A Collatz Core, a Sieve, and a Head–Chain Decomposition for the Odd Dynamics

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Abstract

We study the Collatz iteration restricted to odd integers and exhibit a concrete *core set* X inside the forward-invariant set $Y = \{6n+1,\, 6n+5 : n \in \mathbb{N}_0\}$ (odd integers not divisible by 3). For the odd Collatz map $f_c(u) = (3u+1)/2^{v_2(3u+1)}$ we prove that the restriction $f_c|_X$ is a bijection onto Y. This yields a redundant-free "Collatz sieve": every value in Y has a unique *core predecessor* in X. For this particular core, the induced dynamics *inside* X is strictly decreasing. As a consequence, X decomposes (without duplicates) into disjoint infinite one-sided chains indexed by a set of *heads* $H \subset X$: every element of $X \setminus \{1\}$ lies on exactly one head chain, and moving one step up a chain increases the time spent inside X by one.

1 Introduction

The Collatz map $C : \mathbb{N} \to \mathbb{N}$ is defined by

$$C(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even,} \\ 3n+1, & \text{if } n \text{ is odd.} \end{cases}$$

It is convenient to pass to the *odd-only* dynamics: from an odd integer u we apply $3u+1$ and then divide by the maximal power of 2 until the next odd integer is reached. This defines the odd Collatz map

$$f_c(u) := \frac{3u+1}{2^{v_2(3u+1)}},$$

where $v_2(k)$ denotes the exponent of 2 in the prime factorization of k.

A classical observation is that if u is odd and not divisible by 3, then $f_c(u)$ is again odd and not divisible by 3. We therefore work on the forward-invariant set

$$Y := \{6n+1,\, 6n+5 : n \in \mathbb{N}_0\}.$$

Our first goal is to exhibit an explicit subset $X \subset Y$ (a *core*) such that $f_c|_X$ is bijective onto Y. This gives a "sieve": each value of Y is generated exactly once from X.

The second goal is structural. For the particular core X used here, every core step uses at least two halving operations, forcing $f_c(x) < x$ for all $x \in X$ with $x > 1$. This implies that the directed graph induced by f_c *within* X contains no cycles other than the fixed point 1. Combining strict decrease with the bijectivity of $f_c|_X$ yields a strong normal form: X decomposes into disjoint infinite one-sided chains indexed by "heads" (elements of X whose next odd Collatz value leaves X). This head–chain decomposition is the main organizational device of the paper.

For general background on the $3x+1$ problem and the odd-only ("accelerated") formulation used here, see the survey of Lagarias [1] and the monograph of Wirsching [4]. Classical probabilistic and stopping-time heuristics go back at least to Terras [2] and Crandall [3]; more recent analytic progress includes Tao's "almost all" result [6].

2 The invariant odd set Y

Lemma 2.1. *The set $Y = \{6n+1, 6n+5 : n \in \mathbb{N}_0\}$ is forward invariant under f_c.*

This standard invariance property is well known (see, e.g., [1, 4]).

Proof. Let $u \in Y$. Then u is odd and $3 \nmid u$. Hence $3u+1 \equiv 1 \pmod 3$, so $3 \nmid (3u+1)$. Dividing by a power of 2 does not introduce a factor of 3, because 2 is invertible modulo 3. Thus $3 \nmid f_c(u)$, and by construction $f_c(u)$ is odd. Therefore $f_c(u) \in Y$. □

3 A concrete core set and a Collatz sieve

Define the explicit core set

$$\begin{aligned} X = \{24n+1 : n \in \mathbb{N}_0\} &\cup \{24n+17 : n \in \mathbb{N}_0\} \cup \{48n+13 : n \in \mathbb{N}_0\} \\ &\cup \{48n+29 : n \in \mathbb{N}_0\} \cup \{96n+37 : n \in \mathbb{N}_0\} \cup \{192n+181 : n \in \mathbb{N}_0\}. \end{aligned} \tag{1}$$

It is convenient to also write Y as residue classes modulo 18:

$$\begin{aligned} Y = \{18n+1 : n \in \mathbb{N}_0\} &\cup \{18n+5 : n \in \mathbb{N}_0\} \cup \{18n+7 : n \in \mathbb{N}_0\} \\ &\cup \{18n+11 : n \in \mathbb{N}_0\} \cup \{18n+13 : n \in \mathbb{N}_0\} \cup \{18n+17 : n \in \mathbb{N}_0\}. \end{aligned} \tag{2}$$

(Indeed, these are exactly the odd residues not divisible by 3.)

Theorem 3.1 (Core bijection / sieve)**.** *Let X be as in (1) and Y as in (2). Define f_c on X branchwise by*

$$f_c(u) = \frac{3u+1}{2^i},$$

where i is chosen according to the residue class of u:

$$\begin{aligned} u &= 24n+1 &\Rightarrow&\quad i = 2, \\ u &= 24n+17 &\Rightarrow&\quad i = 2, \\ u &= 48n+13 &\Rightarrow&\quad i = 3, \\ u &= 48n+29 &\Rightarrow&\quad i = 3, \\ u &= 96n+37 &\Rightarrow&\quad i = 4, \\ u &= 192n+181 &\Rightarrow&\quad i = 5. \end{aligned}$$

Then $f_c|_X : X \to Y$ is a bijection. Equivalently, every $y \in Y$ has a unique predecessor in X.

Proof. Each branch is a direct computation:

$$f_c(24n+1) = \frac{72n+4}{4} = 18n+1, \qquad f_c(24n+17) = \frac{72n+52}{4} = 18n+13,$$

$$f_c(48n+13) = \frac{144n+40}{8} = 18n+5, \qquad f_c(48n+29) = \frac{144n+88}{8} = 18n+11,$$

$$f_c(96n+37) = \frac{288n+112}{16} = 18n+7, \qquad f_c(192n+181) = \frac{576n+544}{32} = 18n+17.$$

Thus the six disjoint progressions that form X map bijectively onto the six disjoint progressions that form Y. □

Corollary 3.2 (Collatz sieve). *Every value $y \in Y$ is generated exactly once as $y = f_c(x)$ with $x \in X$.*

4 Heads and the chain decomposition of the core

AI assistance disclosure. The proofs in this section were drafted with the assistance of the AI language model OpenAI ChatGPT 5.2 Thinking and subsequently reviewed and edited by the author.

4.1 The core inverse map

Definition 4.1 (Core inverse). Let $g : Y \to X$ be the unique map satisfying

$$f_c(g(y)) = y \qquad (y \in Y).$$

Equivalently, g is the inverse of the bijection $f_c|_X$ from Theorem 3.1.

Lemma 4.2 (Explicit formula for g). *Let $y \in Y$ and write $r \equiv y \pmod{18}$, so that $r \in \{1, 5, 7, 11, 13, 17\}$. Then*

$$g(y) = \begin{cases} \dfrac{4y-1}{3}, & r \in \{1, 13\}, \\ \dfrac{8y-1}{3}, & r \in \{5, 11\}, \\ \dfrac{16y-1}{3}, & r = 7, \\ \dfrac{32y-1}{3}, & r = 17. \end{cases}$$

Proof. This is the branchwise inversion of the computations in Theorem 3.1. For instance, $y = 18n + 1$ comes from $x = 24n + 1$, and $24n + 1 = (4y - 1)/3$. The remaining residue classes are identical. □

4.2 Strict decrease inside X

Lemma 4.3 (Strict decrease on the core). *For all $x \in X$ with $x > 1$ one has $f_c(x) < x$. In particular, the only periodic point of f_c contained in X is the fixed point 1.*

Proof. By Theorem 3.1, every $x \in X$ satisfies $f_c(x) = (3x + 1)/2^i$ with $i \geq 2$. Hence for $x > 1$,

$$f_c(x) \leq \frac{3x+1}{4} < x,$$

because $3x+1 < 4x$ for $x > 1$. Also $f_c(1) = 1$. Therefore no cycle inside X can exist other than $\{1\}$. □

Lemma 4.4 (Core inverse is strictly increasing). *For all $y \in Y$ with $y > 1$ one has $g(y) > y$. Consequently, for any $y > 1$ the iterates $y, g(y), g^2(y), \ldots$ are strictly increasing and pairwise distinct.*

Proof. By Lemma 4.2, $g(y) = (2^t y - 1)/3$ for some $t \in \{2, 3, 4, 5\}$. Thus

$$g(y) - y = \frac{(2^t - 3)y - 1}{3} \geq \frac{y - 1}{3} > 0$$

because $2^t - 3 \geq 1$ and $y > 1$. The strict increase of iterates follows. □

4.3 Heads and infinite head chains

Definition 4.5 (Heads and head chains)**.** Define the set of *heads*

$$H := \{h \in X \setminus \{1\} : f_c(h) \notin X\}.$$

For $h \in H$ define the (core) *head chain* rooted at h by

$$C(h) := \{g^k(h) : k \in \mathbb{N}_0\} = \big(h,\ g(h),\ g^2(h), \dots\big).$$

4.4 Heads as arithmetic progressions

Because X is a finite union of congruence classes modulo 192, the head condition $f_c(h) \notin X$ can be decided by a finite congruence computation inside each branch.

Proposition 4.6 (Explicit description of the head set)**.** *For the core X in* (1)*, the head set H from Definition 4.5 is the disjoint union*

$$\begin{aligned} H = \{48n + 25\} &\cup \{768n + 625\} \cup \{48n + 41\} \cup \{768n + 113\} \\ &\cup \{96n + 61\} \cup \{384n + 13\} \cup \{96n + 29\} \cup \{384n + 269\} \\ &\cup \{192n + 37\} \cup \{3072n + 1477\} \cup \{384n + 373\} \cup \{1536n + 565\}, \end{aligned}$$

where all unions range over $n \in \mathbb{N}_0$. Equivalently, within each defining progression of X the head condition is:

$$\begin{aligned} x = 24n + 1 &\ :\ n \text{ odd or } n \equiv 26 \pmod{32}, \\ x = 24n + 17 &\ :\ n \text{ odd or } n \equiv 4 \pmod{32}, \\ x = 48n + 13 &\ :\ n \text{ odd or } n \equiv 0 \pmod{8}, \\ x = 48n + 29 &\ :\ n \text{ even or } n \equiv 5 \pmod{8}, \\ x = 96n + 37 &\ :\ n \text{ even or } n \equiv 15 \pmod{32}, \\ x = 192n + 181 &\ :\ n \text{ odd or } n \equiv 2 \pmod{8}. \end{aligned}$$

Proof. We illustrate the method on the first branch. If $x = 24n + 1$, then $f_c(x) = 18n + 1$. Membership in X is determined modulo 192, and since $\gcd(18, 192) = 6$ the residue of $18n + 1$ modulo 192 depends only on $n \bmod 32$. A direct check shows that $18n + 1 \in X$ exactly for

$$n \bmod 32 \in \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 28, 30\},$$

hence x is a head iff n is odd or $n \equiv 26 \pmod{32}$. Rewriting these congruence conditions in terms of x yields the two progressions $\{48m + 25\}$ and $\{768m + 625\}$. The remaining five branches are analogous. □

Remark 4.7 (Origin of the finite progression description)**.** The fact that H splits into finitely many arithmetic progressions is not accidental. On the present core X, the 2–adic exponent $i = v_2(3x + 1)$ is constant on each branch and satisfies $i \in \{2, 3, 4, 5\}$. Consequently, on each branch one has an affine formula $f_c(x) = 18n + r$ with fixed $r \in \{1, 5, 7, 11, 13, 17\}$. Deciding whether $f_c(x) \in X$ is therefore a purely congruence–theoretic problem: since X is a union of residue classes modulo 192, membership of $18n + r$ in X depends only on n modulo a power of two (in the proof above, $n \bmod 32$ or $n \bmod 8$). Hence the head condition $f_c(x) \notin X$ is periodic in the parameter n with dyadic period, and rewriting the permitted congruence classes yields a finite union of arithmetic progressions.

Corollary 4.8 (Density of heads)**.** *The head set has natural density $\delta(H) = 81/1024$. Consequently, a proportion $\delta(H)/\delta(X) = 9/16$ of the elements of X are heads.*

Proof. The progressions in Proposition 4.6 are disjoint, so their natural densities add. Hence

$$\delta(H) = \frac{1}{48} + \frac{1}{768} + \frac{1}{48} + \frac{1}{768} + \frac{1}{96} + \frac{1}{384} + \frac{1}{96} + \frac{1}{384} + \frac{1}{192} + \frac{1}{3072} + \frac{1}{384} + \frac{1}{1536} = \frac{81}{1024}.$$

Dividing by $\delta(X) = 9/64$ (see Proposition 5.1) gives $\delta(H)/\delta(X) = 9/16$. □

Lemma 4.9 (Each head chain is infinite)**.** *For every $h \in H$ the chain $C(h)$ is infinite and contains no duplicates.*

Proof. Since $h \in X \subset Y$ and $g : Y \to X \subset Y$, all iterates $g^k(h)$ are defined and lie in X. By Lemma 4.4, the sequence $h, g(h), g^2(h), \ldots$ is strictly increasing, hence infinite and without repetition. □

Theorem 4.10 (Head–chain decomposition of X)**.** *Let X be the concrete core set* (1) *and H the head set from Definition 4.5. Then:*

1. ***Existence.*** *For every $x \in X \setminus \{1\}$ there exist $h \in H$ and $k \in \mathbb{N}_0$ such that $x = g^k(h)$.*
2. ***Uniqueness / no duplicates.*** *If $g^k(h) = g^{k'}(h')$ with $h, h' \in H$ and $k, k' \in \mathbb{N}_0$, then $h = h'$ and $k = k'$.*

In particular,

$$X = \{1\} \,\dot\cup\, \bigsqcup_{h \in H} C(h)$$

is a disjoint union of the head chains.

Proof. Existence. Let $x \in X \setminus \{1\}$. Consider the forward iterates $x, f_c(x), f_c^2(x), \ldots$. By Lemma 4.3, as long as the orbit stays inside X it strictly decreases, hence cannot remain in X forever. Let $t \geq 1$ be minimal with $f_c^t(x) \notin X$, and set $h := f_c^{t-1}(x)$. Then $h \in X \setminus \{1\}$ and $f_c(h) \notin X$, so $h \in H$. Because g is the inverse of $f_c|_X$, we have $g(h) \in X$ and $f_c(g(h)) = h$, and inductively $f_c(g^j(h)) = g^{j-1}(h)$. Reversing $t-1$ steps yields $x = g^{t-1}(h)$.

Uniqueness. Assume $g^k(h) = g^{k'}(h')$. Apply $f_c^{\min(k,k')}$ to cancel common g-iterates (using $f_c(g(y)) = y$). This reduces to an equality of the form $g^m(h) = h'$ or $h = g^m(h')$ for some $m \geq 0$. If $m > 0$, then applying f_c gives $f_c(h') \in X$ or $f_c(h) \in X$, contradicting that h, h' are heads. Thus $m = 0$ and $h = h'$. Since g is injective (it is the inverse of a bijection), we then also have $k = k'$. □

Corollary 4.11 (Core escape time along a chain)**.** *Let $h \in H$ and $k \in \mathbb{N}_0$. Then the forward orbit of $g^k(h)$ stays inside X for exactly $k+1$ steps:*

$$g^k(h),\ g^{k-1}(h),\ \ldots,\ g(h),\ h \in X, \qquad f_c(h) \notin X.$$

In particular, moving one step upward along a head chain increases the time spent inside X by one.

Proof. From $f_c(g(y)) = y$ we get $f_c(g^j(h)) = g^{j-1}(h)$ for $j \geq 1$. Iterating, $f_c^j(g^k(h)) = g^{k-j}(h)$ for $0 \leq j \leq k$. The head condition gives $f_c^{k+1}(g^k(h)) = f_c(h) \notin X$. □

Remark 4.12 (Example: the chain above 41)**.** Since $f_c(41) = 31 \notin X$, the value 41 is a head. Iterating g gives

$$41,\ 109,\ 145,\ 193,\ 257,\ 685,\ 913,\ 1217, \ldots$$

and Corollary 4.11 implies that the n-th term stays inside X for exactly $n+1$ core steps.

5 Brief remarks on density and other cores

Proposition 5.1 (Density of Y and this core X)**.** *The natural density of Y in $\mathbb{N}$ is $\delta(Y) = 1/3$ and the natural density of X is $\delta(X) = 9/64$.*

Proof. Y consists of two residue classes modulo 6, hence $\delta(Y) = 2/6 = 1/3$. The set X is a disjoint union of arithmetic progressions with moduli $24, 24, 48, 48, 96, 192$, so

$$\delta(X) = \frac{1}{24} + \frac{1}{24} + \frac{1}{48} + \frac{1}{48} + \frac{1}{96} + \frac{1}{192} = \frac{9}{64}.$$

□

Remark 5.2 (Many other cores)**.** The construction is not unique: for each residue class $r \in \{1, 5, 7, 11, 13, 17\}$ modulo 18 there are infinitely many exponents $i \geq 1$ for which $u = (2^i(18n+r)-1)/3$ is an arithmetic progression in Y and maps bijectively onto $18n+r$ under f_c. Choosing one admissible i for each r produces a new core set $X' \subset Y$ with $f_c|_{X'}$ bijective onto Y. Different choices trade off density versus the internal structure of the induced dynamics; compare the discussion of inverse-branch structure in [1,4].

6 Discussion and outlook

The sieve (Theorem 3.1) replaces the many-to-one inverse structure of the odd Collatz map on Y by a canonical choice of one predecessor per $y \in Y$. For the particular core (1), the induced subgraph on X becomes exceptionally rigid: strict decrease forces a union of one-sided chains, and Theorem 4.10 shows that these chains partition X without duplicates.

A natural next step is to study the distribution of head chains (their "basins" in finite windows) and to understand which residue patterns along the inverse map g produce unusually long chains. While this does not by itself resolve the Collatz conjecture, it provides a clean reduction and a concrete combinatorial object (heads and chains) whose behavior can be investigated both experimentally and theoretically.

References

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