ 6.53

Table 1: Total stopping times and maximal values for $x_0 = 27$ corresponding to Fig. 3. Here $\max\{f^t(27) : 0 \le t \le \tau_f(27)\}$ denotes the maximum value attained along the orbit under f before reaching 1.

In particular, one block step represents about 2 *Syracuse steps,* 4 *Terras steps, and* 6 *Collatz steps on average.*

Proof. The exact step counts are proved in Corollary 4.12. The expectations follow from the intrinsic geometric laws for $n(x)$ and $r(x)$ (Propositions 5.1 and 5.2), which give $\mathbb{E}[n] = \mathbb{E}[r] = 2$. □

Heuristic consequence. If successive block starts sample (n, r) with weak dependence as in Heuristic 5.3, then for typical x_0 with large $\tau_B(x_0)$ one expects $\tau_S(x_0) \approx 2\tau_B(x_0)$, $\tau_T(x_0) \approx 4\tau_B(x_0)$, and $\tau_C(x_0) \approx 6\tau_B(x_0)$ (see 5.7).

Remark 2.13 (Degenerate "no-wedge" class)*.* The equality case $\tau_S(x_0) = \tau_B(x_0)$ in Lemma 2.9 is structurally degenerate and occurs if and only if every Mersenne block along the B-orbit of x_0 has length $n(x_r) = 1$. In this case each block step is a single Syracuse step, so the Syracuse and block time scales coincide.

Equivalently, every odd iterate of the Syracuse orbit lies in the residue class $1 \pmod 4$, so the orbit never visits an odd integer $\equiv 3 \pmod 4$. In this "non-compressing" class of cases, the Mersenne wedges are degenerate, and the coarse-grained dynamics of B offers no time compression beyond the original Syracuse dynamics.

3 Mersenne Intra-Block Dynamics

We now describe the Syracuse dynamics inside a single Mersenne block in closed form.

Write $x = (P+1)2^n - 1$ with $P = P(x)$ even and $n = n(x) \ge 1$ as in Lemma 1.3 and Definition 1.4. Since P is even, the shifted factor $P + 1$ is odd, so it is natural to set

$$a := P + 1 = a(x) = \frac{x+1}{2^{n(x)}} \quad \text{(odd)}.$$

Equivalently, $x = 2^n a - 1$ and $P = a - 1$. We regard $P(x)$ as the canonical prefix parameter, and use $a(x) = P(x) + 1$ only as a notational convenience when it streamlines formulas.

Theorem 3.1 (Local Mersenne block dynamics for S)**.** *Let $x = (P+1)2^n - 1$ with P even and $n \ge 1$. Then for $0 \le j \le n-1$,*

$$S^j(x) = 3^j (P+1)\, 2^{n-j} - 1. \tag{2}$$

Proof. We use induction on j. For $j = 0$ we have $S^0(x) = x = (P+1)2^n - 1$, so (2) holds.

Assume (2) holds for some j with $0 \le j \le n-2$, i.e.,

$$S^j(x) = 3^j(P+1)2^{n-j} - 1.$$

Then

$$\begin{aligned} 3S^j(x) + 1 &= 3\big(3^j(P+1)2^{n-j} - 1\big) + 1 \\ &= 3^{j+1}(P+1)2^{n-j} - 3 + 1 \\ &= 3^{j+1}(P+1)2^{n-j} - 2 \\ &= 2\big(3^{j+1}(P+1)2^{n-j-1} - 1\big). \end{aligned}$$

The factor in parentheses is odd, so $\nu_2(3S^j(x)+1) = 1$ and

$$S^{j+1}(x) = \frac{3S^j(x)+1}{2} = 3^{j+1}(P+1)2^{n-j-1} - 1.$$

This is exactly (2) with j replaced by $j+1$. □

Corollary 3.2 (Bit-string structure along a wedge)**.** *In the setting of Theorem 3.1, for each $0 \le j \le n-1$ we can write*

$$S^j(x) = P_j\, 2^{n-j} + \big(2^{n-j} - 1\big),$$

where

$$P_j := 3^j(P+1) - 1$$

is an even integer. Thus:

- *the lower $n-j$ bits of $S^j(x)$ form a Mersenne tail of 1-bits;*
- *the bit immediately above the tail is 0 (since P_j is even);*
- *the higher bits encode the evolving even prefix segment P_j.*

As j increases from 0 to $n-1$, the tail length decreases from n down to 1. In particular,

$$S^j(x) + 1 = 2^{n-j}(P_j + 1),$$

and since P_j is even, $P_j + 1$ is odd. Hence

$$n\big(S^j(x)\big) = \nu_2\big(S^j(x) + 1\big) = n - j \qquad (0 \le j \le n-1),$$

so the Mersenne-tail length decreases deterministically by one at each step inside the block.

Remark 3.3 (Explicit evolution of the even prefix)*.* The sequence of even prefixes (P_j) satisfies the linear recurrence

$$P_{j+1} = 3P_j + 2, \qquad P_0 = P,$$

which solves to $P_j = 3^j(P+1) - 1$ as in Corollary 3.2. Thus the entire Mersenne wedge and, equivalently, the entire Mersenne block rooted at x is determined by the pair $(P(x), n(x))$.

Proposition 3.4 (Strict increase inside a Mersenne block)**.** *Let* $x = (P+1)2^n - 1$ *with* P *even and* $n \geq 1$. *Then*

$$S^0(x) < S^1(x) < \cdots < S^{n-1}(x).$$

Proof. From (2),

$$S^j(x) = 3^j(P+1)2^{n-j} - 1.$$

For $0 \leq j \leq n-2$,

$$\begin{aligned} S^{j+1}(x) - S^j(x) &= 3^{j+1}(P+1)2^{n-j-1} - 1 - \big(3^j(P+1)2^{n-j} - 1\big) \\ &= 3^j(P+1)\big(3 \cdot 2^{n-j-1} - 2^{n-j}\big) \\ &= 3^j(P+1)2^{n-j-1}(3-2) \\ &= 3^j(P+1)2^{n-j-1} > 0. \end{aligned}$$

□

Remark 3.5 (Local regularity vs global complexity). Within a single Mersenne block, the dynamics of S is as regular as one could hope for:

- the odd values are given by a simple explicit formula;
- they form a strictly increasing sequence;
- their tails shrink in a perfectly predictable way, forming a Mersenne wedge.

The intra-block evolution is explicit; the global difficulty is pushed into the exit exponent r(x) and its behavior along block orbits.

3.1 Stairs inside a block

Definition 3.6 (Step valuations, stairs, and exits). For odd $x \geq 1$ define

$$s(x) := \nu_2(3x+1) \ (\geq 1), \qquad x^+ := S(x) = \frac{3x+1}{2^{s(x)}}.$$

We say:

- the step $x \mapsto x^+$ is a *stair step* if $s(x) = 1$;
- the step $x \mapsto x^+$ is an *exit step* if $s(x) \geq 2$.

Lemma 3.7 (Stair valuations inside a block)**.** *Let* $x = (P+1)2^n - 1$ *with* $n = n(x) \geq 2$. *Then for* $0 \leq j \leq n-2$,

$$s\big(S^j(x)\big) = 1.$$

That is, the first $n(x) - 1$ *Syracuse steps starting at* x *are all stair steps.*

Proof. In the proof of Theorem 3.1 we saw that for $0 \leq j \leq n-2$,

$$3S^j(x) + 1 = 2\big(3^{j+1}(P+1)2^{n-j-1} - 1\big),$$

with the factor in parentheses odd. Thus $\nu_2(3S^j(x)+1) = 1$, i.e. $s(S^j(x)) = 1$. □

3.2 Matrix formulation of the intra-block stair dynamics

In this subsection we recast the rigid *intra-block* dynamics as the iteration of a single 2×2 integer matrix acting on integer pairs that encode rationals as ratios. This eliminates the need for induction and makes the "3^n expansion vs. 2^{n+r} contraction" balance completely transparent.

The odd-branch Terras map. Recall the Terras map T from Definition 1.1. On odd inputs it acts by

$$T_{\mathrm{odd}}(x) := \frac{3x+1}{2}. \tag{3}$$

Whenever $s(x) = \nu_2(3x+1) = 1$, the Syracuse map agrees with this odd branch:

$$S(x) = T(x) = T_{\mathrm{odd}}(x),$$

and the output is again odd. By Lemma 3.7, if x begins a Mersenne block of length $n(x)$ then $s\big(S^j(x)\big) = 1$ for $0 \le j \le n(x) - 2$, so along the Mersenne wedge we have

$$S^j(x) = T_{\mathrm{odd}}^j(x) \qquad (0 \le j \le n(x) - 1). \tag{4}$$

A matrix model via integer pairs. We encode a rational number as an integer pair (u, v) with $v \neq 0$, interpreted as the ratio u/v. In particular, we represent an integer x by the column vector

$$\mathbf{v}(x) := \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Then (3) is encoded by the matrix update

$$A := \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \in M_2(\mathbb{Z}), \qquad A\,\mathbf{v}(x) = \begin{pmatrix} 3x+1 \\ 2 \end{pmatrix}. \tag{5}$$

Interpreting $\binom{u}{v}$ as the rational number u/v recovers $T_{\mathrm{odd}}(x) = (3x+1)/2$. (Scaling $\binom{u}{v}$ by a nonzero integer does not change the ratio u/v.) This matrix viewpoint is purely bookkeeping: it just tracks the numerator and denominator of the rational iterate simultaneously, and the underlying map is recovered by taking the ratio of the two entries.

Closed form for A^j and for T_{odd}^j. Since A is triangular, its powers admit a closed form. For all $j \ge 0$,

$$A^j = \begin{pmatrix} 3^j & 3^j - 2^j \\ 0 & 2^j \end{pmatrix}. \tag{6}$$

Applying (6) to $\mathbf{v}(x)$ yields

$$T_{\mathrm{odd}}^j(x) = \frac{3^j x + (3^j - 2^j)}{2^j} = \frac{3^j(x+1) - 2^j}{2^j}. \tag{7}$$

Recovery of the intra-block formula. If x begins a Mersenne block of length $n = n(x) = \nu_2(x+1)$, write

$$x = 2^n a - 1 \qquad \text{with } a \text{ odd.} \tag{8}$$

Then (7) simplifies for every $0 \le j \le n$ to

$$T_{\text{odd}}^{j}(x) = 3^j a\, 2^{\,n-j} - 1. \tag{9}$$

In particular, for $0 \le j \le n-1$ we are still within the block and $S^j(x) = T_{\text{odd}}^{j}(x)$ by (4), so (9) reproduces Theorem 3.1 without induction.

Exit exponent and residue classes. The nth application of the odd branch produces the *raw* exit numerator

$$T_{\text{odd}}^{n}(x) = 3^n a - 1, \tag{10}$$

which is even. The true block exit divides by at least an additional power of 2:

$$r(x) := \nu_2\big(3^n a - 1\big), \qquad B(x) = S^n(x) = \frac{3^n a - 1}{2^{r(x)}}. \tag{11}$$

Equivalently, if we iterate the Terras map T (Definition 1.1) starting at odd x, then the first n iterates apply the odd branch, landing at $3^n a - 1$, and the next $r(x)$ iterates simply divide by 2. Thus

$$B(x) = T^{\,n(x)+r(x)}(x). \tag{12}$$

Expansion cost vs. total contraction across a block. The representative matrix A in (5) has eigenvalues 3 and 2; after n odd-branch steps the numerator scales like 3^n while the baseline denominator scales like 2^n, producing the familiar factor $(3/2)^n$. The additional division by $2^{r(x)}$ in (11) is what turns a typically-expanding stair regime into an overall-contracting block transition.

4 Mersenne Trans-Block Dynamics

The rigid intra-block phase described in §3 ends when the Mersenne tail has been stripped down to a single trailing 1. The next Syracuse step then performs a division by a (potentially large) power of 2 and lands at the start of the next Mersenne block. This *block exit* can be viewed at two resolutions:

- *coarsely*, as the direct start-to-start transition $x \mapsto B(x)$ (one step of the block map);
- *finely*, as the single Syracuse step from the block peak $L(x)$ to the next block start $B(x)$.

We treat the fine-grained picture first.

4.1 Block peak and exit exponent

Definition 4.1 (Block peak and exit exponent). Let x be odd and write

$$x = 2^{n(x)}a(x) - 1$$

with $a(x)$ odd. The *block peak* is

$$L(x) := S^{n(x)-1}(x),$$

and the *exit exponent* is

$$r(x) := \nu_2\big(3^{n(x)}a(x) - 1\big).$$

Lemma 4.2 (From block peak to next block start)**.** *Let x be odd with $n = n(x)$ and $a = a(x)$. Then*

$$L(x) = 2 \cdot 3^{n-1}a - 1, \qquad B(x) = \frac{3^n a - 1}{2^{r(x)}}.$$

Proof. By Theorem 3.1 with $j = n - 1$ (equivalently, (9) with $j = n - 1$), we have

$$L(x) = S^{n-1}(x) = 3^{n-1}a\, 2^{n-(n-1)} - 1 = 2 \cdot 3^{n-1}a - 1.$$

Then

$$S(L(x)) = \frac{3L(x) + 1}{2^{\nu_2(3L(x)+1)}} = \frac{3(2 \cdot 3^{n-1}a - 1) + 1}{2^{\nu_2(3^n a - 1)}} = \frac{3^n a - 1}{2^{r(x)}}.$$

But $S(L(x)) = S^n(x) = B(x)$ by Definition 1.6. □

Example 4.3 (A single block: $x = 15$). We have $15+1 = 16 = 2^4$, so $n(15) = 4$ and $a(15) = 1$. The Mersenne block rooted at 15 is

$$15 \xrightarrow{S} 23 \xrightarrow{S} 35 \xrightarrow{S} 53,$$

so the block peak is $L(15) = 53$. The raw exit numerator is $3^4 \cdot 1 - 1 = 80$, which has $r(15) = \nu_2(80) = 4$, hence the next block start is

$$B(15) = S^4(15) = \frac{80}{2^4} = 5.$$

In this example the block ratio is $B(15)/15 = 1/3$, while the dominant factor $3^n/2^{n+r} = 3^4/2^8 = 81/256 \approx 0.316$ differs from $1/3$ only by the small correction factor in (14).

4.2 Direct block transition theorem

Theorem 4.4 (Direct block transition theorem)**.** *Let $x_k = B^k(x_0)$ denote the sequence of block starts along a block orbit. Write each $x_k = 2^{n_k}a_k - 1$ with a_k odd, and set $r_k := r(x_k) = \nu_2(3^{n_k}a_k - 1)$. Then*

$$x_{k+1} = B(x_k) = \frac{3^{n_k}(x_k + 1) - 2^{n_k}}{2^{n_k + r_k}} = \frac{3^{n_k}a_k - 1}{2^{r_k}}. \tag{13}$$

Proof. By Lemma 4.2 applied to x_k, we have $x_{k+1} = B(x_k) = (3^{n_k}a_k - 1)/2^{r_k}$. Using $a_k = (x_k + 1)/2^{n_k}$ gives the alternative expression

$$x_{k+1} = \frac{3^{n_k}(x_k + 1) - 2^{n_k}}{2^{n_k+r_k}}.$$

□

Corollary 4.5 (Exact block ratio formula and error bound)**.** *Let x be odd with $x = 2^n a - 1$ and exit exponent $r = r(x)$. Then the block ratio is exactly*

$$\frac{B(x)}{x} = \frac{3^n}{2^{n+r}} \cdot \frac{1 - \frac{1}{3^n a}}{1 - \frac{1}{2^n a}}. \tag{14}$$

Moreover, the logarithmic deviation from the heuristic model is bounded by the reciprocal of the input size:

$$\left| \log \left(\frac{1 - \frac{1}{3^n a}}{1 - \frac{1}{2^n a}} \right) \right| \leq \frac{C}{2^n a} \tag{15}$$

for an absolute constant C. This confirms that the approximation $B(x)/x \approx 3^n/2^{n+r}$ becomes exponentially accurate as x grows.

Proof. The identity (14) follows by dividing $B(x) = (3^n a - 1)/2^r$ by $x = 2^n a - 1$ and factoring out the dominant terms. For the bound, let $\delta_1 = (3^n a)^{-1}$ and $\delta_2 = (2^n a)^{-1}$. For large x, these are small, and we apply the estimate $|\log(1 - y)| \leq 2|y|$ for small $|y|$. The logarithmic correction is

$$\Delta_{\text{err}} = \log(1 - \delta_1) - \log(1 - \delta_2).$$

Using the mean value theorem or Taylor expansion, $|\Delta_{\text{err}}| \approx |\delta_2 - \delta_1| < \delta_2 = \frac{1}{2^n a}$. Thus, the error is bounded by C/x for a suitable constant C, vanishing rapidly for large orbits. □

Remark 4.6 (Cost–revenue viewpoint for a block step). The direct transition formula (13) isolates two exponents associated with a block start x_k: the *Mersenne-tail length* $n_k = n(x_k)$ and the *exit exponent* $r_k = r(x_k)$. These govern the dominant multiplicative factor $3^{n_k}/2^{n_k+r_k}$ appearing in (14), while the remaining correction factor in (14) tends to 1 when $a_k = (x_k + 1)/2^{n_k}$ is large.

Heuristically, one may think of each block step as "paying" n_k multiplications by 3 (coming from the stair regime) and then "collecting" $n_k + r_k$ divisions by 2 at the boundary. The block map tends to contract when the exit exponent r_k is frequently large enough to offset the typical 3/2 growth inside the block.

4.3 The Mersenne block drop

The Collatz conjecture asserts that orbits fall to 1 eventually. This is stronger than merely requiring that the orbit's size decreases at some point. However, any finite total stopping time must include at least one drop below the starting value. The next theorem gives a sufficient criterion for when a drop in size occurs across a block.

Theorem 4.7 (Mersenne block drop theorem)**.** *Let $x > 1$ be odd with Mersenne tail length $n(x) = n$, odd factor $a(x) = a$, and exit exponent $r(x) = r$. If*

$$r > n \log_2(3/2) + 1, \tag{16}$$

then the block map contracts: $B(x) < x$.

Proof. We seek a sufficient condition for the contraction $B(x) < x$. First, observe the strict upper bound on the block map:

$$B(x) = \frac{3^n a - 1}{2^r} < \frac{3^n a}{2^r}.$$

Second, for $x > 1$, we have $2^n a \geq 2$, which implies the strict lower bound on x:

$$x = 2^n a - 1 > \frac{1}{2}(2^n a) = 2^{n-1} a.$$

Combining these, it is sufficient to enforce the condition $\frac{3^n a}{2^r} \leq 2^{n-1} a$, which ensures

$$B(x) < \frac{3^n a}{2^r} \leq 2^{n-1} a < x.$$

Dividing by a and rearranging $\frac{3^n}{2^r} \leq 2^{n-1}$ yields

$$3^n \leq 2^{n+r-1} \iff n \log_2 3 \leq n + r - 1.$$

Solving for r gives $r \geq n(\log_2 3 - 1) + 1 = n \log_2(3/2) + 1$. The hypothesis (16) satisfies this inequality strictly. ☐

Remark 4.8 (Zig-zag behavior). Within a nondegenerate Mersenne block, the orbit increases monotonically:

$$x < S(x) < \cdots < S^{n(x)-1}(x) = L(x).$$

The drop occurs in the exit step from $L(x)$ to $B(x) = S^{n(x)}(x)$, followed by the start of the next block. Viewed on the Syracuse time scale, this creates a characteristic "zig-zag" pattern: steady growth inside each wedge, followed by a sharp descent at each block boundary.

Corollary 4.9 (Downward block-to-block drift condition)**.**

$$r \geq \lfloor n \log_2(3/2) + 1 \rfloor + 1 \implies B(x) < x$$

Proof. Let $\alpha := n \log_2(3/2) + 1$. Since $\alpha < \lfloor \alpha \rfloor + 1$ and $r \in \mathbb{Z}$, the hypothesis

$$r \geq \lfloor \alpha \rfloor + 1$$

implies $r > \alpha$, i.e.

$$r > n \log_2(3/2) + 1.$$

Therefore $B(x) < x$ by Theorem 4.7. ☐

4.4 Stairs and exits

We now make the staircase nature of intra-block dynamics precise. Recall the stair/exit terminology from Definition 3.6.

Lemma 4.10 (Exit valuations)**.** *Let x be odd with Mersenne tail length $n(x) = n$ and exit exponent $r(x) = r$. Then the exit step completing the block transition,*

$$L(x) = S^{n-1}(x) \;\mapsto\; B(x) = S^n(x),$$

has valuation

$$s(L(x)) = 1 + r.$$

Proof. By Lemma 4.2, we have $3L(x) + 1 = 2(3^n a(x) - 1)$. Since $r = \nu_2(3^n a(x) - 1)$, it follows that

$$\nu_2(3L(x) + 1) = 1 + r.$$

But $s(L(x)) = \nu_2(3L(x) + 1)$ by definition. □

Proposition 4.11 (Stairs per block)**.** *Let x be odd with Mersenne tail length $n = n(x) \geq 2$, and define $J(x) := n(x) - 1$. Then:*

- *the Mersenne block rooted at x contains exactly $J(x)$ stairs, namely the steps $S^j(x) \mapsto S^{j+1}(x)$ for $0 \leq j \leq n - 2$;*
- *the block transition from this block to the next is completed by a single exit step $L(x) \mapsto B(x)$, whose valuation is $1 + r(x)$.*

Proof. The first claim is exactly Lemma 3.7. The second claim is Lemma 4.10. □

Corollary 4.12 (Exact time cost of one block transition)**.** *Let x be odd and set $n := n(x)$ and $r := r(x)$. Then the start-to-start block transition $x \mapsto B(x)$ consists of:*

- *exactly n Syracuse steps;*
- *exactly $n + r$ Terras steps;*
- *exactly $2n + r$ Collatz steps.*

Equivalently,

$$B(x) = S^{\,n}(x) = T^{\,n+r}(x) = C^{\,2n+r}(x).$$

Proof. Write the odd Syracuse segment completing the block transition as

$$x_0 := x, \quad x_{j+1} := S(x_j) \ (0 \leq j \leq n - 1), \quad \text{so } x_n = B(x).$$

By Lemma 3.7, the first $n - 1$ steps are stairs, so $s(x_j) = \nu_2(3x_j + 1) = 1$ for $0 \leq j \leq n - 2$. By Lemma 4.10, the exit step has valuation $s(x_{n-1}) = 1 + r$.

For an odd input u, the relation $3u + 1 = 2^{s(u)}S(u)$ shows that the odd-to-odd segment $u \mapsto S(u)$ consists of $s(u)$ Terras steps (one odd Terras step followed by $s(u) - 1$ halving steps) and $1 + s(u)$ Collatz steps (one $3u + 1$ step followed by $s(u)$ halvings). Summing over the n Syracuse steps in the block gives

$$\tau_T(x \to B(x)) = \sum_{j=0}^{n-1} s(x_j) = (n-1) \cdot 1 + (1+r) = n + r,$$

and

$$\tau_C(x \to B(x)) = \sum_{j=0}^{n-1} \big(1 + s(x_j)\big) = (n-1) \cdot 2 + (2+r) = 2n + r.$$

Hence $B(x) = T^{\,n+r}(x)$ and $B(x) = C^{\,2n+r}(x)$, while $B(x) = S^{\,n}(x)$ holds by definition. □

4.5 Residue-class constraints and the image of B

The block coordinates $(n(x), a(x), r(x))$ make several modular features essentially one-line consequences of the transition formula $B(x) = (3^{n(x)}a(x) - 1)/2^{r(x)}$ (Lemma 4.2).

Lemma 4.13 (Congruence characterization of the exit exponent)**.** *Let x be odd and write $x = 2^n a - 1$ with $n = n(x) \geq 1$ and $a = a(x)$ odd, and let $r = r(x) = \nu_2(3^n a - 1)$. Then for any integer $t \geq 1$:*

(i) $r \geq t$ if and only if $3^n a \equiv 1 \pmod{2^t}$ (equivalently, $a \equiv 3^{-n} \pmod{2^t}$);

(ii) $r = t$ if and only if $3^n a \equiv 1 + 2^t \pmod{2^{t+1}}$ (equivalently, $a \equiv 3^{-n}(1+2^t) \pmod{2^{t+1}}$),

where 3^{-n} denotes the (unique) inverse of 3^n modulo 2^t (or 2^{t+1}).

Proof. By definition, $r = \nu_2(3^n a - 1)$ means that $2^r \mid (3^n a - 1)$ but $2^{r+1} \nmid (3^n a - 1)$. Thus $r \geq t$ is equivalent to $2^t \mid (3^n a - 1)$, i.e. $3^n a \equiv 1 \pmod{2^t}$, proving (i). Similarly, $r = t$ is equivalent to $3^n a - 1 \equiv 2^t \pmod{2^{t+1}}$, i.e. $3^n a \equiv 1 + 2^t \pmod{2^{t+1}}$, proving (ii). Since $\gcd(3^n, 2^m) = 1$ for all m, the inverse $3^{-n} \pmod{2^m}$ exists and is unique. □

Corollary 4.14 (A geometric residue-class distribution for r at fixed n)**.** *Fix $n \geq 1$. For each $t \geq 1$ there is exactly one* odd *residue class $a \pmod{2^{t+1}}$ for which $r(2^n a - 1) = t$. Equivalently, there is exactly one residue class $x \pmod{2^{n+t+1}}$ such that $n(x) = n$ and $r(x) = t$.*

Consequently, among the 2^t odd residue classes modulo 2^{t+1}, exactly one yields $r = t$. In particular, within the set $\{x : \ n(x) = n\}$ the relative frequency of $r = t$ (per full modulus period) is 2^{-t}, and the relative frequency of $r \geq t$ is $2^{-(t-1)}$.

Proof. By Lemma 4.13(ii), the condition $r = t$ is the single congruence $3^n a \equiv 1 + 2^t \pmod{2^{t+1}}$. Since 3^n is invertible modulo 2^{t+1}, this congruence has a unique solution $a \pmod{2^{t+1}}$. The right-hand side $1 + 2^t$ is odd, and 3^n is odd, so the unique solution class is also odd.

There are 2^t odd residue classes modulo 2^{t+1}, so selecting one class gives relative frequency 2^{-t}. Similarly, $r \geq t$ corresponds by Lemma 4.13(i) to a single odd class modulo 2^t, and among the 2^{t-1} odd classes modulo 2^t this gives relative frequency $2^{-(t-1)}$. □

Proposition 4.15 (A mod 3 law for the block map)**.** *Let x be odd with $n := n(x)$ and exit exponent $r := r(x)$. Then*

$$3 \nmid B(x), \qquad B(x) \equiv (-1)^{r+1} \pmod 3.$$

Equivalently, $B(x) \equiv 1 \pmod 3$ if r is odd, and $B(x) \equiv 2 \pmod 3$ if r is even.

Proof. By Lemma 4.2, we have $B(x) = (3^n a(x) - 1)/2^r$. Since $n \geq 1$,

$$3^n a(x) - 1 \equiv -1 \pmod 3,$$

so the numerator is not divisible by 3, and neither is the denominator 2^r; hence $3 \nmid B(x)$.

Modulo 3, we may divide by 2^r since 2 is invertible mod 3 and $2 \equiv -1 \pmod 3$, so

$$B(x) \equiv \frac{-1}{2^r} \equiv (-1) \cdot (2^{-1})^r \equiv (-1) \cdot 2^r \equiv (-1) \cdot (-1)^r = (-1)^{r+1} \pmod 3.$$

□

Corollary 4.16 (Block starts modulo 6 are determined by the parity of $r(x)$)**.** *For every odd x we have $B(x) \equiv 1$ or $5 \pmod 6$, and more precisely*

$$B(x) \equiv \begin{cases} 1 \pmod 6, & r(x) \text{ odd}, \\ 5 \pmod 6, & r(x) \text{ even}. \end{cases}$$

Proof. $B(x)$ is odd by definition of the Syracuse (hence block) map. By Proposition 4.15, $B(x) \equiv 1$ or $2 \pmod 3$, with the parity of $r(x)$ selecting which. Combining the unique odd lifts of 1 and 2 modulo 3 yields the claimed residues modulo 6. □

Theorem 4.17 (Universal surjectivity of B across all block lengths)**.** *The image of B consists exactly of the odd integers not divisible by 3:*

$$B(\{\textit{odd } x\}) = \{y \in \mathbb{N} : \ y \textit{ odd and } 3 \nmid y\}.$$

Moreover, the map is surjective for every fixed block length. Specifically, for every $n \geq 1$ and every admissible y, there exist infinitely many preimages x such that $n(x) = n$ and $B(x) = y$.

Proof. Let y be odd with $3 \nmid y$, and fix a target block length $n \geq 1$. We seek an integer x of the form $x = 2^n a - 1$ (with a odd) such that

$$B(x) = \frac{3^n a - 1}{2^r} = y,$$

for some exit exponent r. Rearranging for a gives the condition

$$3^n a = 1 + y2^r.$$

This equation requires finding an exponent $r \geq 1$ such that

$$y2^r \equiv -1 \pmod{3^n}.$$

Since 2 is a primitive root modulo 9, standard lifting results (e.g., [3],[14]) imply that 2 remains a primitive root modulo 3^n for all $n \geq 1$; consequently, the powers of 2 generate the entire multiplicative group $(\mathbb{Z}/3^n\mathbb{Z})^\times$. Because $3 \nmid y$, the residue $-y^{-1}$ exists and lies in this group. Therefore, the congruence has solutions for r, forming an arithmetic progression $r \equiv r_0 \pmod{\phi(3^n)}$.

For any such solution r, let $a = (1+y2^r)/3^n$. Note that the numerator $1+y2^r$ is the sum of an odd integer (1) and an even integer ($y2^r$ for $r \geq 1$), so the numerator is odd. Since the denominator 3^n is odd, the quotient a is an odd integer.

Finally, set $x = 2^n a - 1$. By construction:

- $n(x) = \nu_2(x+1) = \nu_2(2^n a) = n$.
- $r(x) = \nu_2(3^n a - 1) = \nu_2(y2^r) = r$ (since y is odd).
- $B(x) = (3^n a - 1)/2^r = y$.

Thus, every admissible y pulls back to infinitely many block starts x for every possible Mersenne wedge size n. □

Corollary 4.18 (Empty preimage for multiples of 3)**.** *Let y be an odd integer divisible by 3. Then there exists no odd integer x such that $B(x) = y$.*

Remark 4.19 (The barrier at multiples of 3)*.* The exclusion of multiples of 3 from the image of B (Theorem 4.17) reflects a fundamental property of the Collatz map C. Since $3x+1 \not\equiv 0 \pmod 3$, a multiple of 3 can never be the result of an odd-step; it can only be reached via division by 2. Consequently, in the full Collatz graph, the preimages of any $y \equiv 0 \pmod 3$ are strictly even (specifically $n = 2y \equiv 0 \pmod 6$). The Mersenne block map, being an odd-to-odd contraction, naturally filters out these values, treating them as "Garden of Eden" points with no odd predecessors.

Example 4.20 (The Mersenne Towers)*.* The geometric regularity visible in Figure 4 is universal. Just as every integer y heads an infinite tower of even preimages ($2^k y$), Theorem 4.17 implies that every odd integer $y \not\equiv 0 \pmod 3$ heads an infinite tower of odd preimages, which we term the *Mersenne Tower* of y. The "trivial" levels of this tower (where $n(x) = 1$) are given by

$$x = \frac{2^k y - 1}{3}, \quad \text{for } k \text{ such that } 2^k y \equiv 1 \pmod 3.$$

For k sufficiently large (specifically $r(x) \geq 2$), these solutions satisfy $x \equiv 5 \pmod 8$. Geometrically, this establishes that the inverse block map B^{-1} lifts every admissible odd y into a linear lattice that scales by powers of 4 ($x_{i+1} \approx 4x_i$), a structural property of the Syracuse inverse graph noted in classical surveys [8]. This provides the odd-integer counterpart to the trivial scaling of the even Collatz map. (Note: If y is a multiple of 3, no such odd tower exists, as discussed in Remark 4.19.)

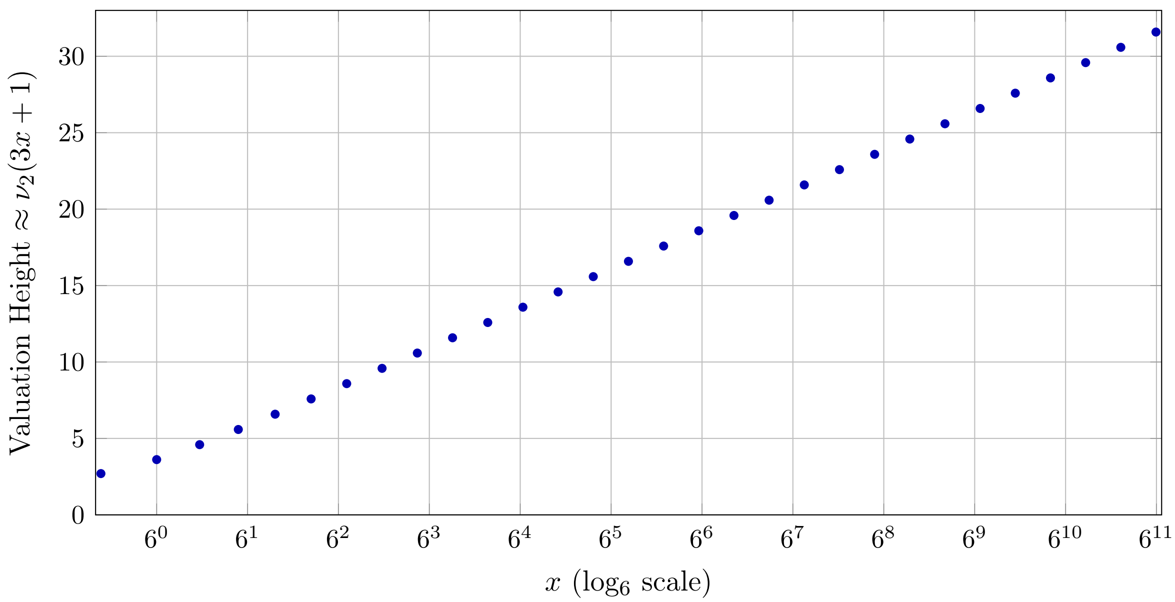

Figure 4: The "Mersenne Tower" for $y = 1$. Just as even numbers form vertical towers $2^k y$, the Mersenne block map lifts $y = 1$ into an infinite linear lattice of odd preimages. The points lie on a line of slope $1/\log_2 6$, and for $x > 1$, they satisfy $x \equiv 5 \pmod 8$, illustrating the structural symmetry between even and odd inverse dynamics.

Corollary 4.21 (Mod 3 rigidity inside a nondegenerate wedge)**.** *Let x be odd with $n := n(x) \geq 2$, and write $x = 2^n a - 1$ with a odd. Then for every j with $1 \leq j \leq n - 1$,*

$$S^j(x) \equiv -1 \equiv 2 \pmod 3.$$

In particular, every nondegenerate block peak satisfies

$$L(x) = S^{n-1}(x) \equiv 2 \pmod 3, \qquad L(x) \equiv 1 \pmod 4, \qquad \text{hence } L(x) \equiv 5 \pmod{12}.$$

Proof. By Theorem 3.1, for $0 \leq j \leq n - 1$ we have

$$S^j(x) = 3^j a\, 2^{n-j} - 1.$$

If $1 \leq j \leq n - 1$, then the term $3^j a\, 2^{n-j}$ is divisible by 3, so $S^j(x) \equiv -1 \pmod 3$.

For the peak, $L(x) = S^{n-1}(x) = 2 \cdot 3^{n-1}a - 1$ (Lemma 4.2). Since $3^{n-1}a$ is odd, we have $2 \cdot 3^{n-1}a \equiv 2 \pmod 4$, hence $L(x) \equiv 1 \pmod 4$. Combining $L(x) \equiv 2 \pmod 3$ with $L(x) \equiv 1 \pmod 4$ yields $L(x) \equiv 5 \pmod{12}$. □

5 Intrinsic Statistics and Logarithmic Drift

The structural framework developed in Sections 1–4 partitions the dynamics into rigid intra-block mechanics and the exit exponent $r(x)$. We now analyze the statistical behavior of the parameters $(n(x), r(x))$ to derive the expected growth rate of the Mersenne block map.

5.1 Exact distribution of $n(x)$

The distribution of the Mersenne-tail length $n(x)$ is determined purely by congruence conditions modulo powers of 2.

Proposition 5.1 (Distribution of Mersenne-tail length)**.** *For each $k \geq 1$, the set of odd integers $\{x \geq 1 : x \text{ odd and } n(x) = k\}$ has natural density 2^{-k} within the odd integers. Equivalently, if x is chosen uniformly from the odd residue classes modulo 2^m with $m \geq k+1$, then*

$$\Pr\bigl(n(x) = k\bigr) = 2^{-k}.$$

In particular, $n(x)$ is geometric$(1/2)$ in the natural density sense and $\mathbb{E}[n(x)] = 2$.

Proof. An odd integer has $n(x) = k$ if and only if $x + 1$ is divisible by 2^k but not by 2^{k+1}, i.e., $x \equiv 2^k - 1 \pmod{2^{k+1}}$. Among the 2^k odd residue classes modulo 2^{k+1}, exactly one class satisfies this congruence. □

5.2 Exact joint distribution of (n, r)

While the trajectory of a specific integer is deterministic, the distribution of the parameters $(n(x), r(x))$ across the set of all odd integers is governed by exact residue-class statistics.

Proposition 5.2 (Joint independence in natural density)**.** *Let x be chosen uniformly from the set of odd integers (in the sense of natural density). Then the Mersenne tail length $n(x)$ and the exit exponent $r(x)$ are independent geometric$(1/2)$ random variables. Specifically, for any $n, r \geq 1$:*

$$\Pr\bigl(n(x) = n,\ r(x) = r\bigr) = 2^{-(n+r)}.$$

Proof. Recall from Corollary 4.14 that for a fixed n, the condition $r(x) = r$ corresponds to exactly one odd residue class modulo 2^{n+r+1}. Specifically, there is a unique solution for $x \pmod{2^{n+r+1}}$ that satisfies both $n(x) = n$ and $r(x) = r$. The set of odd integers modulo 2^{n+r+1} has cardinality 2^{n+r}. Since exactly one of these 2^{n+r} classes satisfies the condition, the natural density is:

$$\frac{1}{2^{n+r}} = 2^{-n} \cdot 2^{-r}.$$

Summing over r gives the marginal probability $\Pr(n(x) = n) = 2^{-n}$, and summing over n gives $\Pr(r(x) = r) = 2^{-r}$. Thus, the joint distribution factors, establishing independence. □

5.3 The orbit-mixing heuristic

Proposition 5.2 establishes that a *randomly selected* odd integer behaves according to the independent geometric law. The central difficulty of the Collatz problem lies in the fact that the block orbit $x_0, x_1, x_2, \ldots$ is not a sequence of independent random integers, but a deterministically generated chain.

To derive predictive statistics for the map, we employ a standard heuristic assumption regarding the mixing properties of the dynamics.

Heuristic 5.3 (Orbit-mixing assumption)**.** *Along "typical" block orbits $x_{k+1} = B(x_k)$, the sequence of parameter pairs (n_k, r_k) behaves statistically like a sequence of independent, identically distributed samples from the exact joint distribution derived in Proposition 5.2.*

This heuristic motivates the formal probabilistic model used to calculate drift.

Definition 5.4 (Independent geometric model). Based on Heuristic 5.3, we model the Mersenne block transition by independent random variables $N, R \sim \mathrm{Geom}(1/2)$ with joint probability

$$\Pr(N = n, R = r) = 2^{-(n+r)}.$$

Under this model, the block ratio is approximated by the random variable:

$$\frac{B(x)}{x} \approx \frac{3^N}{2^{N+R}}. \tag{17}$$

Remark 5.5. As a sanity check, empirical frequencies of $n(x)$ and $r(x)$ over all odd integers $x \leq 200{,}000$ match the exact geometric laws predicted by Propositions 5.1–5.2 (Figure 5). While this static computation does not directly test the dynamical dependence along a single B-orbit Heuristic 5.3, we provide explicit empirical validation of the underlying residue mixing in **Appendix A.2**, where a trajectory of 10^5 steps is shown to exhibit uniform distribution of the odd factor $a(x)$ modulo 64 (Figure 7).

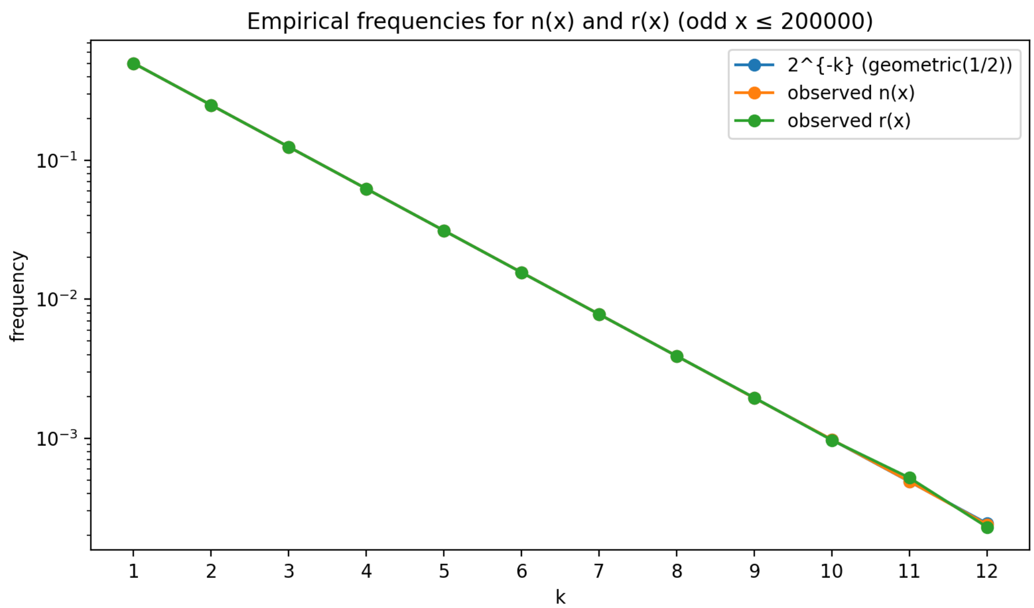

Figure 5: Empirical frequencies of the Mersenne tail length $n(x)$ and exit exponent $r(x)$ computed for all odd integers $x \leq 200,000$. The data (orange and green lines) tracks the theoretical geometric distribution 2^{-k} (blue line) with high precision, supporting the independent geometric model.

5.4 Expected logarithmic drift

We now compute the expected growth of a Mersenne block step in the logarithmic scale. Let $\Delta := \log_2(B(x)/x)$ denote the logarithmic drift. In our model, this becomes the random variable

$$\Delta \approx N \log_2 \frac{3}{2} - R.$$

Proposition 5.6 (Expected logarithmic drift)**.** *In the independent geometric model, the expected logarithmic drift is negative:*

$$\mathbb{E}[\Delta] = 2 \log_2 \frac{3}{2} - 2 \approx -0.83007.$$

Proof. By independence, $\mathbb{E}[\Delta] = \mathbb{E}[N] \log_2(3/2) - \mathbb{E}[R]$. Since N, R are geometric(1/2), we have $\mathbb{E}[N] = \mathbb{E}[R] = 2$. Thus,

$$\mathbb{E}[\Delta] = 2 \log_2(1.5) - 2 \approx 2(0.58496) - 2 \approx -0.83007.$$

□

Proposition 5.7 (Expected time compression)**.** *In the independent geometric model, the expected number of Collatz steps corresponding to a single Mersenne block transition is* 6*. Specifically,*

$$\mathbb{E}[\tau_C(x \to B(x))] = \mathbb{E}[2N + R] = 6.$$

Proof. From Corollary 4.12, the number of Collatz steps in a block transition is exactly $2n(x) + r(x)$. Under the independent geometric model (Definition 5.4), we have $\mathbb{E}[N] = 2$ and $\mathbb{E}[R] = 2$. By linearity of expectation,

$$\mathbb{E}[2N + R] = 2\mathbb{E}[N] + \mathbb{E}[R] = 2(2) + 2 = 6.$$

□

Remark 5.8 (Operational utility of coarse-graining). This result quantifies the efficiency of the Mersenne Block framework. On average, iterating the block map B compresses the dynamics by a factor of roughly 6 compared to the original Collatz map C. This explains the significant reduction in total stopping times observed in Table 1 (where $\tau_C(27) = 111$ and $\tau_B(27) = 17$, a ratio of ≈ 6.5) and validates B as a computationally effective acceleration.

Remark 5.9 (Arithmetic vs. logarithmic contraction). This result recovers the classical probabilistic heuristic for the $3x + 1$ map but in block coordinates. Note the dichotomy:

- In *arithmetic* mean, the model is critical: $\mathbb{E}[3^N/2^{N+R}] = 1$.
- In *logarithmic* mean, the model is contractive: $\mathbb{E}[\Delta] \approx -0.83$.

This tension mirrors the phenomenon utilized by Tao [15] in the analysis of typical orbits using logarithmic density; the distribution of these logarithmic densities is visualized in Figure 6. In the Mersenne block framework, this drift arises from simple intrinsic statistics: an average block has length 2 (adding ≈ 1.17 bits of growth via 3^2) but an average exit adds 2 divisions by 2 (removing 2 bits), resulting in a net loss of ≈ 0.83 bits.

Proposition 5.10 (Frequency of upward steps)**.** *The probability that a random block step increases the value (i.e., $\Delta > 0$) is*

$$\Pr(\Delta > 0) = \sum_{n \geq 1} 2^{-n} \left(1 - 2^{-\lfloor n \log_2(1.5) \rfloor}\right) \approx 0.286.$$

Thus, roughly 71% of Mersenne block transitions are contractive.

Proof. For each n, $\Delta > 0$ iff $R \leq \lfloor n \log_2(3/2) \rfloor$, so $\Pr(\Delta > 0 \mid N = n) = 1 - 2^{-\lfloor n \log_2(3/2) \rfloor}$, then sum over n with weights 2^{-n}. □

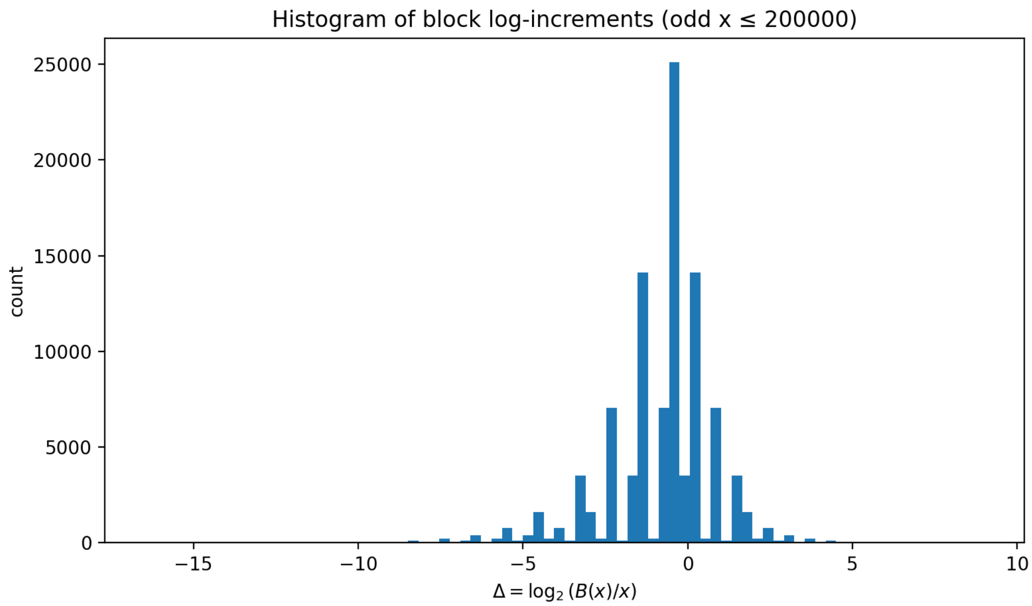

Figure 6: Histogram of block log-increments $\Delta = \log_2(B(x)/x)$ for odd $x \leq 200,000$. The distribution is centered near the theoretical expected drift of ≈ -0.83 bits per step, illustrating the net contraction of the Mersenne block map.

Conclusion, outlook, and prospects

Mersenne block dynamics reorganizes the accelerated Collatz map into two sharply separated phases: a completely deterministic *intra-block* regime and an irregular *exit* at the block boundary. Every odd integer x admits the canonical tail decomposition

$$x = (a(x) - 1)2^{n(x)} + \left(2^{n(x)} - 1\right), \qquad n(x) = \nu_2(x + 1),$$

and this partitions the Syracuse orbit into Mersenne blocks of length $n(x)$. Inside a block we have the closed form

$$S^j(x) = 3^j a(x)\, 2^{n(x)-j} - 1 \qquad (0 \leq j \leq n(x) - 1),$$

so the odd values increase strictly while the Mersenne-tail length decreases by one at each step, producing the rigid "wedge" pattern in the trailing bits.

All nontrivial behavior is therefore concentrated in the single exit step, measured by the exponent

$$r(x) = \nu_2\!\left(3^{n(x)}a(x) - 1\right),$$

and encoded by the induced block map $B(x) = S^{n(x)}(x)$. The explicit transition identity

$$B(x) = \frac{3^{n(x)}a(x) - 1}{2^{r(x)}}$$

(and its equivalent ratio form in Corollary 4.5) isolates the dominant multiplicative factor $3^{n(x)}/2^{n(x)+r(x)}$ together with an exact correction term. At the level of time scales, Lemma 2.9 relates total stopping times across the hierarchy $C \to T \to S \to B$, and Theorem 2.6 restates Collatz equivalently as eventual absorption at 1 under iteration of B.

Finally, the intrinsic statistics analyzed in Section 5 demonstrate the predictive power of this framework. We established that the block length $n(x)$ follows an exact geometric distribution, and that a heuristic independence model for the pair $(n(x), r(x))$ successfully recovers the standard probabilistic prediction of the Collatz map, yielding an expected logarithmic drift of ≈ -0.83 bits per block step.

Outlook. From the Mersenne block perspective, the central obstacle is no longer intra-block complexity (which is rigid and explicit), but rather understanding how the exit exponent $r(x)$ behaves *along* B-orbits $x_{k+1} = B(x_k)$. Concrete directions suggested by the framework include:

- *Residue-class dynamics and mixing for the induced map B.* Both $n(x) = \nu_2(x+1)$ and $r(x) = \nu_2(3^{n(x)}a(x) - 1)$ are defined by 2-adic valuations, so a natural target is to study the induced action of B on odd residue classes modulo 2^m (for example via the evolution of $a(x) = (x+1)/2^{n(x)} \bmod 2^m$). Establishing equidistribution or mixing statements at fixed 2-adic scales along typical block orbits would provide a rigorous pathway to orbit-level statistics in block coordinates.

- *From intrinsic (static) statistics to orbit statistics.* Section 5 proves an *exact* residue-class fact: for a uniformly random odd integer x (in natural density), the block parameters satisfy
$$\Pr(n(x) = n,\ r(x) = r) = 2^{-(n+r)},$$
so $n(x)$ and $r(x)$ are independent geometric(1/2) random variables at a single block start (Proposition 5.2). The remaining heuristic input needed for drift predictions is therefore *dynamical*: one seeks conditions under which successive block starts along a typical B-orbit sample these residue classes with sufficiently weak dependence (Heuristic 5.3).

- *Deterministic descent criteria.* The sufficient condition in Theorem 4.7 shows how large exits can overwhelm the systematic $(3/2)^{n(x)}$ growth inside a block. Strengthening such criteria—for example, by relating $r(x)$ to constraints on $a(x)$ or on residues modulo 2^m—could yield new deterministic mechanisms for forcing net contraction across blocks, complementing probabilistic drift heuristics.

- *Refining the probabilistic bridge and computational experiments.* The exact ratio identity (Corollary 4.5) expresses $B(x)/x$ as a dominant factor $3^{n(x)}/2^{n(x)+r(x)}$ times an explicit correction term. Future work could incorporate this correction and possible residue-class dependencies into more refined (e.g. Markov) block models, and test such refinements using statistics gathered *along long B-orbits* rather than only over all odd integers in an interval.

Prospects.

Problem 5.11 (Finite-time block mixing conditional on long orbits). Fix $m \geq 1$. For "typical" starting values x_0 with large block stopping time $\tau_B(x_0)$, the sequence of residues $a(x_k)$ mod 2^m for $0 \leq k \leq K$ becomes close to uniform on odd residue classes as $K \to \infty$ with $K \ll \tau_B(x_0)$. In particular, the empirical distribution of $(n(x_k), r(x_k))$ over $0 \leq k \leq K$ approaches $\mu(n,r) = 2^{-(n+r)}$.

Problem 5.12 (Diophantine Constraints on Block Cycles). A non-trivial cycle corresponds to a finite sequence of block parameters $((n_0, r_0), \ldots, (n_{k-1}, r_{k-1}))$ satisfying the exact closure condition $x_k = x_0$. While (n_i, r_i) are the primary discrete parameters, the cycle constraint depends explicitly on the evolving odd factors $a(x_i)$. Using the exact ratio identity (Corollary 4.5), the closure condition becomes:

$$\prod_{i=0}^{k-1} \frac{3^{n_i}}{2^{n_i+r_i}} \cdot \prod_{i=0}^{k-1} \frac{1 - \frac{1}{3^{n_i} a(x_i)}}{1 - \frac{1}{2^{n_i} a(x_i)}} = 1.$$

This structurally separates the dominant scaling factor from the small arithmetic corrections. A key prospect is to utilize the modular rigidities of $r(x)$ (Corollary 4.14) to derive strong restrictions on admissible parameter sequences and to obstruct broad families of potential block cycles.

Problem 5.13 (Diffusion and Maximum Excursion). While the expected logarithmic drift $\mathbb{E}[\Delta] \approx -0.83$ in the heuristic model suggests global descent, the *variance* of the block transitions determines the rate of diffusion. A further goal is to establish a central limit theorem for $\tau_B(x)$ when x is sampled uniformly from odd integers up to X (or in logarithmic density), as $X \to \infty$, thereby providing probabilistic bounds on the maximum excursion $\max_k B^k(x)$ relative to the starting value x.

Ultimately, the Mersenne Block Dynamics Framework offers more than a change of variables: it provides a canonical coarse-graining of the dynamics determined directly by the binary structure of the input. By packaging forced "stair" runs into the return-time $n(x)$ and isolating the arithmetic complexity into the single exit exponent $r(x)$, MBDF disentangles the deterministic wedge dynamics from the irregular behavior of the orbit. This separation yields explicit closed forms for block transitions and a rigorous modular description of the image and inverse families (Theorem 4.17), features that are obscured in step-by-step valuation encodings. These coordinates thus supply a natural interface between exact congruence structure and probabilistic heuristics, offering a tractable vantage point for future analytic bounds on the $3x + 1$ problem.

Acknowledgements

I acknowledge the use of Large Language Models (LLMs) in assisting me in bringing my intuitions, insights, and ideas in this manuscript to fruition. Responsibility for the conceptual and analytical content and for any errors or omissions herein is mine.

References

[1] Bernstein, D. J., & Lagarias, J. C., *The* $3x + 1$ *conjugacy map*, Canad. J. Math. **48** (1996), no. 6, 1154–1169.

[2] Chamberland, M., *A continuous extension of the* $3x+1$ *problem to the real line*, Dyn. Contin. Discrete Impuls. Syst. **2** (1996), no. 4, 495–509.

[3] Cohen, H., *A Course in Computational Algebraic Number Theory*, Graduate Texts in Mathematics **138**, Springer-Verlag, Berlin, 1993.

[4] Conway, J. H., *Unpredictable Iterations*, in *Proc. 1972 Number Theory Conference* (Univ. of Colorado, Boulder), 1972, pp. 49–52.

[5] Hercher, C., *There are no Collatz* m*-cycles with* $m \leq 91$, J. Integer Seq. **26** (2023), Article 23.3.5.

[6] Konstadinidis, P. B., *The real* $3x + 1$ *problem*, Acta Arith. **122** (2006), no. 1, 35–44.

[7] Krasikov, I., & Lagarias, J. C., *Bounds for the* $3x + 1$ *problem using difference inequalities*, Acta Arith. **109** (2003), no. 3, 237–258.

[8] Lagarias, J. C., *The* $3x + 1$ *problem and its generalizations*, Amer. Math. Monthly **92** (1985), no. 1, 3–23.

[9] Lagarias, J. C. (ed.), *The Ultimate Challenge: The* $3x + 1$ *Problem*, American Mathematical Society, 2010.

[10] Leavens, G. T., & Vermeulen, M., $3x + 1$ *search programs*, Comput. Math. Appl. **24** (1992), no. 11, 79–99.

[11] Letherman, S., Schleicher, D., & Wood, R., *The* $3n+1$*-problem and holomorphic dynamics*, Experiment. Math. **8** (1999), no. 3, 241–251.

[12] Möller, H., *Über Hasses Verallgemeinerung des Syracuse-Algorithmus*, Arch. Math. **31** (1978), no. 1, 21–33.

[13] Oliveira e Silva, T., *Empirical verification of the* $3x + 1$ *and related conjectures*, in *The Ultimate Challenge: The* $3x + 1$ *Problem* (J. C. Lagarias, ed.), AMS, 2010, pp. 189–207.

[14] Rosen, K. H., *Elementary Number Theory and Its Applications*, 6th ed., Pearson, 2011.

[15] Tao, T., *Almost all orbits of the Collatz map attain almost bounded values*, Forum Math. Pi **10** (2022), e12.

[16] Terras, R., *A stopping time problem on the positive integers*, Acta Arith. **30** (1976), no. 3, 241–252.

[17] Wirsching, G. J., *The Dynamical System Generated by the* $3n + 1$ *Function*, Lecture Notes in Math. **1681**, Springer, 1998.

[18] Yolcu, E., Aaronson, S., & Heule, M. J. H., *An Automated Approach to the Collatz Conjecture*, in *Automated Deduction – CADE 28*, Lecture Notes in Comput. Sci. **12699**, Springer, 2021, pp. 428–445.

Appendix: Computational Implementation and Reproducibility

A.1 Mersenne Block Algorithm

The Mersenne Block framework allows for efficient orbit calculation by skipping the intra-block "stair" steps. A standard Syracuse implementation computes every odd step; a Block implementation computes only the block starts.

Algorithm 1 Mersenne Block Step

Input: Odd integer x
$n \leftarrow \text{count\_trailing\_zeros}(x+1)$
$a \leftarrow (x+1) \gg n$
$y \leftarrow 3^n \cdot a - 1$
$r \leftarrow \text{count\_trailing\_zeros}(y)$
$B(x) \leftarrow y \gg r$
return $B(x)$

For large x, the dominant cost is the multiplication $3^n \cdot a$. Because $n(x)$ is typically small (mean 2), this step is computationally inexpensive compared to iterating the underlying $2n + r$ Collatz steps individually.

A.2 Orbit-level diagnostics for residue mixing and weak dependence

To empirically probe the orbit-level hypotheses central to the probabilistic bridge (Heuristic 5.3 and Problem 5.11), we computed simple diagnostics along a single B-orbit $x_{k+1} = B(x_k)$ of length $N = 100{,}000$ block steps.

Computational setup. The starting value x_0 was chosen as a random odd integer of bitlength $L \approx 200{,}000$ (to ensure the orbit did not collapse to 1 within the first N block steps due to logarithmic drift). We sampled x_0 uniformly from the odd integers in $[2^{L-1}, 2^L)$ by choosing a uniform integer $u \in \{0, \ldots, 2^{L-1} - 1\}$ and setting $x_0 = 2^{L-1} + 2u + 1$.

Residue mixing. We tested equidistribution of the odd factor $a(x) := (x+1)/2^{n(x)}$ modulo 2^m along the orbit. For $m = 6$ (so $2^m = 64$, with 32 odd residue classes), uniformity predicts an expected count $E = N/32 = 3125$ per bin. Using the observed counts from Figure 7, Pearson's chi-square statistic is

$$\chi^2 = \sum_{i=1}^{32} \frac{(O_i - E)^2}{E} \approx 26.64 \qquad (df = 31),$$

with $p \approx 0.69$ (interpreted here as a descriptive goodness-of-fit measure rather than a formal i.i.d. significance test), consistent with uniform sampling of odd residue classes modulo 64 along this orbit.

Lag-1 linear dependence checks. To probe the i.i.d. modeling assumption in Definition 5.4, we computed lag-1 sample correlations for block parameters $(n_k, r_k) = (n(x_k), r(x_k))$:

$$\rho(n_k, n_{k+1}) \approx -0.0018, \quad \rho(r_k, r_{k+1}) \approx 0.0023, \quad \rho(n_k, r_k) \approx -0.0005, \quad \rho(n_k, r_{k+1}) \approx \cdots$$

As a scale reference, under a simple null of negligible dependence one expects typical fluctuations of order $1/\sqrt{N} \approx 0.0032$ for correlations at this sample size. Thus, we detect no meaningful lag-1 *linear* dependence in these basic statistics on this trajectory.

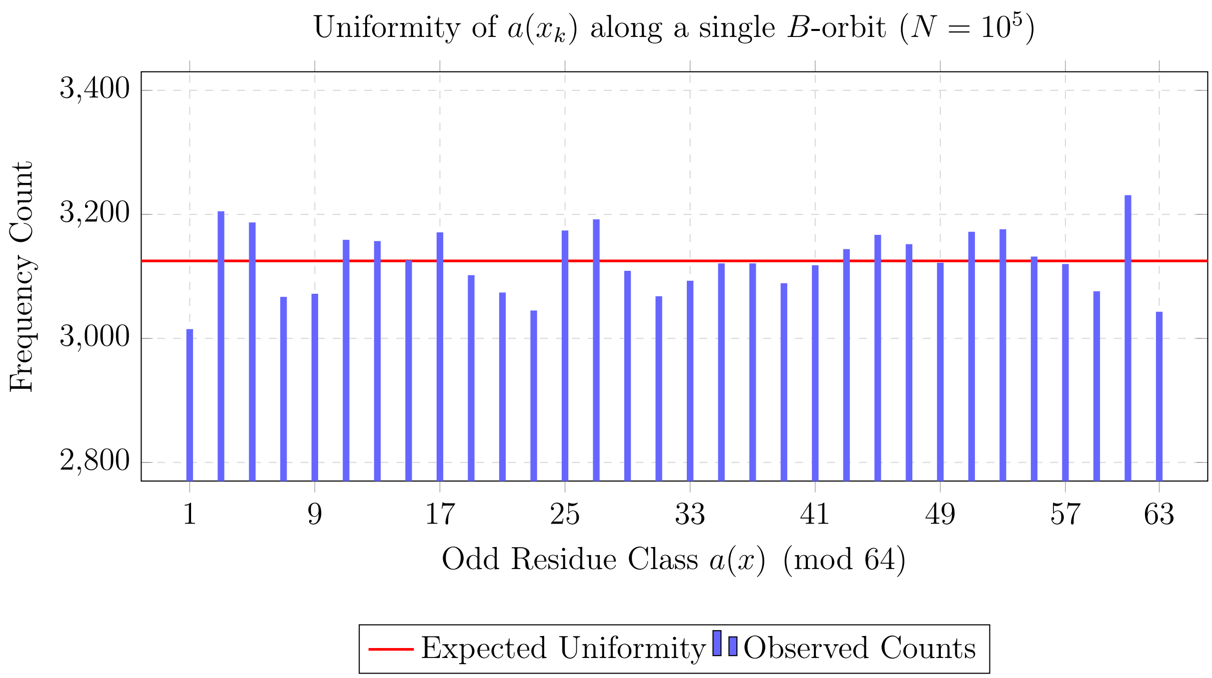

Figure 7: Empirical histogram of $a(x_k) \pmod{64}$ along a single block orbit of $N = 100{,}000$ steps. The counts are consistent with uniformity ($\chi^2 \approx 26.64$, $p \approx 0.69$, $df = 31$), and basic lag-1 correlations for (n_k, r_k) are near zero (see text). Vertical axis truncated for readability.

A.3 Data and Code Availability

All computations were performed using Python 3.8 with numpy; some figures are rendered directly in LaTeX (PGFPlots) from the computed data. The full reproduction code, including the static distribution analysis (Section 5) and the dynamical orbit validation (Appendix A.2), is provided in the supplementary material file `mbd_supplementary.py`.