A Note on Affine-like Invariance in Finite Collatz Segments

Kevin Fidelis
kevinfidelis995@gmail.com

December 6, 2025

Abstract

We observe an explicit algebraic relationship between certain initial values under the Collatz map. For a starting integer X and a chosen step count m, numbers of the form

$$Y = X + k \cdot 3^u \cdot 2^e$$

share the same parity sequence as X for the first m steps, where e is the number of even steps in X's first m iterations. The difference $\Delta_s = C^{(s)}(Y) - C^{(s)}(X)$ evolves as

$$\Delta_s = k \cdot 3^{u+o_s} \cdot 2^{e-e_s},$$

where o_s, e_s count odd/even steps up to s. This relationship persists until the exponent of 2 in Δ_s becomes negative. The result is an elementary algebraic identity with no implication for the Collatz conjecture.

1 Introduction

The Collatz function $C : \mathbb{N} \to \mathbb{N}$ is defined as

$$C(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

It is known that trajectories beginning in certain arithmetic progressions share initial parity patterns. Here we make this relationship explicit through a closed-form expression that tracks the difference between two such trajectories. The purpose of this note is to record a verifiable algebraic fact, not to claim structural novelty.

2 Why "Affine-like" and Not "Affine Isomorphism"

The term **"affine"** is used here in a restricted, non-structural sense:

- The transformation from X to Y is affine in form: $Y = X + \text{constant}$.
- The difference Δ_s evolves under **affine-like rules**:

$$\Delta_{s+1} = \begin{cases} 3\Delta_s & \text{if } X_s \text{ is odd (scaling),} \\ \frac{1}{2}\Delta_s & \text{if } X_s \text{ is even (scaling).} \end{cases}$$

This is **not an isomorphism** in the algebraic or categorical sense. There is no bijective structure-preserving map between trajectories—only a predictable difference propagation. We avoid the stronger term "isomorphism" to prevent misinterpretation.

3 Definitions and Construction

Let $X = X_0, X_1, \ldots, X_m$ be a finite trajectory segment. Define:

$$o = \#\{i < m : X_i \text{ is odd}\}, \quad e = \#\{i < m : X_i \text{ is even}\}.$$

Clearly $o + e = m$. For integers $k \geq 1$ and $u \geq 0$, define

$$Y = X + k \cdot 3^u \cdot 2^e.$$

4 Theorem (Difference Propagation)

For $0 \leq s \leq m$,

$$\Delta_s \equiv C^{(s)}(Y) - C^{(s)}(X) = k \cdot 3^{u+o_s} \cdot 2^{e-e_s},$$

where o_s, e_s are the counts of odd/even steps among the first s steps of X.

In particular, X_s and Y_s have the same parity for all $s \leq m$, and

$$C^{(m)}(Y) = X_m + k \cdot 3^{u+o}.$$

Proof. By induction on s:

- **Base case** $s = 0$: $\Delta_0 = Y - X = k \cdot 3^u \cdot 2^e$.
- **Inductive step**: If X_s is odd, then

  $$\Delta_{s+1} = 3\Delta_s = k \cdot 3^{u+o_s+1} \cdot 2^{e-e_s}.$$

  If X_s is even, then

  $$\Delta_{s+1} = \Delta_s/2 = k \cdot 3^{u+o_s} \cdot 2^{e-e_s-1}.$$

Updating o_s and e_s accordingly confirms the formula. □

5 Termination Condition (2-Depletion)

Let $t_s = e - e_s$, the exponent of 2 in Δ_s. The parity match holds while $t_s \geq 0$. When $t_s < 0$, Δ_s is no longer divisible by $2^{|t_s|}$, and the trajectories diverge in parity. This is guaranteed to occur after enough even steps.

6 Example

Let $X = 1$, $m = 3$. Trajectory: $1 \xrightarrow{O} 4 \xrightarrow{E} 2 \xrightarrow{E} 1$. Here $o = 1$, $e = 2$. Take $k = 1, u = 0$:

$$Y = 1 + 1 \cdot 3^0 \cdot 2^2 = 5.$$

Predicted: $C^{(3)}(5) = 1 + 1 \cdot 3^1 = 4$. Actual: $5 \to 16 \to 8 \to 4$ — confirmed.

7 Discussion

7.1 What This Is

- A closed-form expression for constructing numbers with shared initial parity patterns.
- A deterministic rule for difference evolution under C.
- A finite, local property with no infinite guarantees.

7.2 What This Is Not

- A discovery about the Collatz conjecture.
- A structural isomorphism.
- A new modular invariance—it is a repackaging of known residue class behavior.

7.3 Related Work

This observation of local affine structure is situated within the broader context of modular dynamics in the Collatz problem. It relates to established research on the statistical behavior of trajectories and congruence classes, as discussed in modern surveys and recent analytic results.

References

[1] Terence Tao, *Almost all Collatz orbits attain almost bounded values*, arXiv preprint arXiv:1909.03562, 2019.

[2] Jeffrey C. Lagarias, *The 3x+1 Problem: An Overview*, in: *The Ultimate Challenge: The 3x+1 Problem*, J. C. Lagarias (ed.), American Mathematical Society, 2010, pp. 3–29. Also available as arXiv preprint arXiv:2111.02635, 2021.