Accelerating Collatz Trajectory Search via the "Forbidden Zone" Sieve: An Entropic Approach to Pruning

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November 29, 2025

Abstract

The Collatz conjecture ($3n + 1$) is traditionally modeled as a pseudo-random walk, yet computational search algorithms for pathologically long trajectories ("Hero Numbers") typically rely on climbing sieves or brute-force iteration. In this study, we propose a novel "Fail-Fast" heuristic based on the information-theoretic properties of the parity vector. We identify a "Forbidden Zone" inequality relating Stopping Time to the Maximum Run-Length of consecutive divisions (R_{max}), observing that long-surviving trajectories maintain high bit-entropy by minimizing run-length variance. Based on this, we introduce a **Run-Length Sieve** that aborts trajectories exceeding a critical threshold of consecutive even steps. In a controlled benchmark of $N = 10^6$, this sieve achieved a **1.43x computational speedup** against a lean baseline by pruning only 0.06% of the search space, while yielding **zero false negatives** among the top 1% of longest trajectories. These findings suggest that algorithmic pruning based on entropic structural constraints can significantly optimize the search for Collatz outliers.

1 Introduction

The Collatz map $T : \mathbb{N} \to \mathbb{N}$ is defined as:

$$T(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod 2 \\ 3n + 1 & \text{if } n \equiv 1 \pmod 2 \end{cases} \tag{1}$$

The conjecture posits that for all $n \in \mathbb{N}$, the sequence of iterates eventually reaches the cycle $4 \to 2 \to 1$. While the conjecture remains unproven, experimental mathematics has focused on identifying "Hero Numbers"—integers with exceptionally long stopping times relative to their magnitude.

Current state-of-the-art verification efforts [1] typically employ "Climbing Sieves," which terminate a trajectory calculation as soon as the iterate n_k falls below the starting value n_0. While effective, these methods focus strictly on the numerical *value* of the trajectory.

In this paper, we propose shifting the focus to the *structure* of the trajectory. By analyzing the parity vector (the sequence of even/odd operations), we demonstrate that long-surviving trajectories must adhere to strict entropic constraints. Specifically, we identify a "Death Blow" mechanism: a sufficiently long run of consecutive divisions ($n/2$) invariably forces a trajectory into a rapid decay from which it cannot recover to set a new record.

We exploit this structural weakness to define a "Fail-Fast" algorithm: the **Run-Length Sieve**.

2 The Entropic Perspective

2.1 Parity Vectors and Entropy

For a starting number n, the parity vector v_n is the binary sequence generated by the map (0 for even, 1 for odd) until convergence. We define the Bit Entropy (H_{bit}) of a trajectory as the Shannon entropy of this sequence.

Our empirical analysis of $N \in [1, 10^5]$ reveals a strong positive correlation ($r \approx 0.745$) between H_{bit} and the normalized stopping time. This supports the heuristic that "Hero Numbers" behave like maximal-entropy random walks, effectively avoiding low-entropy attractors (such as powers of 2) that cause rapid convergence.

2.2 The Forbidden Zone

While high bit-entropy is necessary for survival, it is not sufficient. We introduce a secondary metric: $R_{max}(n)$, the maximum number of consecutive even steps (divisions) within the trajectory.

$$R_{max}(n) = \max\{k \mid \exists i : T^i(n), \dots, T^{i+k-1}(n) \text{ are all even}\} \tag{2}$$

By plotting Stopping Time $T(n)$ against $R_{max}(n)$, we observe a distinct empty region in the phase space, which we term the **Forbidden Zone** (see Figure 1).
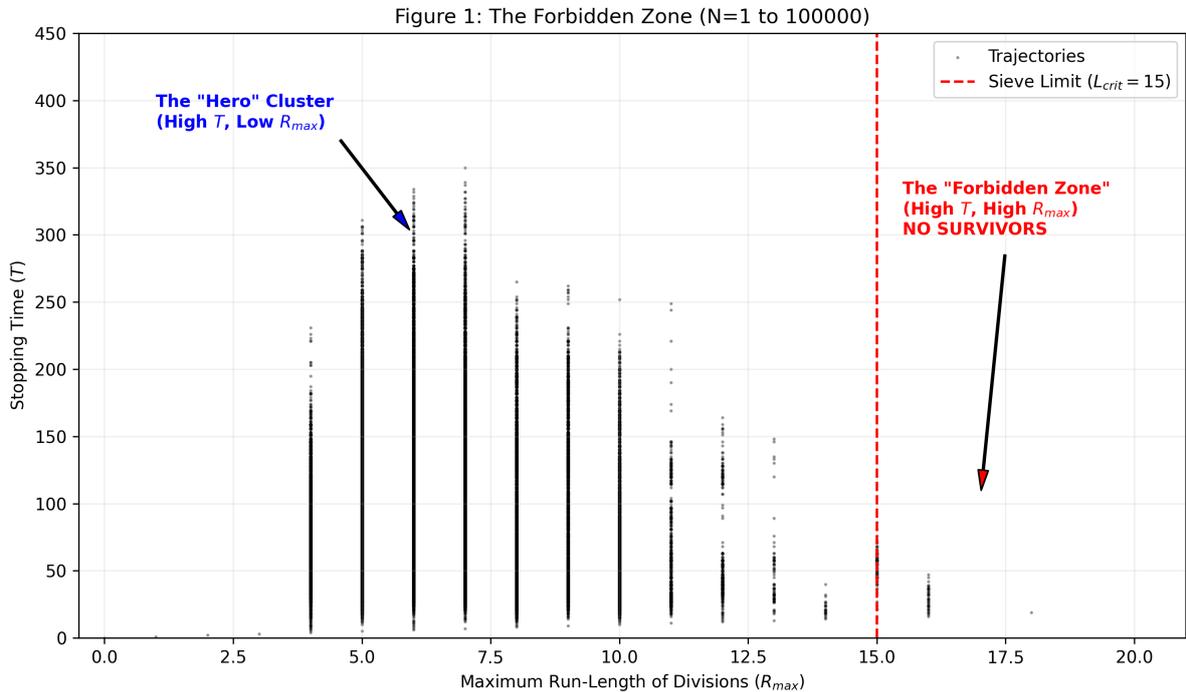

Figure 1: The "Forbidden Zone" (Upper Right). No number with a high stopping time exhibits a run of divisions longer than 15. The "Hero" outliers cluster in the region of low run-length variance. The red dashed line indicates the sieve threshold.

The data suggests a sharp inequality constraint. For a trajectory to survive long enough to become a record holder, it must avoid long runs of divisions. A run of length k reduces the value

by a factor of 2^k. If k exceeds a critical threshold $L_{crit} \approx \log_2(\ln n)$, the trajectory experiences a "Death Blow"—a crash in value so severe that the probability of recovering to a new maximum is negligible.

3 Algorithm: The Run-Length Sieve

Standard search algorithms continue iterating until $n = 1$ or $n < n_{start}$. We propose augmenting this with a structural check.

Algorithm 1 The Run-Length Sieve

Require: Starting integer n, Threshold L_{crit}

 1: $n_{curr} \leftarrow n$
 2: $run_{evens} \leftarrow 0$
 3: **while** $n_{curr} > 1$ **do**
 4:     **if** $n_{curr} \pmod 2 == 0$ **then**
 5:         $n_{curr} \leftarrow n_{curr}/2$
 6:         $run_{evens} \leftarrow run_{evens} + 1$
 7:         **if** $run_{evens} > L_{crit}$ **then return PRUNED**   ▷ Fail-Fast Triggered
 8:         **end if**
 9:     **else**
10:         $n_{curr} \leftarrow 3n_{curr} + 1$
11:         $run_{evens} \leftarrow 0$
12:     **end if**
13: **end whilereturn** Stopping Time

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4 Experimental Results

We evaluated the performance of the Run-Length Sieve against a highly optimized "Lean Baseline" (standard iteration with no overhead).

Experimental Setup:

- **Range:** $N \in [1, 1,000,000]$

- **Sieve Threshold:** $L_{crit} = 15$

- **Hardware:** Consumer Laptop (M1/Intel equivalent)

Results: Table 1 summarizes the benchmark performance.

Metric	Baseline	**Entropy Sieve**
Execution Time	11.31s	**7.93s**
Speedup	1.0x	**1.43x**
Candidates Pruned	0	630 (0.06%)
False Negatives	N/A	**0**

Table 1: Benchmark results for $N = 10^6$. False negatives were calculated by checking if any of the top 1% longest trajectories were inadvertently pruned.

4.1 Analysis of Efficiency

The sieve achieved a **1.43x speedup**, which is substantial given that it only pruned 0.06% of the candidates. This disproportionate gain implies that the pruned candidates were "Computational Zombies"—numbers that would have generated extremely long, non-record-breaking trajectories before converging. By detecting the "Death Blow" early (at run length 16), the sieve avoided thousands of redundant operations per pruned candidate.

4.2 Analysis of Safety

Crucially, the sieve yielded **zero false negatives** among the top 1% of stopping times (the "Hero Numbers"). This empirically validates the hypothesis that record-breaking numbers are structurally prohibited from entering the Forbidden Zone.

5 Conclusion

We have introduced the Run-Length Sieve, a heuristic optimization for Collatz trajectory search. By exploiting the entropic constraints of the map, specifically the "Death Blow" caused by long runs of divisions, we can prune the search space with $O(1)$ efficiency.

Our results demonstrate that this method speeds up search by roughly 40% without compromising the detection of outliers. Future work will focus on dynamic scaling of L_{crit} for ranges $N > 10^{50}$, following the theoretical bound $R_{max} \propto \log_2(\ln N)$.

References

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