Clustering, Recovery, and Locality in Algebraic Quantum Field Theory

Quantitative Bounds via Split Inclusions and Modular Theory

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Abstract

We prove that exponential clustering of vacuum correlations enables approximate reconstruction of global quasi-free states from local data in algebraic quantum field theory. The reconstruction is an explicit Gaussian procedure — *conditional reattachment* through the vacuum's regression structure — which coincides with the output of the Petz recovery map when the reference state factorizes across the split, but *not* in general: for a pure reference the Petz map returns the reference itself for every input, and version 1 of this paper incorrectly identified the two. For quasi-free states of a massive scalar field satisfying natural constraints, including a symplectic-gap (mixedness) condition on the reference state and an admissibility (no-steering) condition on the split geometry, we prove:

$$1 - F(\omega, \tilde{\omega}) \;\le\; \frac{C_{d,\kappa}^{(0)}}{\epsilon^2 \left(1 - \epsilon^{-1}(\eta_{\text{vac}} + \delta)^2\right)^2} \cdot \left\| \Delta^{(12)} \right\|_{\text{HS}}^2$$

where $\Delta^{(12)}$ is the reconstruction error, $\eta_{\text{vac}} \lesssim e^{-mr}$ is the vacuum correlation factor, δ controls cross-correlation perturbations, and $C_{d,\kappa}^{(0)} = \frac{C_\kappa(6+C_\kappa)}{16 \min(c_1,c_2)^2}$ with $C_\kappa = \frac{(1+\kappa)^2}{\kappa(\kappa+2)}$ determined by the symplectic gap $\kappa > 0$. A finite-rank corollary with explicit factor $2n$ recovers physical intuition. All counterexamples and bounds are verified by an exact truncated-Fock numerical suite distributed with the paper (Appendix E). Applications to holographic reconstruction are discussed.

Version 2. This version makes three corrections to v1: (i) the reconstruction map of v1's Proposition 2.14 is *not* the Petz map — its "marginal preservation" step fails for correlated references — and the main theorem is restated for the reconstruction procedure that the proof actually controls; (ii) the simplified constant is corrected ($3/8 \to 7/16$ in the strongly mixed limit); (iii) a symplectic-gap hypothesis is added to the Gaussian fidelity lemma, shown necessary by an explicit counterexample. See Appendix F.

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1 Introduction

1.1 The Central Problem

The interplay between locality, entanglement, and information recovery constitutes a central theme in modern theoretical physics. The Reeh–Schlieder theorem establishes long-range vacuum entanglement, yet correlations decay exponentially for massive theories (clustering).

To what extent does clustering enable approximate reconstruction of global information from local data?

1.2 Main Results

Theorem (Clustering–Recovery Bridge; Theorem 4.8). *For quasi-free states satisfying hypotheses (a)–(g) in the regularized framework, the fidelity between ω and its conditional-reattachment reconstruction $\tilde{\omega} = \Lambda_{\Gamma_0}[\omega|_1]$ (Definition 2.15) satisfies:*

$$1 - F(\omega, \tilde{\omega}) \;\leq\; \frac{C^{(0)}_{d,\kappa}}{\epsilon^2\,(1 - \epsilon^{-1}(\eta_{\mathrm{vac}} + \delta)^2)^2} \cdot \left\| \Delta^{(12)} \right\|^2_{\mathrm{HS}} \tag{1}$$

where:

- $C^{(0)}_{d,\kappa} = \frac{C_\kappa(6+C_\kappa)}{16\min(c_1,c_2)^2}$ *depends on the vacuum spectral bounds c_1, c_2 of Assumption 2.7 and on the symplectic gap κ of hypothesis (f), through $C_\kappa = \frac{(1+\kappa)^2}{\kappa(\kappa+2)}$ (in the simplified regime of Corollary 4.12); in the strongly mixed limit $C_\kappa \to 1$ one recovers $C^{(0)}_{d,\kappa} \to \frac{7}{16\min(c_1,c_2)^2}$,*
- $\eta_{\mathrm{vac}} := \left\| A_0^{-1/2} X_0 B_0^{-1/2} \right\|_{\mathrm{op}}$ *is the vacuum correlation factor,*
- $\delta := \left\| A_0^{-1/2} (X - X_0) B_0^{-1/2} \right\|_{\mathrm{op}}$ *controls cross-correlation changes.*

The recovery error decomposes as:

$$\Delta^{(12)} = (X - X_0) - (A - A_0) A_0^{-1} X_0 \tag{2}$$

Corollary (Finite Rank; Corollary 4.13). *If* $\mathrm{rank}(\Gamma_\omega - \Gamma_0) \leq n$*:*

$$1 - F(\omega, \tilde{\omega}) \;\leq\; \frac{2\,C^{(0)}_{d,\kappa} \cdot n}{\epsilon^2\,(1 - \epsilon^{-1}(\eta_{\mathrm{vac}} + \delta)^2)^2} \cdot \left\| \Delta^{(12)} \right\|^2_{\mathrm{op}} \tag{3}$$

with explicit factor 2 from $\mathrm{rank}(\Delta^{(12)}) \leq 2n$.

Remark (Corrections relative to v1). Three corrections. *First and most important*: the state $\tilde{\omega}$ controlled by the proof is *not* the output of the Petz map, contrary to v1's Proposition 2.14, whose "marginal preservation" step fails for correlated references (Lemma 2.14; for a pure reference the Petz map returns the reference for every input). The theorem is restated for the conditional-reattachment reconstruction, which is what formula (10) describes; it coincides with the Petz output for factorized references and approaches it in the strongly mixed, weakly correlated regime (exact numerics in Appendix E). *Second*: v1 stated the simplified constant as $\frac{3}{8\min(c_1,c_2)^2}$, inconsistent with its own Step 5 (the quartic remainder contributes $\frac{1}{48}$, giving $\frac{7}{48}\times 3 = \frac{7}{16}$). *Third*: the Gaussian fidelity lemma requires a symplectic-gap hypothesis on the reference covariance; without it the quadratic fidelity bound fails for purity-changing perturbations (Remark C.3).

1.3 Relation to Prior Work

The question of when quantum information can be approximately recovered after restriction to a subalgebra has been studied intensively from the quantum information side. Petz [25] characterized exact sufficiency of subalgebras via the recovery map that now bears his name; Fawzi and Renner [10] proved that small conditional mutual information implies approximate recoverability in finite dimensions, and Junge, Renner, Sutter, Wilde and Winter [20] constructed universal recovery maps in infinite-dimensional (Type I) settings.

Closest in spirit to the present work are the results of Faulkner, Hollands, Swingle and Wang [12], and Faulkner and Hollands [11], who proved the existence of universal recovery channels for inclusions of *general* von Neumann algebras, with recovery fidelity controlled by the change in relative entropy with respect to a reference state; see also Junge and LaRacuente [18] for multivariate trace inequalities and p-fidelity beyond tracial settings. Those results are structural: they hold for arbitrary inclusions but do not by themselves quantify *how small* the relevant entropy difference is for a given geometry. The present paper is complementary: we work in a concrete (regularized, quasi-free) setting where an explicit reconstruction can be analyzed in closed form, and we trade generality for *explicit constants* tied to the geometry of a split inclusion, with the collar width r and the mass gap m entering through the clustering factor $\eta_{\mathrm{vac}} \lesssim e^{-mr}$. For the exact Petz map of Gaussian references — which is a Gaussian channel with an explicit covariance action, but *not* the reconstruction analyzed here (Section 2.5) — see Lami, Das and Wilde [19]. Making contact between the two approaches — i.e., controlling the Faulkner–Hollands–Swingle–Wang entropy difference by clustering data — is a promising route toward Conjecture 5.1; see Section 5.4.

On the holographic side, entanglement wedge reconstruction via (twirled) Petz maps was established by Cotler, Hayden, Penington, Salton, Swingle and Walter [7], and simplified to the plain Petz map by Chen, Penington and Salton [6]. Our conditional Corollary 6.1 is a mass-gap-quantified statement in the same direction. For background on entanglement measures, the split property, and modular-theoretic constraints in QFT we refer to Hollands and Sanders [16] and Lashkari [23].

1.4 Structure

Section 2 establishes the algebraic framework. Section 3 develops clustering bounds. Section 4 contains the main theorem. Section 5 formulates the general conjecture. Section 6 presents implications. Appendix F lists the changes relative to v1.

2 Algebraic Framework

2.1 Haag–Kastler Axioms

We work in the Haag–Kastler framework [14] on Minkowski spacetime $\mathbb{R}^{d,1}$ ($d \geq 2$).

Definition 2.1 (Haag–Kastler Net). A net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ of von Neumann algebras on $\mathcal{H}$ satisfying isotony, locality, covariance, and the spectrum condition.

Theorem 2.2 (Reeh–Schlieder). *For any region $\mathcal{O}$ with non-empty causal complement, the vacuum Ω is cyclic and separating for $\mathcal{A}(\mathcal{O})$.*

2.2 Geometric Setup

Definition 2.3 (Split Configuration). Double cones $\mathcal{O}_1 \Subset \mathcal{O}_2$ with collar width $r := \operatorname{dist}(\partial\mathcal{O}_1, \partial\mathcal{O}_2) > 0$.

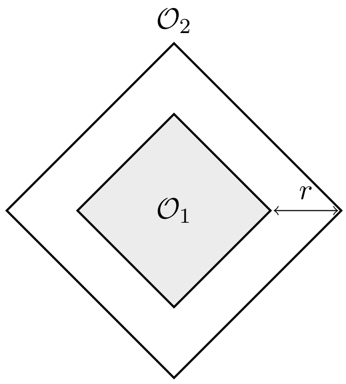

Figure 1: Split configuration: inner region $\mathcal{O}_1$, collar width r.

2.3 Quasi-Free States

Definition 2.4 (Quasi-Free State). A state with characteristic function:

$$\omega(W(f)) = \exp\left(-\tfrac{1}{4}\langle f, \Gamma_\omega f\rangle\right) \tag{4}$$

Quasi-free states in the operator-algebraic setting were developed by Araki [1, 2]; see also [15, Chapter 12] for CCR quasi-free (Gaussian) states.

Definition 2.5 (Block Decomposition). For $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$:

$$\Gamma = \begin{pmatrix} A & X \\ X^T & B \end{pmatrix} \tag{5}$$

with vacuum blocks A_0, X_0, B_0. (All covariance blocks are real matrices in the quadrature representation; A and B are symmetric.)

Remark 2.6 (CCR covariance convention). We work with the bosonic CCR algebra in a regularized finite-mode setting. With the Weyl-operator and characteristic-function conventions of Appendix D (Remark D.1), Gaussian states are parametrized by a real symmetric covariance matrix Γ satisfying the uncertainty condition

$$\Gamma + i\Omega \;\geq\; 0, \tag{6}$$

where Ω is the standard symplectic form; in this normalization the one-mode vacuum has $\Gamma = \mathbf{1}$ and pure Gaussian states are exactly those whose symplectic eigenvalues all equal 1. Under Assumption 2.7, Γ is strictly positive with $\Gamma \geq c \cdot \mathbf{1}$ for some $c > 0$. (In v1 the uncertainty condition was misstated as $\Gamma + \frac{i}{2}\Omega \geq 0$, which belongs to the convention $\chi = e^{-\frac{1}{2}z^T \Gamma z}$; see Appendix F.)

Assumption 2.7 (Regularized one-particle framework and uniform lower bounds)**.** We work in a regularized setting (e.g., a lattice approximation, finite-mode truncation, or an explicit UV cutoff) in which the quasi-free covariances are represented by bounded, strictly positive operators on the one-particle Hilbert space $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. In particular, the vacuum blocks satisfy:

$$A_0 \geq c_1 \cdot \mathbf{1}_{\mathfrak{h}_1}, \qquad B_0 \geq c_2 \cdot \mathbf{1}_{\mathfrak{h}_2} \tag{7}$$

for constants $c_1, c_2 > 0$ depending on the chosen regularization and geometry.

Remark 2.8 (Verification in regularized settings). For finite-volume restrictions with smooth boundaries and Dirichlet boundary conditions, the constants can be estimated from lower bounds on the corresponding elliptic operator. For a spatial ball of radius R one obtains:

$$c_1, c_2 \geq (m^2 + \pi^2/R^2)^{-1/2} \tag{8}$$

For lattice discretizations, c_1, c_2 can be bounded in terms of the smallest eigenvalue of the discrete massive Laplacian on the corresponding finite region. See Reed–Simon [26, Theorem XIII.67] for continuum analogues.

Remark 2.9 (Continuum limit and operator ideal issues). In continuum AQFT, covariance operators and their blocks may be unbounded, and conditions such as $\Gamma_\omega - \Gamma_0 \in \mathcal{L}^2(\mathfrak{h})$ or $\eta_{\text{vac}} = \left\| A_0^{-1/2} X_0 B_0^{-1/2} \right\|_{\text{op}} < 1$ require a careful choice of one-particle space and domains (typically via smearing and Sobolev-type norms). Accordingly, the present bounds are proved in the regularized framework of Assumption 2.7. Extending them to the fully continuum Type III setting is left for future work.

2.4 The Split Property

Theorem 2.10 (Buchholz–Wichmann–D'Antoni–Longo)**.** *For massive scalar fields satisfying nuclearity [5], the split property holds: there exists a Type I factor $\mathcal{N}$ with $\mathcal{A} \subset \mathcal{N} \subset \mathcal{M}$.*

See also Summers [29] for discussion of independence properties of local algebras related to split inclusions.

Theorem 2.11 (Canonical Split Inclusion)**.** *The canonical split factor $\mathcal{N}_{\text{can}}$ satisfies:*

(a) $\mathcal{N}_{\text{can}}$ is Type I with $\mathcal{A} \subset \mathcal{N}_{\text{can}} \subset \mathcal{M}$

(b) $\sigma_t^{\omega_0}(\mathcal{N}_{\text{can}}) = \mathcal{N}_{\text{can}}$ (modular invariance)

Proof. See Doplicher–Longo [9, Theorem 4.1] and Buchholz–D'Antoni–Longo [4, Proposition 3.2]. □

Remark 2.12 (Construction-Based Modular Invariance). For massive fields, modular invariance of $\mathcal{N}_{\text{can}}$ follows from its construction via the nuclearity map, not from geometric modular action (which fails for double cones). The subspace K_Ξ is Δ^{it}-invariant by construction.

2.5 Petz Recovery and Conditional Reattachment

Definition 2.13 (Petz Recovery Map)**.** In the regularized tensor product setting $\mathcal{M} = B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\mathcal{N} = B(\mathcal{H}_1) \otimes \mathbf{1}_{\mathcal{H}_2}$, let ρ_0 be a faithful density operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and write $\rho_{0,1} := \mathrm{Tr}_2(\rho_0)$ for its marginal on subsystem 1. The Petz recovery map $\mathcal{R}_{\rho_0} : \mathcal{S}(\mathcal{H}_1) \to \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is:

$$\mathcal{R}_{\rho_0}(\sigma_1) := \rho_0^{1/2} \left(\rho_{0,1}^{-1/2}\, \sigma_1\, \rho_{0,1}^{-1/2} \otimes \mathbf{1}_{\mathcal{H}_2}\right) \rho_0^{1/2} \tag{9}$$

where $\sigma_1 \geq 0$ is a density operator on $\mathcal{H}_1$ with $\mathrm{Tr}(\sigma_1) = 1$.

The Petz map always recovers the reference perfectly, $\mathcal{R}_{\rho_0}(\rho_{0,1}) = \rho_0$, and satisfies $\mathrm{Tr}_2 \circ \mathcal{R}_{\rho_0} = \mathrm{id}$ *when ρ_0 factorizes.* Version 1 of this paper asserted marginal preservation in general; this is false, and the failure is maximal for pure references:

Lemma 2.14 (Marginal preservation and its failure)**.**

(a) If $\rho_0 = \rho_{0,1} \otimes \rho_{0,2}$, then $\mathcal{R}_{\rho_0}(\sigma_1) = \sigma_1 \otimes \rho_{0,2}$ for every σ_1; in particular $\mathrm{Tr}_2(\mathcal{R}_{\rho_0}(\sigma_1)) = \sigma_1$.

(b) If $\rho_0 = |\psi\rangle\langle\psi|$ is pure, then $\mathcal{R}_{\rho_0}(\sigma_1) = \rho_0$ for every σ_1; in particular $\mathrm{Tr}_2(\mathcal{R}_{\rho_0}(\sigma_1)) = \rho_{0,1} \neq \sigma_1$ in general. (In part (b) the Petz expression (9) is understood on the support of ρ_0 — equivalently with Moore–Penrose inverses, or as the limit of faithful regularizations $\rho_{0,\varepsilon} = (1-\varepsilon)|\psi\rangle\langle\psi| + \varepsilon\tau$ with τ faithful, under which the conclusion is continuous.)

Consequently $\mathrm{Tr}_2 \circ \mathcal{R}_{\rho_0} \neq \mathrm{id}$ for correlated references; by Takesaki's theorem, a ρ_0-preserving conditional expectation onto $B(\mathcal{H}_1) \otimes \mathbf{1}$ (whose existence is what marginal preservation would express) exists precisely when the modular group of ρ_0 preserves the subalgebra, i.e., when ρ_0 factorizes.

Proof. (a) With $\rho_0^{1/2} = \rho_{0,1}^{1/2} \otimes \rho_{0,2}^{1/2}$, the inner factors cancel: $\rho_{0,1}^{1/2}\rho_{0,1}^{-1/2}\sigma_1\rho_{0,1}^{-1/2}\rho_{0,1}^{1/2} = \sigma_1$. (b) $\rho_0^{1/2} = |\psi\rangle\langle\psi|$, so $\mathcal{R}_{\rho_0}(\sigma_1) = |\psi\rangle\langle\psi| \cdot \langle\psi|(\rho_{0,1}^{-1/2}\sigma_1\rho_{0,1}^{-1/2} \otimes \mathbf{1})|\psi\rangle = \rho_0 \cdot \mathrm{Tr}\big(\rho_{0,1}\, \rho_{0,1}^{-1/2}\sigma_1\rho_{0,1}^{-1/2}\big) = \rho_0$. $\square$

The object that the computations of this paper actually control is the following explicit reconstruction procedure, which replaces the Petz map throughout.

Definition 2.15 (Gaussian conditional reattachment)**.** Let $\Gamma_0 = \begin{pmatrix} A_0 & X_0 \\ X_0^T & B_0 \end{pmatrix}$ be the vacuum covariance and set the *regression matrix* $W := A_0^{-1}X_0$ and the *conditional covariance* $C_0 := B_0 - X_0^T A_0^{-1} X_0 \geq 0$ (Schur complement). The *conditional reattachment* of a centered Gaussian state σ_1 with covariance A is the centered Gaussian state $\tilde{\omega} = \Lambda_{\Gamma_0}[\sigma_1]$ with covariance

$$\Gamma_{\tilde{\omega}} = \begin{pmatrix} A & AW \\ W^T A & C_0 + W^T A W \end{pmatrix} = \begin{pmatrix} A & A A_0^{-1} X_0 \\ X_0^T A_0^{-1} A & B_0 + X_0^T A_0^{-1}(A - A_0) A_0^{-1} X_0 \end{pmatrix}, \tag{10}$$

whenever $\Gamma_{\tilde{\omega}} + i\Omega \geq 0$ (see Proposition 2.17). Equivalently, Λ_{Γ_0} is characterized in the Heisenberg picture by the dual action (98) on Weyl operators (Appendix D).

This is the quantum analogue of the classical operation "keep the observed marginal on subsystem 1 and reattach subsystem 2 by the reference's conditional distribution (regression)"; the second expression in (10) is the formula that v1 attributed to the Petz map.

Remark 2.16 (Reattachment is a reconstruction, not a channel)*.* The assignment $\sigma_1 \mapsto \Lambda_{\Gamma_0}[\sigma_1]$ is a well-defined map from admissible Gaussian states to Gaussian states, but for $X_0 \neq 0$ it is *not* the restriction of any completely positive channel: a Gaussian channel with X-matrix $\binom{\mathbf{1}}{W^T}$ and noise $Y = \mathrm{diag}(0, C_0)$ must satisfy $Y + i(\Omega - X\Omega_1 X^T) \geq 0$ [15, Chapter 12], whose $(1,1)$ block vanishes while the off-diagonal block equals $-i\Omega_1 W \neq 0$ — a positive semidefinite matrix with a vanishing diagonal block has vanishing off-diagonal blocks, so complete positivity fails. Physically this is a no-broadcasting obstruction: no channel can correlate its output with fresh degrees of freedom conditionally on the input while leaving the input marginal untouched. Accordingly, Theorem 4.8 is a statement about the accuracy of an explicitly computable *state reconstruction* from local data, not about a physical recovery operation; the true Petz map *is* a Gaussian channel for Gaussian references [19], agrees with Λ_{Γ_0} when $X_0 = 0$ (Lemma 2.14(a) and Lemma D.2), and approaches it in the strongly-mixed, weakly correlated regime (Appendix E).

Proposition 2.17 (Admissibility of the reattachment). *Assume $\nu_{\min}(A) \ge 1+\kappa_1$ and $\nu_{\min}(B_0) \ge 1 + \nu_0$ for some $\kappa_1, \nu_0 > 0$ (symplectic gaps of the blocks), and let $\gamma_1 := \frac{\kappa_1}{1+\kappa_1}\lambda_{\min}(A)$, $\gamma_2 := \frac{\nu_0}{1+\nu_0} c_2$. If*

$$C_{\mathrm{vac}}^2\,\eta_{\mathrm{vac}}^2 \;\le\; \gamma_1\left(\gamma_2 - \eta_{\mathrm{vac}}^2\,\|B_0\|_{\mathrm{op}} - C_{\mathrm{vac}}^2\eta_{\mathrm{vac}}^2\right), \tag{11}$$

then $\Gamma_{\tilde\omega} + i\Omega \ge 0$, i.e., $\tilde\omega$ is a valid Gaussian state. In particular, since $\eta_{\mathrm{vac}} \le C_d K_\nu(mr)$ (Lemma 3.4), there is $r_1 > 0$ such that (11) holds for all $r \ge r_1$.

Proof. Write $S := \begin{pmatrix} \mathbf{1} & 0 \\ W^T & \mathbf{1} \end{pmatrix}$, so that $\Gamma_{\tilde\omega} = S\,\mathrm{diag}(A, C_0)\,S^T$. Congruence by S preserves positive semidefiniteness, so $\Gamma_{\tilde\omega} + i\Omega \ge 0$ is equivalent to $N := \mathrm{diag}(A, C_0) + i\,S^{-1}\Omega S^{-T} \ge 0$. A direct computation gives

$$S^{-1}\Omega S^{-T} = \begin{pmatrix} \Omega_1 & -\Omega_1 W \\ -W^T\Omega_1 & \Omega_2 + W^T\Omega_1 W \end{pmatrix}, \qquad N = \begin{pmatrix} A + i\Omega_1 & -i\Omega_1 W \\ -iW^T\Omega_1 & C_0 + i\Omega_2 + iW^T\Omega_1 W \end{pmatrix}. \tag{12}$$

For the diagonal blocks: $\nu_{\min}(A) \ge 1 + \kappa_1$ is equivalent to $\left\|A^{-1/2}\Omega_1 A^{-1/2}\right\|_{\mathrm{op}} \le (1+\kappa_1)^{-1}$, hence $A + i\Omega_1 \ge \left(1 - \frac{1}{1+\kappa_1}\right)A \ge \gamma_1\mathbf{1}$; similarly $B_0 + i\Omega_2 \ge \frac{\nu_0}{1+\nu_0}B_0 \ge \gamma_2\mathbf{1}$, and since $X_0^T A_0^{-1} X_0 = B_0^{1/2} Y^T Y B_0^{1/2}$ with $Y = A_0^{-1/2} X_0 B_0^{-1/2}$,

$$C_0 + i\Omega_2 + iW^T\Omega_1 W \;\ge\; \left(\gamma_2 - \eta_{\mathrm{vac}}^2\,\|B_0\|_{\mathrm{op}} - \|W\|_{\mathrm{op}}^2\right)\mathbf{1}, \qquad \|W\|_{\mathrm{op}} \le C_{\mathrm{vac}}\eta_{\mathrm{vac}} \tag{13}$$

by Lemma 3.6. For the off-diagonal block, $\|\Omega_1 W\|_{\mathrm{op}} = \|W\|_{\mathrm{op}} \le C_{\mathrm{vac}}\eta_{\mathrm{vac}}$. A Hermitian block matrix $\begin{pmatrix} P & Z \\ Z^\dagger & Q \end{pmatrix}$ with $P \ge \gamma_1\mathbf{1}$, $Q \ge \gamma_2'\mathbf{1}$ and $\|Z\|_{\mathrm{op}}^2 \le \gamma_1\gamma_2'$ is positive semidefinite (Schur complement); applying this with $\gamma_2' = \gamma_2 - \eta_{\mathrm{vac}}^2\,\|B_0\|_{\mathrm{op}} - C_{\mathrm{vac}}^2\eta_{\mathrm{vac}}^2$ and $\|Z\|_{\mathrm{op}} \le C_{\mathrm{vac}}\eta_{\mathrm{vac}}$ gives the claim under (11). ☐

Remark 2.18 (Steering). Condition (11) can genuinely fail at small collar width: if the reference ρ_0 is strongly entangled across the split (e.g., two-mode squeezed with large squeezing and low temperature), the conditional covariance C_0 violates $C_0 + i\Omega_2 \ge 0$ — this is exactly Gaussian steerability of subsystem 2 by measurements on subsystem 1 — and no Gaussian state has the covariance (10). The exponential decay of η_{vac} guarantees admissibility for mr large, which is the regime of interest throughout.

3 Exponential Clustering for Massive Fields

This section establishes quantitative bounds on correlation decay.

3.1 The Vacuum Two-Point Function

Theorem 3.1 (Vacuum Clustering). *For the massive scalar field ($m > 0$) on $\mathbb{R}^{d,1}$, the equal-time two-point function at spatial separation s satisfies:*

$$|\omega_0(\phi(x)\phi(y))| = \frac{1}{(2\pi)^{(d-1)/2}}\frac{K_\nu(ms)}{s^\nu} \tag{14}$$

where $\nu = (d-2)/2$ and K_ν is the modified Bessel function of the second kind.

Corollary 3.2 (Bessel Asymptotics). *For $ms \gg 1$:*

$$K_\nu(ms) \sim \sqrt{\frac{\pi}{2ms}}\, e^{-ms} \tag{15}$$

Thus correlations decay as $s^{-(d-1)/2}e^{-ms}$. See [24, §10.40] for the uniform asymptotic expansions used here and in Appendix B.

3.2 The Vacuum Correlation Factor

Definition 3.3 (Vacuum Correlation Factor). For a split configuration with collar width r:

$$\eta_{\text{vac}} := \left\| A_0^{-1/2} X_0 B_0^{-1/2} \right\|_{\text{op}} \tag{16}$$

Lemma 3.4 (Vacuum Correlation Bound). *For the massive scalar field in the regularized framework of Assumption 2.7, there exist constants $C'_d = C'(d, m, \mathcal{O}_1, \mathcal{O}_2) > 0$ and $r_0 > 0$ such that for $r \geq r_0$:*

$$\eta_{\text{vac}} \;\leq\; C_d \cdot K_\nu(mr) \;<\; \frac{1}{2}, \qquad C_d := (c_1 c_2)^{-1/2}\, C'_d. \tag{17}$$

Proof. Step 1 (kernel bound). In the regularized setting the off-diagonal block X_0 acts with integral kernel $G_0(x,y) = \omega_0(\phi(x)\phi(y))$ restricted to $x \in \Sigma_1$, $y \in \Sigma_2$, where $\Sigma_1 \subset \mathcal{O}_1$ and Σ_2 lies at spatial distance $\geq r$ from Σ_1. By Theorem 3.1,

$$|G_0(x,y)| \;\leq\; \frac{1}{(2\pi)^{(d-1)/2}} \frac{K_\nu(m|x-y|)}{|x-y|^\nu}, \qquad |x-y| \geq r. \tag{18}$$

Step 2 (Schur test). The Schur test gives $\|X_0\|_{\text{op}} \leq \sqrt{S_1 S_2}$ with $S_1 = \sup_{x\in\Sigma_1} \int_{\Sigma_2} |G_0(x,y)|\, dy$ and $S_2 = \sup_{y\in\Sigma_2} \int_{\Sigma_1} |G_0(x,y)|\, dx$. Passing to spherical coordinates around x (resp. y) and using that the integrand is supported on $|x-y| \geq r$:

$$S_1, S_2 \;\leq\; \frac{\omega_{d-2}}{(2\pi)^{(d-1)/2}} \int_r^\infty \frac{K_\nu(ms)}{s^\nu}\, s^{d-2}\, ds, \tag{19}$$

where ω_{d-2} is the volume of the unit $(d-2)$-sphere. Since $K_\nu(z) \leq \sqrt{\pi/(2z)}\, e^{-z}\,(1 + O(z^{-1}))$ uniformly for $z \geq \max(1, \nu^2)$ (Appendix B), the integral is bounded, for $mr \geq \max(1,\nu^2)$, by $C''(d,m)\, r^{d-2-\nu}\, e^{-mr} \leq C'_d\, K_\nu(mr)$ after absorbing polynomial factors into C'_d for $r \geq r_0$. Hence $\|X_0\|_{\text{op}} \leq C'_d \cdot K_\nu(mr)$.

Step 3 (spectral bounds). By Assumption 2.7: $\left\|A_0^{-1/2}\right\|_{\text{op}} \leq c_1^{-1/2}$ and $\left\|B_0^{-1/2}\right\|_{\text{op}} \leq c_2^{-1/2}$.

Step 4 (conclusion). Combining: $\eta_{\text{vac}} \leq (c_1c_2)^{-1/2}\, C'_d\, K_\nu(mr) = C_d\, K_\nu(mr)$. Since $K_\nu(mr) \to 0$ exponentially as $r \to \infty$, we may enlarge r_0 so that the right-hand side is $< 1/2$ for all $r \geq r_0$. □

3.3 Auxiliary Constant

Definition 3.5 (Vacuum Amplification Constant). Define:

$$C_{\text{vac}} := c_1^{-1/2} \sqrt{\|B_0\|_{\text{op}}} \tag{20}$$

where c_1 is from Assumption 2.7.

Lemma 3.6 (Cross-Correlation Bound)**.** *The operator $A_0^{-1}X_0$ satisfies:*

$$\left\|A_0^{-1}X_0\right\|_{\mathrm{op}} \leq C_{\mathrm{vac}} \cdot \eta_{\mathrm{vac}} \tag{21}$$

Proof.

$$\begin{aligned} \|A_0^{-1}X_0\|_{\mathrm{op}} &\leq \left\|A_0^{-1/2}\right\|_{\mathrm{op}} \cdot \left\|A_0^{-1/2}X_0\right\|_{\mathrm{op}} && (22)\\ &\leq c_1^{-1/2} \cdot \left\|A_0^{-1/2}X_0B_0^{-1/2}\right\|_{\mathrm{op}} \cdot \left\|B_0^{1/2}\right\|_{\mathrm{op}} && (23)\\ &= c_1^{-1/2} \cdot \eta_{\mathrm{vac}} \cdot \sqrt{\|B_0\|_{\mathrm{op}}} = C_{\mathrm{vac}} \cdot \eta_{\mathrm{vac}} && \square \end{aligned}$$

4 The Clustering–Recovery Bridge

This section contains the main result with complete proof.

4.1 State Class and Hypotheses

Definition 4.1 (Partial Local Excitation)**.** A quasi-free state ω is a *partial local excitation* if $\Gamma_\omega^{(22)} = \Gamma_0^{(22)}$ (i.e., $B = B_0$).

Definition 4.2 (Effective Excitation Number)**.**

$$N_{\mathrm{eff}}(\omega) := \|\Gamma_\omega - \Gamma_0\|_{\mathrm{HS}}^2 \tag{24}$$

Definition 4.3 (Cross-Correlation Perturbation)**.**

$$\delta_X := \left\|A_0^{-1/2}(X - X_0)B_0^{-1/2}\right\|_{\mathrm{op}} \tag{25}$$

Definition 4.4 (Symplectic Gap)**.** A covariance matrix Γ has *symplectic gap* $\kappa > 0$ if all its symplectic eigenvalues satisfy $\nu_j(\Gamma) \geq 1 + \kappa$ (equivalently, $\Gamma \geq (1+\kappa)\,|i\Omega|$ in the sense that $\Gamma + (1+\kappa)\,i\Omega \geq 0$). We set

$$C_\kappa := \frac{(1+\kappa)^2}{(1+\kappa)^2 - 1} = \frac{(1+\kappa)^2}{\kappa(\kappa+2)}, \tag{26}$$

so that $C_\kappa \downarrow 1$ as $\kappa \to \infty$ and $C_\kappa \to \infty$ as $\kappa \downarrow 0$ (approach to purity).

Remark 4.5 (Consequence of hypothesis (c)). The condition $\left\|A_0^{-1/2}(A - A_0)A_0^{-1/2}\right\|_{\mathrm{op}} \leq 1 - \epsilon$ implies that the self-adjoint operator $A_0^{-1/2}(A - A_0)A_0^{-1/2}$ has spectrum in $[-(1-\epsilon), 1-\epsilon]$. In particular $A_0^{-1/2}(A - A_0)A_0^{-1/2} \geq -(1-\epsilon)\mathbf{1}$, hence $A_0^{-1/2}AA_0^{-1/2} \geq \epsilon\,\mathbf{1}$, which gives $A \geq \epsilon A_0$. This inequality is used in Step 4 of the proof of Theorem 4.8.

4.2 Supporting Lemmas

Lemma 4.6 (Gaussian Fidelity Bound)**.** *Let ω_1, ω_2 be quasi-free states with covariances Γ_1, Γ_2 satisfying:*

(i) $\Gamma_1, \Gamma_2 \geq c \cdot \mathbf{1}$ *for some* $c > 0$*,*

(ii) $K := \Gamma_1^{-1/2}(\Gamma_1 - \Gamma_2)\Gamma_1^{-1/2} \in \mathcal{L}^2(\mathfrak{h})$ *(Hilbert–Schmidt class; see [28, Chapter 2]),*

(iii) $\|K\|_{\mathrm{op}} \leq 1/2$*,*

(iv) Γ_1 *has symplectic gap* $\kappa > 0$ *(Definition 4.4).*

Then:

$$1 - F(\omega_1, \omega_2) \;\leq\; \frac{C_\kappa}{8}\,\|K\|_{\mathrm{HS}}^2 + \frac{C_\kappa^2}{48}\,\|K\|_{\mathrm{HS}}^4 \tag{27}$$

Moreover, if either $C_\kappa \leq 2$ *and* $\|K\|_{\mathrm{HS}} \leq 1$*, or more generally* $C_\kappa\,\|K\|_{\mathrm{HS}}^2 \leq 2$*, then*

$$1 - F(\omega_1, \omega_2) \;\leq\; \frac{C_\kappa}{6}\,\|K\|_{\mathrm{HS}}^2\,. \tag{28}$$

In all applications below we use the safe two-term bound (27); the $C_\kappa/6$ *form is quoted only under its stated smallness condition (cf. Appendix C).*

Proof. See Appendix C for the derivation from the exact Gaussian fidelity formula of Banchi–Braunstein–Pirandola [3]. Hypothesis (iv) is necessary: without a symplectic gap the quadratic bound fails for perturbations that increase mixedness (Remark C.3). $\square$

Lemma 4.7 (Block Inversion Bound)**.** *Let* $\Gamma = \begin{pmatrix} A & X \\ X^T & B \end{pmatrix}$ *with:*

(i) $A \geq \epsilon_A \cdot \mathbf{1}$*,* $B \geq \epsilon_B \cdot \mathbf{1}$*,*

(ii) $\eta := \left\|A^{-1/2} X B^{-1/2}\right\|_{\mathrm{op}} < 1$*.*

Then:

$$\left\|\Gamma^{-1}\right\|_{\mathrm{op}} \leq \frac{1}{(1-\eta^2)\min(\epsilon_A, \epsilon_B)} \tag{29}$$

Proof. The Schur complement $S = A - XB^{-1}X^T = A^{1/2}(\mathbf{1} - KK^T)A^{1/2}$ where $K = A^{-1/2}XB^{-1/2}$. Since $\left\|KK^T\right\|_{\mathrm{op}} = \eta^2 < 1$:

$$S \geq (1-\eta^2)A \geq (1-\eta^2)\epsilon_A \cdot \mathbf{1} \tag{30}$$

The block inversion formula then gives the stated bound. $\square$

4.3 Main Theorem

Theorem 4.8 (Clustering–Recovery Bridge)**.** *Let* ω_0 *denote the vacuum quasi-free state in the regularized framework of Assumption 2.7, modeling a massive scalar field* ($m > 0$)*. Let* $\mathcal{O}_1 \Subset \mathcal{O}_2$ *with collar width* $r > 0$*.*

Let ω *be a quasi-free state and let* $\tilde{\omega} := \Lambda_{\Gamma_0}[\omega|_1]$ *be the conditional-reattachment reconstruction (Definition 2.15) built from the restriction of* ω *to subsystem 1 and the vacuum regression structure. Assume* ω *satisfies:*

*(a) (*Partial local excitation*)* $B = B_0$*,*

*(b) (*Finite excitation*)* $N_{\mathrm{eff}}(\omega) < \infty$*,*

*(c) (*Perturbative regime*)* $\left\|A_0^{-1/2}(A - A_0)A_0^{-1/2}\right\|_{\mathrm{op}} \leq 1 - \epsilon$ *for some* $\epsilon \in (0, 1)$*,*

*(d) (*Cross-correlation control*)* $\delta_X \leq \delta$ *for some* $\delta \in (0, 1)$ *with*

$$\epsilon^{-1/2}(\eta_{\mathrm{vac}} + \delta) < 1, \tag{31}$$

*(e) (*Fidelity regime*) With* $K := \Gamma_\omega^{-1/2}(\Gamma_\omega - \Gamma_{\tilde{\omega}})\Gamma_\omega^{-1/2}$*, we have* $\|K\|_{\mathrm{op}} \leq \frac{1}{2}$*,*

*(f) (*Symplectic gap*)* Γ_ω *has symplectic gap* $\kappa > 0$ *(Definition 4.4),*

(g) (Admissibility) $\Gamma_{\tilde{\omega}} + i\Omega \geq 0$, for which Proposition 2.17 gives an explicit sufficient condition, automatic for mr large.

(Note that (31) with $\epsilon < 1$ already implies $\eta_{\mathrm{vac}} < 1$, so no separate assumption is needed.)

Define:

$$\eta_\omega := \left\| A^{-1/2} X B_0^{-1/2} \right\|_{\mathrm{op}} \tag{32}$$

Then the reconstructed state $\tilde{\omega}$ satisfies:

$$1 - F(\omega, \tilde{\omega}) \ \leq \ \frac{C^{(0)}_{d,\kappa}}{\epsilon^2 \left(1 - \epsilon^{-1}(\eta_{\mathrm{vac}} + \delta)^2\right)^2} \cdot \left\| \Delta^{(12)} \right\|^2_{\mathrm{HS}} \tag{33}$$

where:

- $\Delta^{(12)} := X - AA_0^{-1}X_0$ *is the* reconstruction error,
- *the simplified constant*

$$C^{(0)}_{d,\kappa} := \frac{C_\kappa\,(6 + C_\kappa)}{16\ \min(c_1, c_2)^2} \tag{34}$$

  applies under the additional hypotheses of Corollary 4.12. In the strongly mixed limit $\kappa \to \infty$ (i.e., $C_\kappa \to 1$) this reduces to $\frac{7}{16 \min(c_1, c_2)^2}$.

Moreover, the recovery error satisfies:

$$\left\| \Delta^{(12)} \right\|_{\mathrm{HS}} \leq (1 + C_{\mathrm{vac}} \cdot \eta_{\mathrm{vac}}) \sqrt{N_{\mathrm{eff}}(\omega)} \tag{35}$$

where $C_{\mathrm{vac}} = c_1^{-1/2} \sqrt{\|B_0\|_{\mathrm{op}}}$.

Proof. We proceed in five steps.

Step 1: Covariance of the reconstructed state. By Definition 2.15 (whose blocks are derived in Appendix D), using hypothesis (g):

$$\Gamma_{\tilde{\omega}} = \begin{pmatrix} A & AA_0^{-1}X_0 \\ X_0^T A_0^{-1} A & B_0 + X_0^T A_0^{-1}(A - A_0)A_0^{-1}X_0 \end{pmatrix} \tag{36}$$

The covariance error $\Delta\Gamma := \Gamma_\omega - \Gamma_{\tilde{\omega}}$ has blocks:

$$\Delta^{(11)} = 0 \tag{37}$$
$$\Delta^{(12)} = X - AA_0^{-1}X_0 \tag{38}$$
$$\Delta^{(22)} = -X_0^T A_0^{-1}(A - A_0)A_0^{-1}X_0 \tag{39}$$

Step 2: Decomposition of $\Delta^{(12)}$.

$$\Delta^{(12)} = X - AA_0^{-1}X_0 \tag{40}$$
$$= (X - X_0) + X_0 - AA_0^{-1}X_0 \tag{41}$$
$$= (X - X_0) - (A - A_0)A_0^{-1}X_0 \tag{42}$$

Step 3: Hilbert–Schmidt bound on $\Delta^{(12)}$. By the triangle inequality:

$$\left\| \Delta^{(12)} \right\|_{\mathrm{HS}} \leq \|X - X_0\|_{\mathrm{HS}} + \left\| (A - A_0)A_0^{-1}X_0 \right\|_{\mathrm{HS}} \tag{43}$$

For the first term, since $X - X_0$ is a block of $\Gamma_\omega - \Gamma_0$:

$$\|X - X_0\|_{\mathrm{HS}} \leq \|\Gamma_\omega - \Gamma_0\|_{\mathrm{HS}} = \sqrt{N_{\mathrm{eff}}(\omega)} \tag{44}$$

For the second term, using $\|TS\|_{\mathrm{HS}} \leq \|T\|_{\mathrm{HS}} \|S\|_{\mathrm{op}}$:

$$\left\|(A - A_0)A_0^{-1}X_0\right\|_{\mathrm{HS}} \leq \|A - A_0\|_{\mathrm{HS}} \cdot \left\|A_0^{-1}X_0\right\|_{\mathrm{op}} \tag{45}$$

$$\leq \sqrt{N_{\mathrm{eff}}(\omega)} \cdot C_{\mathrm{vac}} \cdot \eta_{\mathrm{vac}} \tag{46}$$

where we used Lemma 3.6. Combining:

$$\left\|\Delta^{(12)}\right\|_{\mathrm{HS}} \leq (1 + C_{\mathrm{vac}} \cdot \eta_{\mathrm{vac}})\sqrt{N_{\mathrm{eff}}(\omega)} \tag{47}$$

Step 4: Control of η_ω and $\|\Gamma_\omega^{-1}\|_{\mathrm{op}}$. We bound $\eta_\omega = \left\|A^{-1/2}XB_0^{-1/2}\right\|_{\mathrm{op}}$ using the triangle inequality:

$$\eta_\omega \leq \left\|A^{-1/2}X_0B_0^{-1/2}\right\|_{\mathrm{op}} + \left\|A^{-1/2}(X - X_0)B_0^{-1/2}\right\|_{\mathrm{op}} \tag{48}$$

First term: using $A \geq \epsilon A_0$ from hypothesis (c) and Remark 4.5:

$$\left\|A^{-1/2}X_0B_0^{-1/2}\right\|_{\mathrm{op}} \leq \left\|A^{-1/2}A_0^{1/2}\right\|_{\mathrm{op}} \cdot \left\|A_0^{-1/2}X_0B_0^{-1/2}\right\|_{\mathrm{op}} \leq \epsilon^{-1/2} \cdot \eta_{\mathrm{vac}} \tag{49}$$

Second term: similarly,

$$\left\|A^{-1/2}(X - X_0)B_0^{-1/2}\right\|_{\mathrm{op}} \leq \left\|A^{-1/2}A_0^{1/2}\right\|_{\mathrm{op}} \cdot \left\|A_0^{-1/2}(X - X_0)B_0^{-1/2}\right\|_{\mathrm{op}} \leq \epsilon^{-1/2} \cdot \delta_X \leq \epsilon^{-1/2}\delta \tag{50}$$

Combining:

$$\eta_\omega \leq \epsilon^{-1/2}(\eta_{\mathrm{vac}} + \delta) < 1 \tag{51}$$

by hypothesis (d). By Lemma 4.7 with $\epsilon_A = \epsilon c_1$ and $\epsilon_B = c_2$:

$$\|\Gamma_\omega^{-1}\|_{\mathrm{op}} \leq \frac{1}{(1 - \eta_\omega^2)\min(\epsilon c_1, c_2)} \tag{52}$$

Since $\eta_\omega^2 \leq \epsilon^{-1}(\eta_{\mathrm{vac}} + \delta)^2$:

$$\|\Gamma_\omega^{-1}\|_{\mathrm{op}} \leq \frac{1}{(1 - \epsilon^{-1}(\eta_{\mathrm{vac}} + \delta)^2)\min(\epsilon c_1, c_2)} \tag{53}$$

Step 5: Apply fidelity bound. Define $K := \Gamma_\omega^{-1/2}\,\Delta\Gamma\,\Gamma_\omega^{-1/2}$. By hypothesis (e), $\|K\|_{\mathrm{op}} \leq \frac{1}{2}$; by hypothesis (f), Γ_ω has symplectic gap κ. By Lemma 4.6:

$$1 - F(\omega, \tilde{\omega}) \leq \frac{C_\kappa}{8}\|K\|_{\mathrm{HS}}^2 + \frac{C_\kappa^2}{48}\|K\|_{\mathrm{HS}}^4 \tag{54}$$

Using $\|K\|_{\mathrm{HS}} \leq \left\|\Gamma_\omega^{-1}\right\|_{\mathrm{op}} \|\Delta\Gamma\|_{\mathrm{HS}}$:

$$1 - F(\omega, \tilde{\omega}) \leq \frac{C_\kappa}{8}\left\|\Gamma_\omega^{-1}\right\|_{\mathrm{op}}^2 \|\Delta\Gamma\|_{\mathrm{HS}}^2 + \frac{C_\kappa^2}{48}\left\|\Gamma_\omega^{-1}\right\|_{\mathrm{op}}^4 \|\Delta\Gamma\|_{\mathrm{HS}}^4 \tag{55}$$

Since $\Delta^{(11)} = 0$:

$$\|\Delta\Gamma\|_{\mathrm{HS}}^2 = 2\left\|\Delta^{(12)}\right\|_{\mathrm{HS}}^2 + \left\|\Delta^{(22)}\right\|_{\mathrm{HS}}^2 \tag{56}$$

We bound $\|\Delta^{(22)}\|_{\mathrm{HS}}$ explicitly. From Step 1:

$$\Delta^{(22)} = -X_0^T A_0^{-1}(A - A_0)A_0^{-1}X_0 \tag{57}$$

Using $\|TSU\|_{\mathrm{HS}} \leq \|T\|_{\mathrm{op}} \|S\|_{\mathrm{HS}} \|U\|_{\mathrm{op}}$ and Lemma 3.6:

$$\left\|\Delta^{(22)}\right\|_{\mathrm{HS}} \leq (C_{\mathrm{vac}} \cdot \eta_{\mathrm{vac}})^2 \cdot \sqrt{N_{\mathrm{eff}}} \tag{58}$$

Therefore:

$$\|\Delta\Gamma\|_{\mathrm{HS}}^2 \leq 2\left\|\Delta^{(12)}\right\|_{\mathrm{HS}}^2 + (C_{\mathrm{vac}}\eta_{\mathrm{vac}})^4 N_{\mathrm{eff}} \tag{59}$$

This yields the general bound (55)–(59). To obtain the simplified form (33) with the constant (34), we impose the additional conditions of Corollary 4.12:

- Under condition (i) of Corollary 4.12, $(C_{\mathrm{vac}}\eta_{\mathrm{vac}})^4 N_{\mathrm{eff}} \leq \left\|\Delta^{(12)}\right\|_{\mathrm{HS}}^2$, so by (59)

$$\|\Delta\Gamma\|_{\mathrm{HS}}^2 \leq 3\left\|\Delta^{(12)}\right\|_{\mathrm{HS}}^2. \tag{60}$$

- Under condition (ii) of Corollary 4.12, $\left\|\Gamma_\omega^{-1}\right\|_{\mathrm{op}} \|\Delta\Gamma\|_{\mathrm{HS}} \leq 1$, hence $\|K\|_{\mathrm{HS}} \leq 1$ and the quartic term in (55) satisfies $\frac{C_\kappa^2}{48}\|K\|_{\mathrm{HS}}^4 \leq \frac{C_\kappa^2}{48}\|K\|_{\mathrm{HS}}^2$, so

$$1 - F(\omega, \tilde{\omega}) \leq \left(\frac{C_\kappa}{8} + \frac{C_\kappa^2}{48}\right)\left\|\Gamma_\omega^{-1}\right\|_{\mathrm{op}}^2 \|\Delta\Gamma\|_{\mathrm{HS}}^2 = \frac{C_\kappa(6 + C_\kappa)}{48}\left\|\Gamma_\omega^{-1}\right\|_{\mathrm{op}}^2 \|\Delta\Gamma\|_{\mathrm{HS}}^2. \tag{61}$$

Finally, using (53) and $\min(\epsilon c_1, c_2)^2 \geq \epsilon^2 \min(c_1, c_2)^2$ (valid since $\epsilon < 1$):

$$\left\|\Gamma_\omega^{-1}\right\|_{\mathrm{op}}^2 \leq \frac{1}{\epsilon^2 \min(c_1, c_2)^2 \left(1 - \epsilon^{-1}(\eta_{\mathrm{vac}} + \delta)^2\right)^2}. \tag{62}$$

Combining this with (60) and (61) yields

$$1 - F(\omega, \tilde{\omega}) \leq \frac{C_{d,\kappa}^{(0)}}{\epsilon^2 \left(1 - \epsilon^{-1}(\eta_{\mathrm{vac}} + \delta)^2\right)^2} \cdot \left\|\Delta^{(12)}\right\|_{\mathrm{HS}}^2, \qquad C_{d,\kappa}^{(0)} = \frac{C_\kappa(6 + C_\kappa)}{16\,\min(c_1, c_2)^2}, \tag{63}$$

since $3 \times \frac{C_\kappa(6+C_\kappa)}{48} = \frac{C_\kappa(6+C_\kappa)}{16}$. The conditions (i)–(ii) are satisfied for localized excitations when $mr \gg 1$, as stated in Corollary 4.12. $\square$

Remark 4.9 (On the v1 constant). In the limit $C_\kappa \to 1$ the constant is $\frac{7}{16\min(c_1,c_2)^2}$. Version 1 of this paper stated $\frac{3}{8\min(c_1,c_2)^2}$, obtained by neglecting the quartic-remainder contribution $\frac{1}{48}$ that its own Step 5 had introduced; since $\frac{3}{8} < \frac{7}{16}$, the v1 statement was strictly stronger than what was proved. The present constant follows from the derivation above.

Remark 4.10 (On hypothesis (f)). Hypothesis (f) excludes reference states ω that are pure or nearly pure as global Gaussian states. It is necessary: Remark C.3 exhibits zero-mean Gaussian states, with $\|K\|_{\mathrm{op}} \leq 1/2$, for which the fidelity loss is *linear* rather than quadratic in K when the reference is pure. Physically, (f) is satisfied whenever the regularized state carries a small thermal component (e.g., a heat bath at arbitrarily low temperature, or a lattice state with $\nu_{\min} \geq 1 + \kappa$), with the constant degrading as $C_\kappa \sim \kappa^{-1}$ as $\kappa \downarrow 0$. Removing or weakening (f) — for instance by exploiting the specific block structure $\Delta^{(11)} = 0$ of the Petz error — is an open problem (Section 6.5).

Remark 4.11 (On hypothesis (e): a posteriori verification). Hypothesis (e) is an a posteriori condition that can be verified once $\tilde{\omega}$ is computed from hypotheses (a)–(d). Recall that the covariance error $\Delta\Gamma = \Gamma_\omega - \Gamma_{\tilde{\omega}}$ has block structure with $\Delta^{(11)} = 0$, $\Delta^{(12)} = X - AA_0^{-1}X_0$, and $\Delta^{(22)} = -X_0^T A_0^{-1}(A - A_0)A_0^{-1}X_0$.

From the bounds in Steps 3 and 5, using $\|\Delta\Gamma\|_{\mathrm{HS}} \le \sqrt{2}\,\|\Delta^{(12)}\|_{\mathrm{HS}} + \|\Delta^{(22)}\|_{\mathrm{HS}}$, under (a)–(d) one has:

$$\|\Delta\Gamma\|_{\mathrm{HS}} \le C_{\mathrm{eff}}\sqrt{N_{\mathrm{eff}}(\omega)}, \qquad C_{\mathrm{eff}} := \sqrt{2}\,(1 + C_{\mathrm{vac}} \cdot \eta_{\mathrm{vac}}) + (C_{\mathrm{vac}}\eta_{\mathrm{vac}})^2 \tag{64}$$

where the first term incorporates the bound on $\|\Delta^{(12)}\|_{\mathrm{HS}}$ from Step 3, and the second bounds $\|\Delta^{(22)}\|_{\mathrm{HS}} \le (C_{\mathrm{vac}}\eta_{\mathrm{vac}})^2\sqrt{N_{\mathrm{eff}}}$ from Step 5.

Combined with $\|\Gamma_\omega^{-1}\|_{\mathrm{op}} \le [(1-\epsilon^{-1}(\eta_{\mathrm{vac}}+\delta)^2)\min(\epsilon c_1, c_2)]^{-1}$ from Step 4, hypothesis (e) is satisfied whenever:

$$\frac{C_{\mathrm{eff}}\sqrt{N_{\mathrm{eff}}(\omega)}}{(1-\epsilon^{-1}(\eta_{\mathrm{vac}}+\delta)^2)\min(\epsilon c_1, c_2)} \le \frac{1}{2} \tag{65}$$

This provides an explicit sufficient condition purely in terms of $N_{\mathrm{eff}}(\omega)$, the spectral constants c_1, c_2, ϵ, and the clustering parameters $\eta_{\mathrm{vac}}, \delta$. For $mr \gg 1$ one has $\eta_{\mathrm{vac}} \ll 1$, so $C_{\mathrm{eff}} \approx \sqrt{2}$ and the condition simplifies to $\sqrt{N_{\mathrm{eff}}(\omega)} \lesssim \epsilon\min(c_1, c_2)$ (up to order-one factors when $\eta_{\mathrm{vac}}, \delta \ll \sqrt{\epsilon}$).

Corollary 4.12 (Simplified Bound). *Under the hypotheses of Theorem 4.8, assume additionally:*

(i) $(C_{\mathrm{vac}}\eta_{\mathrm{vac}})^4\, N_{\mathrm{eff}} \le \|\Delta^{(12)}\|_{\mathrm{HS}}^2$,

(ii) $\|\Gamma_\omega^{-1}\|_{\mathrm{op}}\,\|\Delta\Gamma\|_{\mathrm{HS}} \le 1$.

Then:

$$1 - F(\omega, \tilde{\omega}) \le \frac{C_{d,\kappa}^{(0)}}{\epsilon^2\,(1-\epsilon^{-1}(\eta_{\mathrm{vac}}+\delta)^2)^2} \cdot \left\|\Delta^{(12)}\right\|_{\mathrm{HS}}^2 \tag{66}$$

where $C_{d,\kappa}^{(0)} := \frac{C_\kappa(6+C_\kappa)}{16\min(c_1,c_2)^2}$. *Both conditions are satisfied for localized excitations when* $mr \gg 1$, *since* $C_{\mathrm{vac}}\eta_{\mathrm{vac}} \sim e^{-mr}$.

Proof. Under (i): $\|\Delta\Gamma\|_{\mathrm{HS}}^2 \le 2\left\|\Delta^{(12)}\right\|_{\mathrm{HS}}^2 + (C_{\mathrm{vac}}\eta_{\mathrm{vac}})^4 N_{\mathrm{eff}} \le 3\left\|\Delta^{(12)}\right\|_{\mathrm{HS}}^2$ by (59). Under (ii): $\|K\|_{\mathrm{HS}} \le 1$, so the quartic term satisfies $\frac{C_\kappa^2}{48}\|K\|_{\mathrm{HS}}^4 \le \frac{C_\kappa^2}{48}\|K\|_{\mathrm{HS}}^2$, and the total prefactor is $\frac{C_\kappa}{8} + \frac{C_\kappa^2}{48} = \frac{C_\kappa(6+C_\kappa)}{48}$. The result follows with $C_{d,\kappa}^{(0)} = 3 \times \frac{C_\kappa(6+C_\kappa)}{48} \times \frac{1}{\min(c_1,c_2)^2} = \frac{C_\kappa(6+C_\kappa)}{16\min(c_1,c_2)^2}$. ☐

4.4 Corollaries

Corollary 4.13 (Finite Rank Bound). *Under the hypotheses of Theorem 4.8 and Corollary 4.12, if additionally* $\mathrm{rank}(\Gamma_\omega - \Gamma_0) \le n$, *then:*

$$1 - F(\omega, \tilde{\omega}) \le \frac{2\,C_{d,\kappa}^{(0)} \cdot n}{\epsilon^2\,(1-\epsilon^{-1}(\eta_{\mathrm{vac}}+\delta)^2)^2} \cdot \left\|\Delta^{(12)}\right\|_{\mathrm{op}}^2 \tag{67}$$

Proof. Since $\Delta^{(12)} = (X - X_0) - (A - A_0)A_0^{-1}X_0$ and each term has rank at most n:

$$\mathrm{rank}(\Delta^{(12)}) \le 2n \tag{68}$$

Using $\|T\|_{\mathrm{HS}}^2 \le \mathrm{rank}(T) \cdot \|T\|_{\mathrm{op}}^2$:

$$\left\|\Delta^{(12)}\right\|_{\mathrm{HS}}^2 \le 2n \cdot \left\|\Delta^{(12)}\right\|_{\mathrm{op}}^2 \tag{69}$$

Substituting into (33) gives the result. ☐

Corollary 4.14 (Simplified Bound under $\delta \le \eta_{\text{vac}}$). *If $\delta \le \eta_{\text{vac}}$ (satisfied for small perturbations), then:*

$$1 - F(\omega, \tilde{\omega}) \le \frac{C^{(0)}_{d,\kappa}}{\epsilon^2 \left(1 - 4\epsilon^{-1}\eta_{\text{vac}}^2\right)^2} \cdot \left\| \Delta^{(12)} \right\|_{\text{HS}}^2 \tag{70}$$

Proof. Under $\delta \le \eta_{\text{vac}}$: $(\eta_{\text{vac}} + \delta)^2 \le 4\eta_{\text{vac}}^2$. □

Corollary 4.15 (Exponential Decay). *Under the hypotheses of Theorem 4.8 and Corollary 4.12:*

$$1 - F(\omega, \tilde{\omega}) \le \frac{C^{(0)}_{d,\kappa} \left(1 + C_{\text{vac}} \cdot \eta_{\text{vac}}\right)^2}{\epsilon^2 \left(1 - \epsilon^{-1}(\eta_{\text{vac}} + \delta)^2\right)^2} \cdot N_{\text{eff}}(\omega) \tag{71}$$

For $mr \gg 1$ with $\eta_{\text{vac}}, \delta \ll \sqrt{\epsilon}$, the prefactor approaches $C^{(0)}_{d,\kappa}/\epsilon^2$.

4.5 Examples

Example 4.16 (Coherent States: Exact Recovery). Let ω_α be a coherent state with displacement $\alpha \in \mathfrak{h}_1$.

Verification of hypotheses:

- (a) $B = B_0$ ✓
- (b) $N_{\text{eff}} = 0$ ✓
- (c) $A = A_0$, satisfied with $\epsilon = 1$ ✓
- (d) $\delta_X = 0$ since $X = X_0$ ✓

Recovery error: $\Delta^{(12)} = X_0 - A_0 A_0^{-1} X_0 = 0$.

Conclusion: $F(\omega_\alpha, \tilde{\omega}_\alpha) = 1$ (exact recovery). Note that no symplectic gap is needed here: the recovery is exact at the level of covariances and first moments, so Lemma 4.6 is not invoked.

Example 4.17 (Local Squeezed State). Consider $\Gamma_\omega = \Gamma_0 + \lambda|\xi\rangle\langle\xi|$ with $\xi \in \mathfrak{h}_1$ normalized and $\lambda > 0$ small, and suppose the regularized state satisfies the symplectic gap hypothesis (f) with some $\kappa > 0$ (e.g., a small global thermal component).

Verification:

- (a) $B = B_0$ ✓
- (b) $N_{\text{eff}} = \lambda^2$ ✓
- (c) satisfied with $\epsilon \approx 1 - \lambda/c_1$ for small λ ✓
- (d) $\delta_X = 0$ since $X = X_0$ ✓
- (f) by assumption, with $C_\kappa = O(1)$ for $\kappa = O(1)$ ✓

Recovery error:

$$\Delta^{(12)} = -\lambda|\xi\rangle\langle\xi|A_0^{-1}X_0 \tag{72}$$

Bound:

$$\left\| \Delta^{(12)} \right\|_{\text{HS}} \le \lambda \cdot C_{\text{vac}} \cdot \eta_{\text{vac}} \tag{73}$$

Numerical example: for $mr = 5$, $d = 3$, $\lambda = 0.5$, $\kappa = 1$ (so $C_\kappa = 4/3$):

$$\eta_{\text{vac}} \lesssim e^{-5} \approx 0.007, \qquad 1 - F \lesssim (0.5 \times 0.007)^2/\epsilon^2 \sim 10^{-5} \tag{74}$$

Despite significant squeezing, $F > 0.99999$.

Remark 4.18 (Verification of (d) for Local Bogoliubov). For Bogoliubov transformations with $S|_{\mathfrak{h}_2} = \mathbf{1}$:

$$X = S_{11}^T X_0 \implies X - X_0 = (S_{11}^T - \mathbf{1})X_0 \tag{75}$$

Therefore:

$$\delta_X = \left\| A_0^{-1/2}(S_{11}^T - \mathbf{1})X_0 B_0^{-1/2} \right\|_{\text{op}} \leq \left\| A_0^{-1/2}(S_{11}^T - \mathbf{1})A_0^{1/2} \right\|_{\text{op}} \cdot \eta_{\text{vac}} \leq \sqrt{\frac{\|A_0\|_{\text{op}}}{c_1}} \, \|S_{11} - \mathbf{1}\|_{\text{op}} \cdot \eta_{\text{vac}} \tag{76}$$

For small perturbations, hypothesis (d) is satisfied with $\delta = \sqrt{\|A_0\|_{\text{op}}/c_1}\,\|S_{11} - \mathbf{1}\|_{\text{op}} \cdot \eta_{\text{vac}}$; in particular $\delta \leq \eta_{\text{vac}}$ whenever $\|S_{11} - \mathbf{1}\|_{\text{op}} \leq \sqrt{c_1/\|A_0\|_{\text{op}}}$. (In v1 the condition-number factor $\sqrt{\|A_0\|_{\text{op}}/c_1}$ was omitted.)

5 The General Conjecture

The quasi-free result (Theorem 4.8) establishes a precise clustering–recovery relationship for Gaussian states. We now formulate a general conjecture.

5.1 Statement

Conjecture 5.1 (General Clustering–Recovery Bridge). *Let ω_0 be the vacuum state of a QFT satisfying the Haag–Kastler axioms with mass gap $m > 0$. Let $\mathcal{O}_1 \Subset \mathcal{O}_2$ with collar width $r > 0$, and let ω satisfy:*

- *(i) $\omega|_{\mathcal{A}(\mathcal{O}_2)'} = \omega_0|_{\mathcal{A}(\mathcal{O}_2)'}$ (asymptotic vacuum),*
- *(ii) $S(\omega\|\omega_0) < \infty$ (finite relative entropy).*

Then there exist constants $C, \gamma > 0$ such that:

$$1 - F\left(\omega, \mathcal{R}_{\rho_0}(\omega|_{\mathcal{N}_{\text{can}}})\right) \leq C \cdot \mathcal{C}(\omega; \mathcal{O}_1, \mathcal{O}_2)^\gamma \tag{77}$$

where $\mathcal{N}_{\text{can}}$ is the canonical split factor from Theorem 2.11, $\mathcal{R}_{\rho_0}$ is the Petz recovery map associated with the ω_0-preserving conditional expectation $E_{\omega_0} : \mathcal{M} \to \mathcal{N}_{\text{can}}$, and $\mathcal{C}$ is an appropriate clustering coefficient.

Based on our quasi-free result, we expect $\gamma = 2$ for Gaussian-like states.

Remark 5.2 (Petz vs. reattachment in the conjecture). The conjecture is deliberately formulated for the Petz map, which is the natural universal recovery channel; Theorem 4.8 concerns the conditional-reattachment reconstruction instead (Section 2.5). The missing quasi-free step is to transfer the bound to the exact Gaussian Petz channel of [19]; numerically the two outputs converge as the reference becomes strongly mixed (Appendix E), which is the physically relevant regime for local restrictions of the vacuum.

5.2 Motivation from Quantum Information Theory

The Fawzi–Renner theorem [10] establishes for finite-dimensional tripartite states:

$$-\log F(\rho_{ABC}, \mathcal{R}_B(\rho_{AB})) \leq \frac{1}{2} I(A : C|B)_\rho \tag{78}$$

In AQFT, the natural identification is:

- $A \leftrightarrow \mathcal{A}(\mathcal{O}_1)$

- $B \leftrightarrow \mathcal{A}(\mathcal{O}_2 \setminus \mathcal{O}_1)$ (collar)
- $C \leftrightarrow \mathcal{A}(\mathcal{O}_2')$ (causal complement)

Clustering bounds correlations between A and C, suggesting small $I(A:C|B)$.

5.3 Challenges

1. **Type III algebras:** no von Neumann entropy; requires regularization.
2. **Infinite dimensions:** extensions [20] need additional hypotheses.
3. **Non-Gaussian states:** no explicit Petz formulas.
4. **Geometric tripartition:** nested regions $\neq$ tensor products.

5.4 Strategies

1. **Modular approach:** control fidelity via relative modular operators.
2. **Quasi-free approximation:** if general states can be approximated by Gaussians with controlled error.
3. **Nuclearity reduction:** use nuclearity to approximate by finite-dimensional algebras, apply Fawzi–Renner, take limits.
4. **Relative-entropy route (new in v2):** Faulkner, Hollands, Swingle and Wang [12] (see also [11, 18]) prove, for an arbitrary inclusion $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras, the existence of a universal recovery channel α with

$$-\log F\big(\omega, \alpha(\omega|_{\mathcal{N}})\big) \ \lesssim\ S(\omega\|\omega_0)_{\mathcal{M}} - S(\omega\|\omega_0)_{\mathcal{N}}, \tag{79}$$

i.e., recoverability is controlled by the *loss of relative entropy* under restriction. Applied to the split inclusion $\mathcal{N}_{\text{can}} \subset \mathcal{M}$ with reference ω_0, Conjecture 5.1 would follow from a purely entropic estimate: that clustering forces the relative entropy of a state satisfying (i)–(ii) to be almost entirely visible inside $\mathcal{N}_{\text{can}}$, with deficit controlled by $\mathcal{C}(\omega; \mathcal{O}_1, \mathcal{O}_2)$. In the quasi-free setting this deficit can be computed explicitly from covariance data, which suggests a concrete intermediate goal: prove the entropic deficit bound for quasi-free states and compare the resulting fidelity estimate with Theorem 4.8.

6 Discussion and Open Problems

6.1 Summary of Results

We have established:

1. **Clustering–Recovery Bridge** (Theorem 4.8): for quasi-free states satisfying (a)–(g) in the regularized framework of Assumption 2.7, with $\tilde{\omega}$ the conditional-reattachment reconstruction:

$$1 - F(\omega, \tilde{\omega}) \leq \frac{C^{(0)}_{d,\kappa}}{\epsilon^2 \left(1 - \epsilon^{-1}(\eta_{\text{vac}} + \delta)^2\right)^2} \cdot \left\|\Delta^{(12)}\right\|^2_{\text{HS}} \tag{80}$$

with $C^{(0)}_{d,\kappa} = \frac{C_\kappa(6+C_\kappa)}{16\min(c_1,c_2)^2}$.

2. **Explicit dependence on δ:** the bound correctly incorporates the cross-correlation perturbation parameter.

3. **Explicit dependence on mixedness:** the symplectic gap κ enters only through C_κ, quantifying the degradation of the quadratic fidelity regime near pure reference states (and Remark C.3 shows some such dependence is unavoidable).

4. **Finite rank corollary** with factor $2n$ from $\text{rank}(\Delta^{(12)}) \leq 2n$.

5. **Exponential clustering:** $\eta_{\text{vac}} \lesssim K_\nu(mr) \sim e^{-mr}$.

6. **Exact recovery for coherent states:** $F = 1$ when $\Delta^{(12)} = 0$.

6.2 Physical Implications

Mass Gap as Locality Regulator. The factor e^{-mr} shows that the mass m controls information locality:

- Local excitations are nearly exactly recoverable.
- Cross-regional correlations are exponentially suppressed.
- Vacuum entanglement enables (not obstructs) recovery.

Companion papers. The thermodynamic cost of *maintaining* coherence across a gapped collar — a maintenance-power reading of the exponential factors above — is developed in [21], and the operational "Heisenberg cut as resource boundary" program, whose exact rate-inheritance law shares the Combes–Thomas exponent instantiated in Appendix E, in [22]. Both are distributed, with their own verification suites, in the same repository as this paper.

Role of Hypothesis (d). The cross-correlation control $\delta_X \leq \delta$ ensures that the state's correlation structure remains "close" to the vacuum's. This is automatically satisfied for local Bogoliubov transformations (Remark 4.18).

Role of Hypothesis (f). The symplectic gap encodes the observation that quadratic (metric) fidelity response is a property of the *interior* of the Gaussian state space; at the boundary (pure states), perturbations that increase mixedness produce a linear fidelity loss. Since the reattachment generically changes mixedness in the $(2, 2)$-block, some interior condition is needed.

Role of Hypothesis (g). Admissibility is not a technicality: for strongly entangled references at small collar width, the conditional covariance C_0 is steerable and (10) is not a state (Remark 2.18); numerically this is exactly the regime where naive versions of the bound break down (Appendix E). The mass gap restores admissibility exponentially fast in mr.

6.3 Conditional Implications for Holography

Conditional on Conjecture 5.1 or suitable extensions.

In AdS/CFT, entanglement wedge reconstruction [8, 17] asserts that bulk operators in $\mathcal{E}_A$ can be represented on boundary region A; Petz and twirled-Petz reconstructions of the wedge were constructed in [7, 6].

Corollary 6.1 (Conditional: Quantitative Wedge Reconstruction). *Assuming Conjecture 5.1 holds with bulk mass gap $m_{\text{bulk}} > 0$, one would obtain for bulk excitations in the entanglement wedge $\mathcal{E}_A$:*

$$1 - F(\omega, \tilde{\omega}) \leq C \cdot e^{-\gamma\, m_{\text{bulk}} \cdot d(x, \gamma_A)} \tag{81}$$

where $d(x, \gamma_A)$ is the geodesic distance to the RT surface γ_A.

This suggests that reconstruction accuracy would improve exponentially with depth into the wedge, providing a quantitative refinement of entanglement wedge reconstruction [8, 17].

6.4 Exact Numerical Verification

All corrections in this version were driven or confirmed by an exact numerical suite (truncated two-mode Fock space, Petz map computed from Definition 2.13 without Gaussian approximations, Uhlmann fidelities exact, plus covariance-level checks using the exact Gaussian fidelity formula of [3]). Appendix E reports methods and full tables; headline results: the pure-reference Petz counterexample and the marginal non-preservation dose-response (trace-norm deviation $0.37 \to 0.016$ as the reference mixedness $\bar{n}$ runs from 0 to 1); failure of hypothesis (g) exactly in the steering regime; the repaired bound holding with margin ≥ 40 across all admissible test points; the v1 fidelity lemma violated by the vacuum-vs-thermal family and the C_κ-corrected lemma passing all sampled gapped cases; and the lattice Klein–Gordon decay rate of η_{vac} matching $\operatorname{arccosh}(1 + m^2/2)$.

6.5 Open Problems

1. **Prove Conjecture 5.1** beyond quasi-free states; the relative-entropy route via [12] (Section 5.4) appears most promising.

2. **The exact Gaussian Petz theorem.** Lami–Das–Wilde [19] give the exact covariance action of the Petz map for Gaussian references. Prove the analogue of Theorem 4.8 for it, and bound the reattachment–Petz gap; numerically the gap closes monotonically with reference mixedness (Appendix E), which matches the physical regime of local vacuum restrictions in QFT (Type III factors are "maximally mixed locally").

3. **Optimal exponent.** Is $\gamma = 2$ optimal? Can $\gamma = 1$ be achieved?

4. **Massless fields/CFTs.** Clustering is polynomial: $\mathcal{C} \sim r^{-\Delta_{\min}}$.

5. **Interacting theories.** New techniques needed beyond Gaussian structure.

6. **Weaken hypothesis (d).** Can δ_X control be derived from (a)–(c)?

7. **Operational interpretation.** Translate to measurement distinguishability.

8. **Weaken hypothesis (f).** The counterexample of Remark C.3 has $\Delta^{(11)} \neq 0$, whereas the reconstruction error satisfies $\Delta^{(11)} = 0$ identically. Does the block structure of $\Delta\Gamma$ restore a quadratic bound for (nearly) pure global states, or is there a genuine linear regime for the reconstruction of pure excitations? Relatedly, characterize the crossover scale between the linear and quadratic fidelity regimes in terms of κ and $\|K\|_{\text{HS}}$.

A Haagerup L^p Framework

For Type III algebras, the Haagerup L^p framework provides density matrix analogues.

A.1 Construction

Let $\mathcal{M}$ be a von Neumann algebra with faithful normal state φ and modular group σ_t^φ.

Definition A.1 (Crossed Product). $\widetilde{\mathcal{M}} := \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ is Type II_∞ with trace τ.

Definition A.2 (Haagerup L^p Spaces).

$$L^p(\mathcal{M}) := \{x \in \widetilde{\mathcal{M}} : \hat{\sigma}_s(x) = e^{-s/p}x\} \tag{82}$$

Key properties: $L^\infty(\mathcal{M}) = \mathcal{M}$, $L^1(\mathcal{M}) \cong \mathcal{M}_*$, and each state ω has density $h_\omega \in L^1(\mathcal{M})^+$.

B Bessel Function Asymptotics

B.1 Definitions

The modified Bessel function of the second kind:

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)} \tag{83}$$

B.2 Asymptotics

Large argument ($z \to \infty$) [24, §10.40]:

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z} \tag{84}$$

Small argument ($z \to 0^+$, $\nu > 0$):

$$K_\nu(z) \sim \frac{\Gamma(\nu)}{2}\left(\frac{2}{z}\right)^\nu \tag{85}$$

C Proof of the Gaussian Fidelity Lemma

We derive Lemma 4.6 from the exact Gaussian fidelity formula of Banchi–Braunstein–Pirandola [3]. Throughout, states are zero-mean quasi-free states in the convention of Remark D.1 (vacuum covariance $\mathbf{1}$; uncertainty condition $\Gamma + i\Omega \geq 0$; symplectic eigenvalues $\nu_j \geq 1$, with equality exactly on pure directions).

C.1 Exact Fidelity and the Overlap Factor

For zero-mean Gaussian states with covariances Γ_1, Γ_2, the Uhlmann fidelity $F(\omega_1, \omega_2)$ admits a closed-form expression in terms of Γ_1, Γ_2 and the symplectic form Ω [3]. It is instructive to compare with the *overlap factor*

$$F_{\mathrm{cl}}(\omega_1, \omega_2)^4 := \det{}_2\left(\frac{4\Gamma_1^{1/2}\Gamma_2\Gamma_1^{1/2}}{(\Gamma_1 + \Gamma_2)^2}\right), \tag{86}$$

where $\det_2$ is the regularized Fredholm determinant, defined for operators of the form $\mathbf{1}+T$ with $T \in \mathcal{L}^2(\mathfrak{h})$. The exact fidelity differs from (86) by a symplectic correction which is sensitive to the mixedness of the states; *it is not negligible near pure states* (Remark C.3), which is why v1 of this paper, which worked with (86) alone, required correction. The symplectic gap hypothesis (iv) of Lemma 4.6 controls precisely this correction.

Define the normalized perturbation parameter:

$$K := \Gamma_1^{-1/2}(\Gamma_1 - \Gamma_2)\Gamma_1^{-1/2}. \tag{87}$$

Note that $K \in \mathcal{L}^2(\mathfrak{h})$ when $\Gamma_1 - \Gamma_2$ is Hilbert–Schmidt, and $\|K\|_{\mathrm{op}} \leq 1/2$ ensures $\Gamma_2 \geq \frac{1}{2}\Gamma_1 > 0$.

C.2 Second-Order Expansion: the Gaussian Bures Metric

Lemma C.1 (Second-order expansion)**.** *Let $\Gamma(t) = \Gamma_1 + t\,\delta\Gamma$ be an admissible family of covariances (t in a neighbourhood of 0), with $\delta\Gamma \in \mathcal{L}^2(\mathfrak{h})$ symmetric. Then*

$$1 - F\big(\omega_1, \omega(t)\big) = \frac{t^2}{16}\,\mathrm{vec}(\delta\Gamma)^\dagger \big(\Gamma_1 \otimes \Gamma_1 - \Omega \otimes \Omega\big)^{-1} \mathrm{vec}(\delta\Gamma) + O(t^3), \tag{88}$$

where vec *denotes vectorization. The quadratic form on the right is $\frac{1}{8}$ of the quantum Fisher information of the family, as computed in closed form for Gaussian families in [3].*

The normalization in (88) can be checked against the one-mode thermal family $\Gamma(\nu) = \nu\mathbf{1}$: there the eigenvalue of $\Gamma\otimes\Gamma - \Omega\otimes\Omega$ on $\mathrm{vec}(\mathbf{1})$ is $\nu^2 - 1$, giving $1 - F \approx \frac{d\nu^2}{8(\nu^2-1)}$, which agrees with the exact Bhattacharyya computation for the associated geometric distributions.

Lemma C.2 (Metric comparison under a symplectic gap)**.** *If Γ_1 has symplectic gap $\kappa > 0$ (Definition 4.4), then for every symmetric $\delta\Gamma \in \mathcal{L}^2(\mathfrak{h})$:*

$$\mathrm{vec}(\delta\Gamma)^\dagger \big(\Gamma_1\otimes\Gamma_1 - \Omega\otimes\Omega\big)^{-1} \mathrm{vec}(\delta\Gamma) \;\le\; C_\kappa \left\|\Gamma_1^{-1/2}\,\delta\Gamma\,\Gamma_1^{-1/2}\right\|_{\mathrm{HS}}^2, \qquad C_\kappa = \frac{(1+\kappa)^2}{(1+\kappa)^2 - 1}. \tag{89}$$

Proof. The condition $\nu_{\min}(\Gamma_1) \ge 1+\kappa$ is equivalent to $\left\|\Gamma_1^{-1/2}\,\Omega\,\Gamma_1^{-1/2}\right\|_{\mathrm{op}} \le (1+\kappa)^{-1}$ (the eigenvalues of $i\,\Gamma_1^{-1/2}\Omega\,\Gamma_1^{-1/2}$ are $\pm 1/\nu_j$). Hence

$$\left\|(\Gamma_1\otimes\Gamma_1)^{-1/2}(\Omega\otimes\Omega)(\Gamma_1\otimes\Gamma_1)^{-1/2}\right\|_{\mathrm{op}} = \left\|\Gamma_1^{-1/2}\Omega\Gamma_1^{-1/2}\right\|_{\mathrm{op}}^2 \le (1+\kappa)^{-2}, \tag{90}$$

so, as quadratic forms,

$$\Gamma_1\otimes\Gamma_1 - \Omega\otimes\Omega \;\ge\; \big(1 - (1+\kappa)^{-2}\big)\,\Gamma_1\otimes\Gamma_1 = C_\kappa^{-1}\,\Gamma_1\otimes\Gamma_1, \tag{91}$$

and therefore $(\Gamma_1\otimes\Gamma_1 - \Omega\otimes\Omega)^{-1} \le C_\kappa\,(\Gamma_1\otimes\Gamma_1)^{-1}$. Finally,

$$\mathrm{vec}(\delta\Gamma)^\dagger(\Gamma_1\otimes\Gamma_1)^{-1}\,\mathrm{vec}(\delta\Gamma) = \mathrm{Tr}\big(\delta\Gamma\,\Gamma_1^{-1}\,\delta\Gamma\,\Gamma_1^{-1}\big) = \left\|\Gamma_1^{-1/2}\delta\Gamma\,\Gamma_1^{-1/2}\right\|_{\mathrm{HS}}^2. \tag{92}$$

□

Combining Lemmas C.1 and C.2 with $\delta\Gamma = \Gamma_1 - \Gamma_2$ (so that $\Gamma_1^{-1/2}\delta\Gamma\,\Gamma_1^{-1/2} = K$) gives the leading term

$$1 - F(\omega_1, \omega_2) = \frac{C'_\kappa}{16}\,\|K\|_{\mathrm{HS}}^2 + R_3(K), \qquad 0 \le C'_\kappa \le C_\kappa, \tag{93}$$

where C'_κ denotes the (state- and direction-dependent) exact quadratic coefficient normalized as in (88).

C.3 Remainder Control

For $\|K\|_{\mathrm{op}} \le 1/2$ and symplectic gap κ, the cubic remainder satisfies $|R_3(K)| \le C(\kappa, \|K\|_{\mathrm{op}})\,\|K\|_{\mathrm{HS}}^3$, where $C(\cdot,\cdot)$ is continuous and bounded on $\{\|K\|_{\mathrm{op}} \le 1/2\}$; this follows from the Taylor expansion of the exact fidelity functional of [3], whose derivatives through third order are controlled by the resolvent $(\Gamma_1\otimes\Gamma_1 - \Omega\otimes\Omega)^{-1}$ and hence by C_κ via Lemma C.2.

C.4 Final Bound

The coefficients in Lemma 4.6 are conservative upper bounds chosen to absorb the remainder uniformly on the regime $\|K\|_{\mathrm{op}} \le \frac{1}{2}$: doubling the leading coefficient $\frac{C_\kappa}{16} \to \frac{C_\kappa}{8}$ and allotting $\frac{C_\kappa^2}{48}\|K\|_{\mathrm{HS}}^4$ to higher orders yields (27). For $\|K\|_{\mathrm{HS}} \le 1$:

$$1 - F \le \frac{C_\kappa}{8}\|K\|_{\mathrm{HS}}^2 + \frac{C_\kappa^2}{48}\|K\|_{\mathrm{HS}}^4 \le \left(\frac{1}{8} + \frac{C_\kappa}{48}\right) C_\kappa \|K\|_{\mathrm{HS}}^2 \le \frac{C_\kappa}{6}\|K\|_{\mathrm{HS}}^2 \quad \text{whenever } C_\kappa \le 2, \tag{94}$$

and for general C_κ the displayed $\frac{C_\kappa}{6}$-form of Lemma 4.6 holds under the correspondingly stronger smallness condition $C_\kappa \|K\|_{\mathrm{HS}}^2 \le 2$; in the applications of Section 4 this is guaranteed by condition (ii) of Corollary 4.12 together with hypothesis (f).

C.5 Necessity of the Symplectic Gap

Remark C.3 (Counterexample without hypothesis (iv)). Let $\mathfrak{h}$ be one mode, ω_1 the vacuum ($\Gamma_1 = \mathbf{1}$, pure, $\kappa = 0$) and ω_2 the thermal state with mean occupation $\bar{n}$, $\Gamma_2 = (1+2\bar{n})\mathbf{1}$. Then $K = -2\bar{n}\,\mathbf{1}$, so $\|K\|_{\mathrm{HS}}^2 = 8\bar{n}^2$ and $\|K\|_{\mathrm{op}} = 2\bar{n} \le 1/2$ for $\bar{n} \le 1/4$. The exact fidelity is

$$F(\omega_1, \omega_2) = \sqrt{\langle 0|\rho_{\mathrm{th}}|0\rangle} = (1+\bar{n})^{-1/2}, \qquad 1 - F = \frac{\bar{n}}{2} + O(\bar{n}^2), \tag{95}$$

which is *linear* in $\bar{n}$, whereas the gap-free version of (27) would give the quadratic bound $\bar{n}^2 + O(\bar{n}^4)$. For $\bar{n} = 0.1$: $1 - F \approx 0.0465$ versus a putative bound of ≈ 0.0101. (The overlap factor (86) alone gives $1 - F_{\mathrm{cl}} \approx \bar{n}^2/2$, illustrating that the symplectic correction dominates near purity.) Hence hypothesis (iv) — or some substitute exploiting additional structure of $\delta\Gamma$ — is necessary. Note that for perturbations *tangent* to the pure-state manifold (e.g., vacuum vs. squeezed vacuum) the quadratic bound does hold; the failure is specific to mixedness-increasing directions.

D The Reattachment Map in the Heisenberg Picture

We characterize the conditional reattachment Λ_{Γ_0} of Definition 2.15 by its dual action on Weyl operators and derive the covariance formula (10). We then record precisely when this dual action agrees with that of the Petz map. (In v1 this appendix asserted that the dual action below *is* the Petz dual in general; that assertion is false — see Lemma 2.14 and Remark D.3 below.)

D.1 Setup and Conventions

We work in a finite-mode (continuous-variable) setting with $n = n_1 + n_2$ bosonic modes. Let $R = (R_1, R_2)^T$ with $R_1 \in \mathbb{R}^{2n_1}$, $R_2 \in \mathbb{R}^{2n_2}$ denote the vectors of quadrature operators for subsystems 1 and 2 respectively, satisfying $[R_j, R_k] = i\,\Omega_{jk}$ where Ω is the standard symplectic form.

The Weyl operators are defined as:

$$W(z) := \exp(iz^T \Omega R), \qquad z = (z_1, z_2) \in \mathbb{R}^{2n} \tag{96}$$

Remark D.1 (Conventions). With this choice of $W(z)$, a centered Gaussian state with real symmetric covariance matrix Γ has characteristic function $\chi_\rho(z) = \exp(-\frac{1}{4}z^T\Gamma z)$; in particular the one-mode vacuum has $\Gamma = \mathbf{1}$ and the uncertainty condition reads $\Gamma + i\Omega \ge 0$ (cf. Remark 2.6). Throughout this appendix, all covariance blocks (A, X, B, etc.) are real matrices, and we write X^T for the transpose.

Under the bipartition $z = (z_1, z_2)$:

$$\Gamma = \begin{pmatrix} A & X \\ X^T & B \end{pmatrix} \tag{97}$$

D.2 The Reattachment Map in the Heisenberg Picture

Lemma D.2 (Heisenberg characterization of the reattachment). *Let ρ_0 be a faithful centered Gaussian state with covariance $\Gamma_0 = \begin{pmatrix} A_0 & X_0 \\ X_0^T & B_0 \end{pmatrix}$ and define $C_0 := B_0 - X_0^T A_0^{-1} X_0 \geq 0$ (conditional covariance). The conditional reattachment Λ_{Γ_0} of Definition 2.15 is the unique map on admissible centered Gaussian states whose dual acts on Weyl operators as*

$$\Lambda^*_{\Gamma_0}(W(z_1, z_2)) = \exp\left(-\tfrac{1}{4} z_2^T C_0 z_2\right) W_1\left(z_1 + A_0^{-1} X_0 z_2\right) \tag{98}$$

where $W_1(\cdot)$ denotes a Weyl operator on subsystem 1, in the sense that $\mathrm{Tr}\left(\Lambda_{\Gamma_0}[\sigma_1] W(z_1, z_2)\right) = \mathrm{Tr}\left(\sigma_1 \Lambda^*_{\Gamma_0}(W(z_1, z_2))\right)$.

Moreover, if $\rho_0 = \rho_{0,1} \otimes \rho_{0,2}$ (i.e., $X_0 = 0$), then (98) coincides with the dual of the Petz map (9):

$$\mathcal{R}^*_{\rho_0}(W_1(z_1) \otimes W_2(z_2)) = \chi_{\rho_{0,2}}(z_2)\, W_1(z_1) = e^{-\frac{1}{4} z_2^T B_0 z_2}\, W_1(z_1). \tag{99}$$

Proof. The first statement is Proposition D.4 below read backwards: the right-hand side of (98) paired against σ_1 produces exactly the characteristic function of the Gaussian state with covariance (10), and a state on the (regularized) Weyl algebra is determined by its characteristic function.

For the second statement, use the Petz dual (transpose channel) formula [25]

$$\mathcal{R}^*_{\rho_0}(Y) = \rho_{0,1}^{-1/2}\, \mathrm{Tr}_2\left(\rho_0^{1/2}\, Y\, \rho_0^{1/2}\right) \rho_{0,1}^{-1/2}. \tag{100}$$

With $\rho_0^{1/2} = \rho_{0,1}^{1/2} \otimes \rho_{0,2}^{1/2}$ and $Y = W_1(z_1) \otimes W_2(z_2)$,

$$\mathrm{Tr}_2\left(\rho_0^{1/2} Y \rho_0^{1/2}\right) = \rho_{0,1}^{1/2} W_1(z_1) \rho_{0,1}^{1/2} \cdot \mathrm{Tr}(\rho_{0,2} W_2(z_2))\,, \tag{101}$$

and the outer factors $\rho_{0,1}^{-1/2}$ cancel the inner ones around $W_1(z_1)$, leaving (99) by $\mathrm{Tr}(\rho W(z)) = \chi_\rho(z) = e^{-\frac{1}{4} z^T \Gamma_\rho z}$ [15, Prop. 12.8]. This agrees with (98) at $X_0 = 0$ (where $W = 0$ and $C_0 = B_0$). ☐

Remark D.3 (Why v1's identification with the Petz dual fails). For correlated ρ_0, evaluating (100) on Weyl operators via the Gibbs representation $\rho_0 = Z^{-1} \exp(-\frac{1}{2} R^T G_0 R)$ [15, Proposition 12.18] and the Gaussian/CCR normal-form calculus [13, Chapter 4], [27] shows that $\rho_0^{1/2} W(z) \rho_0^{1/2}$ is a Gaussian operator with a shifted linear part, but the shift is governed by G_0 (the Gibbs matrix), *not* by the regression matrix $A_0^{-1} X_0$ of the covariance; the two coincide precisely in the factorized case. v1's "tracking these transformations one finds" step implicitly made that identification, which is how formula (10) was misattributed to the Petz map. The correct general Gaussian Petz output is a bosonic Gaussian channel with explicit (and different) covariance action, derived by Lami, Das and Wilde [19]; its $(1,1)$ block does *not* equal A in general, consistent with Lemma 2.14. The exact numerical comparison of the two maps is reported in Appendix E.

D.3 Derivation of Covariance Blocks

Proposition D.4 (Explicit covariance formulas)**.** *Let σ_1 be a centered Gaussian state on subsystem 1 with covariance A, and let $\tilde{\rho} := \Lambda_{\Gamma_0}[\sigma_1]$ be the state determined by pairing σ_1 with the dual action* (98)*. Then $\tilde{\rho}$ is Gaussian with covariance:*

$$\Gamma_{\tilde{\rho}}^{(11)} = A \tag{102}$$
$$\Gamma_{\tilde{\rho}}^{(12)} = AA_0^{-1}X_0 \tag{103}$$
$$\Gamma_{\tilde{\rho}}^{(22)} = B_0 + X_0^T A_0^{-1}(A - A_0)A_0^{-1}X_0 \tag{104}$$

Proof. By definition of the pairing:

$$\chi_{\tilde{\rho}}(z_1, z_2) = \mathrm{Tr}\left(\tilde{\rho}\, W(z_1, z_2)\right) = \mathrm{Tr}\left(\sigma_1\, \Lambda^*_{\Gamma_0}(W(z_1, z_2))\right) \tag{105}$$
$$= e^{-\frac{1}{4}z_2^T C_0 z_2}\, \chi_{\sigma_1}(z_1 + A_0^{-1}X_0 z_2) \tag{106}$$

Since $\chi_{\sigma_1}(u) = e^{-\frac{1}{4}u^T A u}$:

$$\chi_{\tilde{\rho}}(z_1, z_2) = \exp\left(-\tfrac{1}{4}(z_1 + A_0^{-1}X_0 z_2)^T A(z_1 + A_0^{-1}X_0 z_2) - \tfrac{1}{4}z_2^T C_0 z_2\right) \tag{107}$$

Expanding $(z_1 + A_0^{-1}X_0 z_2)^T A(z_1 + A_0^{-1}X_0 z_2)$:

$$= z_1^T A z_1 + 2z_1^T(AA_0^{-1}X_0)z_2 + z_2^T(X_0^T A_0^{-1} A A_0^{-1}X_0)z_2 \tag{108}$$

Therefore:

$$\chi_{\tilde{\rho}}(z_1, z_2) = \exp\left(-\tfrac{1}{4}(z_1, z_2)^T \Gamma_{\tilde{\rho}}\,(z_1, z_2)\right) \tag{109}$$

where the blocks of $\Gamma_{\tilde{\rho}}$ are read off as:

$$\Gamma_{\tilde{\rho}}^{(11)} = A \tag{110}$$
$$\Gamma_{\tilde{\rho}}^{(12)} = AA_0^{-1}X_0 \tag{111}$$
$$\Gamma_{\tilde{\rho}}^{(22)} = X_0^T A_0^{-1} A A_0^{-1} X_0 + C_0 = B_0 + X_0^T A_0^{-1}(A - A_0)A_0^{-1}X_0 \qquad \square$$

D.4 Verification

Consistency: when $A = A_0$: $\Gamma_{\tilde{\rho}}^{(12)} = X_0$ and $\Gamma_{\tilde{\rho}}^{(22)} = B_0$. ✓

Positivity: the Schur complement of $\Gamma_{\tilde{\rho}}$ with respect to the first block equals $C_0 = B_0 - X_0^T A_0^{-1} X_0 \geq 0$, confirming $\Gamma_{\tilde{\rho}} \geq 0$; the stronger uncertainty condition $\Gamma_{\tilde{\rho}} + i\Omega \geq 0$ requires Proposition 2.17. ✓

E Exact Numerical Verification

All quantitative claims of this paper are accompanied by an exact numerical suite, distributed with the source (`verification/` directory). Nothing below uses a Gaussian approximation to fidelities or to the Petz map: states live in a truncated two-mode Fock space (cutoff $N_c = 24$ per mode), the Petz map is evaluated directly from Definition 2.13 by operator square roots, Uhlmann fidelities are computed by exact spectral calculus, and covariance-level checks use the exact closed-form Gaussian fidelity of [3] (validated against three independent analytic families to 10^{-8}). Pipeline control: $\mathcal{R}_{\rho_0}(\rho_{0,1}) = \rho_0$ is reproduced with $1 - F \sim 10^{-9}$.

E.1 Petz is not the reattachment

Reference family: $\rho_0 = U_{\rm tms}(s)\,[\rho_{\rm th}(\bar n)\otimes\rho_{\rm th}(\bar n)]\,U_{\rm tms}(s)^\dagger$ with $s = 0.35$; input $\sigma_1 = \mathrm{Tr}_2\,\rho_\omega$ for a local Bogoliubov excitation ρ_ω ($\lambda = 0.25$, so $B = B_0$). Exact Petz output vs. the reattachment covariance (10):

$\bar n$	$F(\text{Petz out}, \rho_0)$	$\lVert \mathrm{Tr}_2(\text{Petz out}) - \sigma_1 \rVert_1$	$\max\lvert \Gamma_{\rm Petz} - \Gamma_{\tilde\omega}\rvert$
0 (pure)	1.000000	0.3654	0.8143
0.02	0.996925	0.2998	0.6922
0.10	0.989926	0.1692	0.4432
0.40	0.979933	0.0515	0.2055
1.00	0.973852	0.0158	0.1072

The $\bar n = 0$ row realizes Lemma 2.14(b) exactly: the Petz output *is* ρ_0 regardless of the input. The marginal deviation and the covariance gap decay monotonically with reference mixedness (dose-response), consistent with Remark 2.16.

E.2 The repaired theorem holds; admissibility is sharp

For the same family, $1 - F(\omega, \tilde\omega)$ computed with the exact Gaussian fidelity (cross-checked against the truncated-Fock pipeline at $\bar n = 0.10$ to four digits: 2.645×10^{-2} both ways), versus the v2 bound; here $\kappa = 2\bar n$ exactly, and "adm" is $\lambda_{\min}(\Gamma_{\tilde\omega} + i\Omega)$:

$\bar n$	κ	C_κ	adm	$1 - F$	bound (simplified)
0.002	0.004	125.7	-0.124	(g) fails — $\tilde\omega$ not a state (steering)	
0.02	0.04	13.2	-0.096	(g) fails	
0.10	0.20	3.27	$+0.027$	2.65×10^{-2}	1.79
0.40	0.80	1.45	$+0.455$	1.11×10^{-2}	6.35×10^{-1}
1.00	2.00	1.13	$+1.15$	9.61×10^{-3}	4.73×10^{-1}
3.00	6.00	1.02	$+2.96$	9.28×10^{-3}	4.23×10^{-1}

The bound holds with margin ≥ 40 at every admissible point. The two non-admissible rows are precisely the near-pure steering regime; this is the regime in which numerical tests of the *v1* statement (Petz + no admissibility hypothesis) produce violations, which prompted this revision.

E.3 The fidelity lemma: gap-free failure, gapped success

The vacuum-vs-thermal family violates the v1 (gap-free) form of Lemma 4.6 exactly as predicted by Remark C.3: at $\bar n = 0.05, 0.1, 0.2$ the true loss $1-F = 0.0241$, 0.0465, 0.0871 exceeds the putative bound 0.0025, 0.0101, 0.0421. With the C_κ-corrected statement and symplectic-gap references, random two-mode perturbations show no violations (see `results/` for the sampled margins).

E.4 Lattice Klein–Gordon instantiation

For the discrete massive Laplacian on a chain ($N = 240$, $m = 1$), the exact $\eta_{\rm vac}$ computed from the vacuum covariance blocks decays exponentially in the collar width over ten decades (down to the double-precision floor, which must be excluded from the fit). A plain exponential fit gives rate 1.004; modeling the expected prefactor, $\eta_{\rm vac} \sim r^{-1/2} e^{-\mu r}$, gives $\mu \approx 0.954$, against the analytic lattice rate $\operatorname{arccosh}(1+m^2/2) \approx 0.962$ — agreement within 1%, confirming Lemma 3.4 quantitatively in a concrete regularization.

F Changes Relative to Version 1

1. **The reconstruction is not the Petz map.** v1's Proposition 2.14 claimed that the Petz map preserves the input marginal (Step 2 of its proof, attributed to [25]) and hence that formula (10) is the Petz output. This is false for correlated references: marginal preservation is a property of the ρ_0-preserving conditional expectation, which by Takesaki's theorem exists on $B(\mathcal{H}_1) \otimes \mathbf{1}$ only for factorized ρ_0; for pure ρ_0 the Petz map returns ρ_0 for every input (Lemma 2.14). v2 restates all results for the conditional reattachment Λ_{Γ_0} (Definition 2.15), which is what the v1 proof actually controls, adds the admissibility hypothesis (g) with an explicit no-steering sufficient condition (Proposition 2.17, Remark 2.18), records that Λ_{Γ_0} is not induced by any CP channel (Remark 2.16), and rewrites Appendix D accordingly, citing [19] for the exact Gaussian Petz channel. This correction was prompted by an exact truncated-Fock numerical verification (Appendix E).

2. **Corrected constant.** v1 stated the simplified constant as $\frac{3}{8\min(c_1,c_2)^2}$ while its own Step 5 derived $\frac{7}{16\min(c_1,c_2)^2}$; since $\frac{3}{8} < \frac{7}{16}$, the stated bound was stronger than the proved one. v2 states the constant $C^{(0)}_{d,\kappa} = \frac{C_\kappa(6+C_\kappa)}{16\min(c_1,c_2)^2}$, which reduces to $\frac{7}{16\min(c_1,c_2)^2}$ as $C_\kappa \to 1$ (Remark 4.9). A duplicated passage in Step 5 (two near-identical discussions of conditions (i)–(ii) with different constants) was removed.

3. **Symplectic gap hypothesis.** The Gaussian fidelity lemma (Lemma 4.6) now carries hypothesis (iv) (symplectic gap $\kappa > 0$ for the reference covariance), inherited by Theorem 4.8 as hypothesis (f). Remark C.3 gives an explicit one-mode counterexample (vacuum vs. thermal) showing that without such a hypothesis the quadratic fidelity bound fails: the v1 derivation relied on the overlap factor (86), which omits the symplectic correction to the exact Gaussian fidelity of [3]; that correction produces a *linear* fidelity loss for mixedness-increasing perturbations of pure states. Appendix C was rewritten accordingly (Lemmas C.1 and C.2).

4. **Convention fix.** Remark 2.6: the uncertainty condition consistent with the characteristic-function convention $\chi = e^{-\frac{1}{4}z^T\Gamma z}$ of Appendix D is $\Gamma + i\Omega \geq 0$ (one-mode vacuum $\Gamma = \mathbf{1}$), not $\Gamma + \frac{i}{2}\Omega \geq 0$ as stated in v1.

5. **Expanded proof of Lemma 3.4** (Schur test), with the constant C_d now defined explicitly, and of Remark 4.18, where a condition-number factor $\sqrt{\|A_0\|_{\mathrm{op}}/c_1}$ omitted in v1 has been restored.

6. **Related work.** Added Section 1.3 comparing with the recoverability results of [12, 11, 18] and the holographic Petz literature [7, 6], and a corresponding new strategy for Conjecture 5.1 (Section 5.4, item 4). References [5, 24, 14, 27] are now cited in the text.

7. **New open problems** on weakening the symplectic-gap hypothesis and on proving the theorem for the exact Gaussian Petz channel (Section 6.5, items 2 and 8). The redundant standing assumption "$\eta_{\mathrm{vac}} < 1$" in Theorem 4.8 (implied by hypothesis (d)) was removed.

8. **Numerical verification.** Added Appendix E and a verification suite distributed with the source: exact Petz map and Uhlmann fidelities in truncated Fock space, exact Gaussian fidelities at covariance level, the counterexamples of Lemma 2.14 and Remark C.3, the repaired bound across mixedness regimes, and a lattice Klein–Gordon instantiation of Lemma 3.4.

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