The Conditional Maintenance Work Theorem:

Operational Power Lower Bounds from Energy Pinching
and a Split-Inclusion Blueprint for Type III AQFT

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Abstract

We derive operational lower bounds on the minimal thermodynamic power required to maintain quantum coherence against uncontrolled open-system dynamics. Work is defined as the increase of non-equilibrium free energy stored in an explicit battery at bath temperature T, namely $W := \Delta\mathcal{F}_W$ with $\mathcal{F}_W(\sigma) = k_B T\, S(\sigma\|\gamma_W)$, and allowed controls are battery-assisted thermal operations implemented by energy-conserving global unitaries. Coherence is quantified as a relative entropy to a conditional expectation: $\mathcal{C}^{\mathcal{E}}(\rho) := S(\rho\|\mathcal{E}[\rho])$. Our main proved results are (i) a total-power maintenance bound for the energy pinching Δ under an explicit diagonal-contraction hypothesis satisfied by Davies generators, and (ii) an assumption-free *extra-power* variant defined operationally via an infimum over *pairs* of strategies (full-state maintenance and population maintenance), bounding the incremental stabilization cost of coherence at fixed populations. We further provide a conditional extension to general γ_S-preserving conditional expectations and a Type III split blueprint. External geometric inputs are encoded as explicit interface assumptions, yielding separate one-shot work bounds and conditional maintenance-power scalings once a collar-dependent relaxation hypothesis is specified. These results provide an operational resource criterion for quantum-to-classical behavior, rather than a collapse theory.

Contents

1 Introduction

1.1 Problem

Maintaining coherence is a dynamical control task: an uncontrolled environment drives a system toward a thermal fixed point, while a controller expends work to hold a target state stationary. This paper asks for operational lower bounds on the minimal *power* required to maintain a target state under explicit thermodynamic control primitives.

1.2 Contribution and architecture

We use a deliberately modular structure:

(C1) From static work to dynamical power. Static free-energy/work inequalities are translated into maintenance-power lower bounds.

(C2) Paired-strategy rigor. The incremental cost of coherence stabilization is formulated as an infimum over *pairs* of strategies, avoiding invalid manipulations of differences of infima.

(C3) Proved/conditional/interface separation. Finite-dimensional results are proved; general-$\mathcal{E}$ and Type III statements are conditional; geometric inputs appear only as explicit interface assumptions.

1.3 Editorial status and dependency map

This manuscript is self-contained at the level of proved results.

(S1) Closed results proved here: Proposition 2.7, Proposition 3.5, Theorems 4.9 and 4.12, and Corollary 5.2, with full appendices.

(S2) Conditional extensions: Assumption 6.1 and Theorem 6.3, and the Type III blueprint in section 7.

(S3) External geometric inputs: encoded by Assumptions 8.1 and 8.4.

2 Operational framework: system–bath–battery

Remark 2.1 (Entropy convention). We use natural logarithms. All entropies are measured in nats.

Definition 2.2 (Thermal state). Let X be finite-dimensional with Hamiltonian H_X and inverse temperature $\beta := 1/(k_B T)$. Define $\gamma_X := e^{-\beta H_X} / \mathrm{Tr}(e^{-\beta H_X})$.

Definition 2.3 (Relative entropy). For states ρ, σ, define $S(\rho\|\sigma) := \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$ when $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ and $S(\rho\|\sigma) = +\infty$ otherwise.

Definition 2.4 (Non-equilibrium free energy). For any finite-dimensional system X at temperature T, define

$$\mathcal{F}_X(\sigma) \;:=\; k_B T\, S(\sigma\|\gamma_X).$$

Definition 2.5 (Battery-assisted thermal operation). A battery-assisted thermal operation on S is any reduced map induced by a global unitary U on SBW such that $[U, H_S + H_B + H_W] = 0$, acting as

$$\rho_S \;\mapsto\; \mathrm{Tr}_{BW}\big(U(\rho_S \otimes \gamma_B \otimes \sigma_W)U^\dagger\big).$$

Definition 2.6 (Work and power)**.** If the battery changes from σ_W to σ'_W, define work

$$W \;:=\; \mathcal{F}_W(\sigma'_W) - \mathcal{F}_W(\sigma_W),$$

and average power over time t as $P := W/t$.

Proposition 2.7 (Work pays for system free energy)**.** *Let a battery-assisted thermal operation map $\rho_S \mapsto \rho'_S$ while the battery changes $\sigma_W \mapsto \sigma'_W$. Then*

$$W \;\geq\; \mathcal{F}_S(\rho'_S) - \mathcal{F}_S(\rho_S).$$

Proof. See Appendix A. □

3 Conditional expectations and coherence

Definition 3.1 (Finite-dimensional conditional expectation)**.** Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_S)$ be a unital *-subalgebra. A map $\mathcal{E} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{A}$ is a conditional expectation if it is completely positive, trace-preserving, unital, idempotent, and satisfies the bimodule property

$$\mathcal{E}(AXB) = A\,\mathcal{E}(X)\,B \quad \text{for all } A, B \in \mathcal{A},\ X \in \mathcal{B}(\mathcal{H}_S).$$

Definition 3.2 (γ_S-preserving conditional expectation)**.** A conditional expectation $\mathcal{E}$ is γ_S-preserving if $\mathcal{E}^\dagger(\gamma_S) = \gamma_S$, equivalently $\mathrm{Tr}(\gamma_S \mathcal{E}(X)) = \mathrm{Tr}(\gamma_S X)$ for all X.

Definition 3.3 (Relative-entropy coherence to $\mathcal{E}$)**.** Define

$$\mathcal{C}^{\mathcal{E}}(\rho) \;:=\; S(\rho \| \mathcal{E}[\rho]).$$

Assumption 3.4 (Faithfulness)**.** In Sections 4–5, the target state ρ is assumed faithful (full rank), so that $\mathcal{C}^{\mathcal{E}}(\rho) < \infty$.

3.1 Energy pinching

Let $H_S = \sum_n E_n \Pi_n$ be the spectral decomposition. Define energy pinching

$$\Delta[\rho] \;:=\; \sum_n \Pi_n \rho \Pi_n.$$

Proposition 3.5 (Pythagorean identity for pinching)**.** *Let $\gamma_S \propto e^{-\beta H_S}$. Then*

$$S(\rho \| \gamma_S) \;=\; S(\Delta[\rho] \| \gamma_S) \;+\; S(\rho \| \Delta[\rho]).$$

Equivalently,

$$\mathcal{F}_S(\rho) \;=\; \mathcal{F}_S(\Delta[\rho]) \;+\; k_{\mathrm{B}} T\, \mathcal{C}^{\Delta}(\rho).$$

Proof. Because $\log \gamma_S = -\beta H_S - \log Z$ is diagonal in the $\{\Pi_n\}$ basis, $\mathrm{Tr}(\rho \log \gamma_S) = \mathrm{Tr}(\Delta[\rho] \log \gamma_S)$, hence

$$S(\rho \| \gamma_S) - S(\Delta[\rho] \| \gamma_S) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\Delta[\rho] \log \Delta[\rho]) = S(\rho \| \Delta[\rho]).$$

□

4 Single-law maintenance bounds

4.1 Uncontrolled dynamics, maintenance, and power

Assumption 4.1 (Markovian setting)**.** The uncontrolled evolution is a quantum dynamical semigroup $\rho_t = e^{t\mathcal{L}}(\rho)$ with a stationary thermal state γ_S, i.e. $\mathcal{L}(\gamma_S) = 0$.

Definition 4.2 (Coherence-loss rate)**.** Let $\rho_t = e^{t\mathcal{L}}(\rho)$. Define

$$\dot{\mathcal{C}}^{\mathcal{E}}_{\text{loss}}(\rho) \;:=\; -\frac{d}{dt}\mathcal{C}^{\mathcal{E}}(\rho_t)\Big|_{t=0}.$$

Definition 4.3 (Maintenance task)**.** A control strategy maintains ρ if the controlled reduced state satisfies $\rho_t = \rho$ for all $t \geq 0$.

Definition 4.4 (Asymptotic power of a strategy)**.** For a strategy s with battery state $\sigma^{\mathsf{s}}_W(t)$, define

$$P(\mathsf{s}) \;:=\; \limsup_{t\to\infty} \frac{1}{t}\Big(\mathcal{F}_W(\sigma^{\mathsf{s}}_W(t)) - \mathcal{F}_W(\sigma^{\mathsf{s}}_W(0))\Big).$$

Definition 4.5 (Minimal maintenance power)**.** Let $\mathsf{Ctrl}(\rho)$ be the set of battery-assisted thermal-operation strategies that maintain ρ. Define

$$P_{\min}(\rho; \mathcal{L}, T) \;:=\; \inf_{\mathsf{s}\in\mathsf{Ctrl}(\rho)} P(\mathsf{s}).$$

Lemma 4.6 (Infinitesimal to asymptotic)**.** *Fix $\delta t > 0$ and consider stroboscopic maintenance: evolve uncontrolled for time δt and then apply a correction. If each cycle costs at least $w(\delta t)$, then any such strategy satisfies*

$$\limsup_{t\to\infty} \frac{W(t)}{t} \;\geq\; \frac{w(\delta t)}{\delta t}.$$

If $w(\delta t) = a\,\delta t + o(\delta t)$ as $\delta t \to 0$, then $P_{\min} \geq a$.

Proof. After N cycles, $t = N\delta t$ and $W(t) \geq N w(\delta t)$, so $W(t)/t \geq w(\delta t)/\delta t$. Take $\limsup$ in t. □

4.2 Total-power theorem (pinching) under diagonal contraction

Assumption 4.7 (Diagonal contraction)**.** Assume that $t \mapsto S(\Delta[\rho_t]\|\Delta[\gamma_S])$ is nonincreasing:

$$\frac{d}{dt}S(\Delta[\rho_t]\|\Delta[\gamma_S]) \;\leq\; 0 \qquad \text{for all } t \geq 0.$$

Since $\Delta[\gamma_S] = \gamma_S$, this equals $\frac{d}{dt}S(\Delta[\rho_t]\|\gamma_S) \leq 0$.

Proposition 4.8 (Davies generator implies diagonal contraction)**.** *If $\mathcal{L}$ is a Davies generator obtained by weak coupling and secular approximation in the energy basis, then populations evolve as a classical detailed-balance Markov semigroup with stationary distribution given by the diagonal of γ_S. Consequently, Assumption 4.7 holds.*

Proof sketch. Under Davies form, the diagonal algebra is invariant and decouples from coherences; the induced classical semigroup satisfies detailed balance. The classical H-theorem gives monotonicity of $S(\Delta[\rho_t]\|\gamma_S)$. See [1, 2]. □

Theorem 4.9 (Single-law bound for total maintenance power (proved, pinching case))**.** *Assume Assumptions 3.4, 4.1, and 4.7. Then*

$$P_{\min}(\rho; \mathcal{L}, T) \;\geq\; k_{\mathrm{B}}T\,\dot{\mathcal{C}}^{\Delta}_{\text{loss}}(\rho).$$

Proof. Fix a small cycle time $\delta t > 0$ and write $\rho_{\delta t} = e^{\delta t \mathcal{L}}(\rho)$. One maintenance cycle must implement $\rho_{\delta t} \mapsto \rho$ via battery-assisted thermal operations. By Proposition 2.7,

$$w(\delta t) \;\geq\; \mathcal{F}_S(\rho) - \mathcal{F}_S(\rho_{\delta t}) = k_{\mathrm{B}} T\Big(S(\rho \| \gamma_S) - S(\rho_{\delta t} \| \gamma_S)\Big).$$

By Proposition 3.5 applied to ρ and $\rho_{\delta t}$,

$$S(\rho\|\gamma_S) - S(\rho_{\delta t}\|\gamma_S) = \Big(S(\Delta[\rho]\|\gamma_S) - S(\Delta[\rho_{\delta t}]\|\gamma_S)\Big) + \Big(\mathcal{C}^{\Delta}(\rho) - \mathcal{C}^{\Delta}(\rho_{\delta t})\Big).$$

By Assumption 4.7, the first bracket is nonnegative, hence

$$w(\delta t) \;\geq\; k_{\mathrm{B}} T\Big(\mathcal{C}^{\Delta}(\rho) - \mathcal{C}^{\Delta}(\rho_{\delta t})\Big).$$

Divide by δt and let $\delta t \to 0$ using Lemma 4.6 to obtain the claim. □

4.3 Assumption-free experimental variant: extra power

Assumption 4.10 (Non-emptiness of control sets)**.** Assume $\mathsf{Ctrl}(\rho) \neq \emptyset$ and $\mathsf{Ctrl}(\Delta[\rho]) \neq \emptyset$.

Definition 4.11 (Extra power (paired strategies))**.** Define the extra power as an infimum over pairs:

$$P_{\text{extra}}(\rho; \mathcal{L}, T) \;:=\; \inf_{\substack{\mathsf{s} \in \mathsf{Ctrl}(\rho) \\ \mathsf{s}_{\text{diag}} \in \mathsf{Ctrl}(\Delta[\rho])}} \Big(P(\mathsf{s}) - P(\mathsf{s}_{\text{diag}})\Big).$$

Operationally, $P(\mathsf{s}) - P(\mathsf{s}_{\text{diag}})$ corresponds to the incremental battery power required when a coherence-stabilizing controller s is run on top of (or compared against) a population-stabilizing baseline controller s_{diag}.

Theorem 4.12 (Single-law bound for extra coherence power (proved, assumption-free diagonal))**.** *Assume Assumptions 3.4, 4.1, and 4.10. Then*

$$P_{\text{extra}}(\rho; \mathcal{L}, T) \;\geq\; k_{\mathrm{B}} T\, \dot{\mathcal{C}}^{\Delta}_{\text{loss}}(\rho).$$

Proof. Fix $\delta t > 0$ and set $\rho_{\delta t} = e^{\delta t \mathcal{L}}(\rho)$. Consider an arbitrary pair of stroboscopic maintenance strategies:

- $\mathsf{s} \in \mathsf{Ctrl}(\rho)$ restores ρ after the step $\rho \mapsto \rho_{\delta t}$;
- $\mathsf{s}_{\text{diag}} \in \mathsf{Ctrl}(\Delta[\rho])$ restores $\Delta[\rho]$ after $\Delta[\rho] \mapsto \Delta[\rho_{\delta t}]$.

Let $w_{\mathsf{s}}(\delta t)$ and $w_{\mathsf{s}_{\text{diag}}}(\delta t)$ denote the corresponding per-cycle works.
By Proposition 2.7,

$$w_{\mathsf{s}}(\delta t) \;\geq\; \mathcal{F}_S(\rho) - \mathcal{F}_S(\rho_{\delta t}) = k_{\mathrm{B}} T\Big(S(\rho\|\gamma_S) - S(\rho_{\delta t}\|\gamma_S)\Big),$$

and similarly

$$w_{\mathsf{s}_{\text{diag}}}(\delta t) \;\geq\; \mathcal{F}_S(\Delta[\rho]) - \mathcal{F}_S(\Delta[\rho_{\delta t}]) = k_{\mathrm{B}} T\Big(S(\Delta[\rho]\|\gamma_S) - S(\Delta[\rho_{\delta t}]\|\gamma_S)\Big).$$

Subtracting and using Proposition 3.5 cancels the diagonal contribution:

$$w_{\mathsf{s}}(\delta t) - w_{\mathsf{s}_{\text{diag}}}(\delta t) \;\geq\; k_{\mathrm{B}} T\Big(\mathcal{C}^{\Delta}(\rho) - \mathcal{C}^{\Delta}(\rho_{\delta t})\Big).$$

Final step (order of $\inf$ **and** $\limsup$**).** Fix $\delta t > 0$ and fix an arbitrary pair $(\mathsf{s}, \mathsf{s}_{\text{diag}})$. Lemma 4.6 implies

$$P(\mathsf{s}) \;\geq\; \frac{w_{\mathsf{s}}(\delta t)}{\delta t}, \qquad P(\mathsf{s}_{\text{diag}}) \;\geq\; \frac{w_{\mathsf{s}_{\text{diag}}}(\delta t)}{\delta t},$$

hence

$$P(\mathsf{s}) - P(\mathsf{s}_{\rm diag}) \;\geq\; \frac{w_{\mathsf{s}}(\delta t) - w_{\mathsf{s}_{\rm diag}}(\delta t)}{\delta t}.$$

From the per-cycle bound above,

$$\frac{w_{\mathsf{s}}(\delta t) - w_{\mathsf{s}_{\rm diag}}(\delta t)}{\delta t} \;\geq\; k_{\rm B}T\,\frac{\mathcal{C}^{\Delta}(\rho) - \mathcal{C}^{\Delta}(\rho_{\delta t})}{\delta t}.$$

The right-hand side is independent of the chosen pair, therefore for each fixed δt,

$$\inf_{(\mathsf{s},\mathsf{s}_{\rm diag})} \big(P(\mathsf{s}) - P(\mathsf{s}_{\rm diag})\big) \;\geq\; k_{\rm B}T\,\frac{\mathcal{C}^{\Delta}(\rho) - \mathcal{C}^{\Delta}(\rho_{\delta t})}{\delta t}.$$

Taking $\limsup_{\delta t\to 0}$ yields

$$P_{\rm extra}(\rho;\mathcal{L},T) \;\geq\; k_{\rm B}T\,\dot{\mathcal{C}}^{\Delta}_{\rm loss}(\rho),$$

which completes the proof. □

5 Qubit specialization

Assumption 5.1 (Qubit pure dephasing)**.** Let S be a qubit in the energy basis such that under uncontrolled dynamics $\frac{d}{dt}\rho_{01}(t) = -\Gamma_\phi\,\rho_{01}(t)$ and populations are constant.

Corollary 5.2 (Qubit law (leading order))**.** *Under Assumption 5.1, in the small-coherence regime,*

$$P_{\rm extra}(\rho;\mathcal{L},T) \;\gtrsim\; 2\,k_{\rm B}T\,\Gamma_\phi\,\mathcal{C}^{\Delta}(\rho).$$

Under Assumption 4.7, the same leading-order scaling also holds as a lower bound for $P_{\rm min}(\rho;\mathcal{L},T)$ via Theorem 4.9.

6 Conditional extension to general conditional expectations

Assumption 6.1 (Pythagorean identity for general $\mathcal{E}$)**.** Let $\mathcal{E}:\mathcal{B}(\mathcal{H}_S)\to\mathcal{A}$ be a γ_S-preserving conditional expectation. Assume the Pythagorean identity holds for all faithful ρ:

$$S(\rho\|\gamma_S) \;=\; S(\mathcal{E}[\rho]\|\gamma_S) \;+\; S(\rho\|\mathcal{E}[\rho]).$$

Equivalently,

$$\mathcal{F}_S(\rho) \;=\; \mathcal{F}_S(\mathcal{E}[\rho]) \;+\; k_{\rm B}T\,\mathcal{C}^{\mathcal{E}}(\rho).$$

Remark 6.2. Assumption 6.1 is automatic for pinching $\mathcal{E}=\Delta$ and is related to sufficiency and Petz recovery structure; see [3, 4].

Theorem 6.3 (Single-law bound (conditional general-$\mathcal{E}$))**.** *Assume Assumptions 3.4, 4.1, and 6.1. Then*

$$P_{\rm min}(\rho;\mathcal{L},T) \;\geq\; k_{\rm B}T\,\dot{\mathcal{C}}^{\mathcal{E}}_{\rm loss}(\rho).$$

Proof. Repeat the proof of Theorem 4.9, replacing Δ by $\mathcal{E}$ and using Assumption 6.1. □

7 Type III split blueprint (conditional)

Remark 7.1 (Blueprint status). This section is a blueprint for a full Type III treatment: no new Type III technical proofs (Haagerup L^p, domain control, energy constraints) are claimed here.

Assumption 7.2 (Split inclusion and vacuum-preserving expectation). Let $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) =: \mathcal{M}$ be a split inclusion with intermediate Type I factor $\mathcal{N}$ and vacuum state ω_0. Assume $\mathcal{N}$ is $\sigma_t^{\omega_0}$-invariant, hence there exists a unique ω_0-preserving conditional expectation $E_{\omega_0} : \mathcal{M} \to \mathcal{N}$.

Definition 7.3 (Split-relative entropy functional (Type III, conditional)). Define

$$\mathcal{C}^{\mathrm{split}}(\omega) \ := \ S(\omega \| \omega \circ E_{\omega_0}),$$

where S is Araki relative entropy.

8 Geometric interface and scaling laws

8.1 One-shot work vs maintenance power

We distinguish:

- **One-shot shaping (work):** reduce a leakage proxy at fixed collar width ϵ, yielding a bound on W.
- **Sustained maintenance (power):** keep a target (or leakage level) stationary in time, yielding a bound on P.

Assumption 8.1 (Geometric leakage amplitude interface). In a bosonic Gaussian split regime with collar width ϵ and mass gap $m > 0$, assume there exists a bounded leakage proxy $\mathrm{Leak}(\epsilon)$ satisfying a baseline bound

$$\mathrm{Leak}(\epsilon) \ \le \ A\,(m\epsilon)^{\alpha}\, K_{\nu}(m\epsilon),$$

for constants $A > 0$, $\alpha \ge 0$, and modified Bessel K_ν. Assume also weak-correlation quadraticity of mutual information in the weak-leakage regime:

$$I(S : \mathrm{Env}) \ \asymp \ \mathrm{Leak}(\epsilon)^2.$$

Remark 8.2 (On the geometric interface assumptions). Assumptions 8.1 and 8.4 are stated here as explicit interfaces to keep the present manuscript logically autonomous. A concrete derivation of Bessel-type collar suppression in Gaussian split configurations (massive free-field regime) and its connection to recoverability via split inclusions is provided in the companion preprint [11]. In [11] the collar width is denoted by r; in this paper we use ϵ.

Corollary 8.3 (One-shot geometric work scaling). *Under Assumption 8.1, any thermal-operation protocol at temperature T that reduces the leakage proxy from its baseline value* $\mathrm{Leak}(\epsilon)$ *to a target level δ satisfies a Landauer-type lower bound of the form*

$$W \ \ge \ k_{\mathrm{B}}T\Big(\underline{\kappa}\,\mathrm{Leak}(\epsilon)^2 - \overline{\kappa}\,\delta^2\Big)_+,$$

hence in particular

$$W \ \gtrsim \ k_{\mathrm{B}}T\,K_{\nu}(m\epsilon)^2$$

up to polynomial prefactors in $m\epsilon$ and state-family dependent constants.

Assumption 8.4 (Bessel-suppressed relaxation rate). Assume the uncontrolled relaxation rate satisfies

$$\kappa(\epsilon) \ \le \ B\,(m\epsilon)^{\beta}\, K_{\nu}(m\epsilon)$$

for constants $B > 0$ and $\beta \ge 0$ within the regime of interest.

Corollary 8.5 (Conditional maintenance-power scaling)**.** *Assume the uncontrolled evolution yields* $\mathcal{C}^{\Delta}(\rho_t) = \mathcal{C}^{\Delta}(\rho)e^{-\kappa(\epsilon)t}$ *for* t *near* 0 *and Assumption 8.4. Then*

$$P_{\text{extra}}(\epsilon) \;\geq\; k_{\text{B}}T\,\kappa(\epsilon)\,\mathcal{C}^{\Delta}(\rho),$$

and, in particular,

$$P_{\text{extra}}(\epsilon) \;\gtrsim\; k_{\text{B}}T\,K_{\nu}(m\epsilon)$$

up to polynomial prefactors in $m\epsilon$*.*

9 Interpretation and scope: an operational "cut" (not a collapse theory)

Remark 9.1 (What this paper does and does not claim)*.* This work does not propose a collapse mechanism, does not derive the Born rule, and does not claim to solve the measurement problem in the strong foundational sense. The results are operational: they quantify minimal work/power requirements for maintaining a target state against uncontrolled dynamics under explicit thermodynamic control restrictions (battery-assisted thermal operations at temperature T).

Remark 9.2 (Quantum-to-classical transition as resource infeasibility (operational))*.* Theorems 4.9 and 4.12 imply that sustaining coherence against uncontrolled noise requires a minimal power budget proportional to $k_{\text{B}}T$ times an intrinsic coherence-loss rate. When this required power exceeds available control resources, coherence cannot be maintained in steady state. This is an operational statement about control feasibility (classical-like behavior in the sense of suppressed maintainable coherence); it is not, by itself, an ontological account of definite outcomes.

9.1 On a proposed "master inequality"

Remark 9.3 (Work vs power, amplitude vs rate)*.* This paper does not prove a universal inequality of the literal form $I \lesssim E\,e^{-2m\epsilon}$. What it supports, under explicit assumptions, is closer to two distinct statements:

- **One-shot work (amplitude suppression):** if $\text{Leak}(\epsilon) \lesssim (m\epsilon)^{\alpha}K_{\nu}(m\epsilon)$ and mutual information is quadratic in leakage in the weak-correlation regime, then one-shot work typically scales like $W \gtrsim k_{\text{B}}T\,K_{\nu}(m\epsilon)^2 \sim k_{\text{B}}T\,e^{-2m\epsilon}$ for $m\epsilon \gg 1$ up to polynomial factors.
- **Maintenance power (rate suppression):** a quantitative power scaling requires an additional dynamical assumption, e.g. $\kappa(\epsilon) \lesssim (m\epsilon)^{\beta}K_{\nu}(m\epsilon)$ for the relevant relaxation rate. In that case one obtains $P_{\text{extra}} \gtrsim k_{\text{B}}T\,K_{\nu}(m\epsilon) \sim k_{\text{B}}T\,e^{-m\epsilon}$ for $m\epsilon \gg 1$ up to polynomial factors, provided the exponential-decay hypothesis holds for the chosen functional.

10 Protocols and reproducibility

Remark 10.1*.* This paper provides theoretical bounds and reproducibility templates; it does not claim new direct measurements of battery-accounted work in hardware.

10.1 Protocol sketch: incremental coherence stabilization

(P1) Stabilize populations (implement a baseline strategy in $\mathsf{Ctrl}(\Delta[\rho])$).

(P2) Turn on phase/coherence stabilization to implement a strategy in $\mathsf{Ctrl}(\rho)$.

(P3) Measure incremental controller power $P(\mathsf{s}) - P(\mathsf{s}_{\text{diag}})$ and compare against the bound $k_{\text{B}}T\,\dot{\mathcal{C}}^{\Delta}_{\text{loss}}(\rho)$.

11 Conclusion

We derived single-law operational lower bounds on minimal coherence-maintenance power under explicit thermodynamic control constraints. The pinching case yields a total-power bound under diagonal contraction (automatic for Davies generators), and an assumption-free extra-power bound aligns directly with incremental stabilization measurements. Conditional extensions to general expectations and Type III split configurations are provided as blueprints, and geometric inputs are incorporated only via explicit interface assumptions separating one-shot work from maintenance power.

A Work bound proof via product-reference decomposition

Definition A.1 (Mutual information and multi-information)**.** For a state ρ_{XY} define mutual information

$$I_\rho(X:Y) \;:=\; S(\rho_{XY}\|\rho_X\otimes\rho_Y) \;\geq\; 0.$$

For ρ_{XYZ} define multi-information

$$I_\rho(X:Y:Z) \;:=\; S(\rho_{XYZ}\|\rho_X\otimes\rho_Y\otimes\rho_Z) \;\geq\; 0.$$

Lemma A.2 (Relative entropy decomposition for product references)**.** *Let σ_X, σ_Y be states and ρ_{XY} any joint state. Then*

$$S(\rho_{XY}\|\sigma_X\otimes\sigma_Y) = S(\rho_X\|\sigma_X) + S(\rho_Y\|\sigma_Y) + I_\rho(X:Y).$$

Similarly, for ρ_{XYZ} and product reference $\sigma_X\otimes\sigma_Y\otimes\sigma_Z$,

$$S(\rho_{XYZ}\|\sigma_X\otimes\sigma_Y\otimes\sigma_Z) = S(\rho_X\|\sigma_X) + S(\rho_Y\|\sigma_Y) + S(\rho_Z\|\sigma_Z) + I_\rho(X:Y:Z).$$

Proof. Expand definitions and use $\mathrm{Tr}(\rho_{XY}\log\sigma_X) = \mathrm{Tr}(\rho_X\log\sigma_X)$, etc., then add and subtract $\mathrm{Tr}(\rho_X\log\rho_X)$ and $\mathrm{Tr}(\rho_Y\log\rho_Y)$ to identify the mutual information terms. □

Proof of Proposition 2.7. Let initial global state be $\rho_{SBW} := \rho_S\otimes\gamma_B\otimes\sigma_W$ and final global state $\rho'_{SBW} := U\rho_{SBW}U^\dagger$, with $[U, H_S + H_B + H_W] = 0$. Let the global product reference be $\gamma_{SBW} := \gamma_S\otimes\gamma_B\otimes\gamma_W$.

Invariance of relative entropy under *-automorphisms gives

$$S(U\rho U^\dagger\|U\sigma U^\dagger) = S(\rho\|\sigma).$$

Since $[U, H_S + H_B + H_W] = 0$ implies $U\gamma_{SBW}U^\dagger = \gamma_{SBW}$, we obtain

$$S(\rho'_{SBW}\|\gamma_{SBW}) \;=\; S(\rho_{SBW}\|\gamma_{SBW}).$$

Apply Lemma A.2 to the final state with product reference:

$$S(\rho'_{SBW}\|\gamma_{SBW}) = S(\rho'_S\|\gamma_S)+S(\rho'_B\|\gamma_B)+S(\rho'_W\|\gamma_W)+I_{\rho'}(S:B:W) \;\geq\; S(\rho'_S\|\gamma_S)+S(\rho'_W\|\gamma_W),$$

since $S(\rho'_B\|\gamma_B)\geq 0$ and $I_{\rho'}(S:B:W)\geq 0$.

Apply Lemma A.2 to the initial product state:

$$S(\rho_{SBW}\|\gamma_{SBW}) = S(\rho_S\|\gamma_S)+S(\gamma_B\|\gamma_B)+S(\sigma_W\|\gamma_W)+I_\rho(S:B:W) = S(\rho_S\|\gamma_S)+S(\sigma_W\|\gamma_W),$$

because $S(\gamma_B\|\gamma_B) = 0$ and $I_\rho(S:B:W) = 0$ for product states.

Combining gives

$$S(\rho'_S\|\gamma_S) + S(\rho'_W\|\gamma_W) \;\leq\; S(\rho_S\|\gamma_S) + S(\sigma_W\|\gamma_W),$$

hence

$$S(\rho'_W\|\gamma_W) - S(\sigma_W\|\gamma_W) \;\geq\; S(\rho'_S\|\gamma_S) - S(\rho_S\|\gamma_S).$$

Multiplying by $k_{\mathrm{B}}T$ yields

$$W \;:=\; \mathcal{F}_W(\rho'_W) - \mathcal{F}_W(\sigma_W) \;\geq\; \mathcal{F}_S(\rho'_S) - \mathcal{F}_S(\rho_S).$$

□

B Exact qubit formula for $\dot{\mathcal{C}}^{\Delta}_{\text{loss}}$ under pure dephasing

Let

$$\rho = \begin{pmatrix} p & c \\ c^* & 1-p \end{pmatrix}, \qquad \Delta[\rho] = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}.$$

Let $z := 2p - 1$ and $r := \sqrt{z^2 + 4|c|^2}$. Under pure dephasing with rate Γ_ϕ, p is constant and $|c(t)| = |c|e^{-\Gamma_\phi t}$. Then

$$\dot{\mathcal{C}}^{\Delta}_{\text{loss}}(\rho) = \frac{2\Gamma_\phi |c|^2}{r} \log\left(\frac{1+r}{1-r}\right).$$

For $|c| \ll 1$ at fixed p,

$$\dot{\mathcal{C}}^{\Delta}_{\text{loss}}(\rho) \;=\; 2\Gamma_\phi \, \mathcal{C}^{\Delta}(\rho) \;+\; O(|c|^4).$$

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