

Constructing Spacetime from Rapidity: Explicit BQ Derivation of the PG Coframe and the metric tensor

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Abstract

This paper presents a compact and fully transparent derivation of the algebraic sequence that links biquaternionic rapidity fields to the Painlevé–Gullstrand (PG) coframe and the corresponding spacetime metric $g_{\mu\nu}$. Building on the gravitational rotor Q_g introduced in earlier work, we show explicitly how a general rapidity field generates the boosted BQ displacement $d\mathcal{R}^G$, and how the PG coframe θ^a can be read off directly as its linear coefficient structure, without any prior metric assumptions. The quadratic metric then follows immediately from the BQ norm of this coframe, placing the PG form as the natural bridge between the linear BQ description of spacetime flow and the standard geometric formulation of general relativity. The presentation resolves the remaining algebraic ambiguities in the rotor-based construction of gravity and provides a clean, reproducible algorithmic path from rapidity to tetrad to metric.

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1 Introduction

The purpose of this paper is to present a fully transparent and technically complete algebraic pathway from biquaternionic rapidity fields to the Painlevé–Gullstrand (PG) coframe and the corresponding spacetime metric $g_{\mu\nu}$. In earlier work [1, 2, 3] the gravitational rotor Q_g was introduced as the central object that closes the biquaternion (BQ) algebra under Lorentz boosts, generates metric structure from rapidity fields, and provides the algebraic backbone for Dirac, Klein–Gordon, and Einstein–Cartan dynamics. More recently, it was shown that the same rotor formalism extends naturally to astrophysical applications, including jet collimation and helical PG geometry in active galactic nuclei [4].

The present work complements and sharpens these developments by giving a streamlined and fully explicit derivation of the sequence

$$\text{rapidity field } \psi(x) \longrightarrow Q_g(x) \longrightarrow d\mathcal{R}^G \longrightarrow \theta^a \longrightarrow g_{\mu\nu},$$

with every algebraic step displayed in detail. A central point of the analysis is that the PG coframe θ^a can be read off directly from the *linear* boosted displacement $d\mathcal{R}^G$, without any prior assumption about the metric or tetrad. The metric $g_{\mu\nu}$ then follows as the quadratic BQ norm of this coframe, so that the PG structure appears as the natural intermediary between the BQ description of gravitational flow and the standard spacetime geometry of general relativity.

In this way, the paper closes the conceptual loop between the BQ representation of Lorentz transformations, the rotor-based encoding of rapidity fields, the PG “river” picture of gravity, and the usual metric formulation of GR, placing the entire construction on a firm algebraic foundation.

2 Basic tools

The purpose of this section is to collect all algebraic ingredients needed for the BQ calculus used throughout the remainder of the paper. All later constructions—gravitational rapidity fields, boosted bases, biquaternionic line elements, tetrads, and finally the derivation of the PG-form of the GR metric—follow directly from the tools established here.

In particular: (i) the Lorentz adjoint action on the Dirac/BQ basis, (ii) the dictionary between the β_μ , γ_μ , α_μ , and Σ_μ sets, (iii) the structure of biquaternionic products, and (iv) the multiplication rule of biquaternionic products. Nothing else is assumed.

2.1 Important Lemma

Using the Lorentz transformation expression of the operator combinations Λ_D and Λ_D^{-1} in terms of the rapidity and the hyperbolic trigonometric expressions,

we can calculate explicitly the adjoint action of Λ_D on the beta matrices. After straightforward Clifford algebra manipulations one finds the central identity:

$$\boxed{\Lambda_D \beta^\mu \Lambda_D^{-1} = \Lambda_\nu{}^\mu \beta^\nu} \quad (1)$$

Thus the β_μ transform as components of a Minkowski vector under Lorentz transformations implemented through the Dirac spinor representation. In particular, for a standard boost with $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = v/c$, the transformed components read

$$(\beta^\mu)^L = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}^L = \Lambda_\nu{}^\mu \beta^\nu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \gamma\beta_0 - \beta\gamma\beta_1 \\ \gamma\beta_1 - \beta\gamma\beta_0 \\ \beta_2 \\ \beta_3 \end{bmatrix}.$$

This lemma is the algebraic cornerstone of the gravitational rotor formalism: it guarantees that a rotor Q_g built from rapidities acts on the β_μ exactly as a Lorentz transformation, thereby defining a boosted basis $\widehat{\beta}_\mu$ and, via the BQ line element, a gravitational coframe.

2.2 Connecting the beta-matrices to the gamma-matrices and to the Dirac alpha and spin matrices

My reversed order of the Pauli spin matrices, with

$$\sigma_I = \sigma_z, \quad \sigma_J = \sigma_y, \quad \sigma_K = \sigma_x, \quad \boldsymbol{\sigma} = (\sigma_I, \sigma_J, \sigma_K),$$

implies that the usual (x, y, z) order of the gamma matrices is reversed correspondingly. Thus

$$\gamma_1 = \gamma_I = \gamma_z, \quad \gamma_2 = \gamma_J = \gamma_y, \quad \gamma_3 = \gamma_K = \gamma_x, \quad \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = (\gamma_I, \gamma_J, \gamma_K).$$

The connection between the usual gamma's and the beta's used here is simply

$$\beta = i\boldsymbol{\gamma}.$$

The Dirac gamma set $\gamma_\mu = (\gamma_0, \boldsymbol{\gamma}) = (\beta, \boldsymbol{\gamma})$ reads

$$\gamma_\mu = \left(\begin{bmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} \right).$$

In the Weyl representation the same set takes the form

$$\gamma_\mu = \left(\begin{bmatrix} 0 & \hat{1} \\ \hat{1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} \right).$$

In my beta-based $(\hat{1}, \boldsymbol{\sigma})$ norm–spin basis, the Dirac set $\alpha_\mu = (\mathbb{1}, \boldsymbol{\alpha})$ is represented as

$$\alpha_\mu = \left(\begin{bmatrix} \hat{1} & 0 \\ 0 & \hat{1} \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} \right).$$

The most straightforward doubling of the Pauli level norm–spin set $(\hat{1}, \boldsymbol{\sigma})$ is the Dirac level norm–spin set $\Sigma_\mu = (\mathbb{1}, \boldsymbol{\Sigma})$ defined as

$$\Sigma_\mu = \left(\begin{bmatrix} \hat{1} & 0 \\ 0 & \hat{1} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} \right).$$

To go back to the tensor $\beta_\mu\beta^\nu = i\gamma_\mu i\gamma^\nu = -\gamma_\mu\gamma^\nu$ we have

$$\beta_\mu\beta^\nu = [\beta_0 \ \beta_1 \ \beta_2 \ \beta_3] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} -\mathbb{1} & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & \mathbb{1} & i\Sigma_{21} & -i\Sigma_{31} \\ -\alpha_2 & -i\Sigma_{12} & \mathbb{1} & i\Sigma_{32} \\ -\alpha_3 & i\Sigma_{13} & -i\Sigma_{23} & \mathbb{1} \end{bmatrix}.$$

Thus, the product $\beta_\mu\beta^\nu$ firmly connects the minquat domain to the pauliquat domain on the Dirac level. The product of two Dirac-level duplex minquats produces a mixture of a duplex minquat and a duplex pauliquat, exactly as happens already on the Pauli level. This observation will be central later when the gravitational rotor acts on the basis and the BQ line element is squared to obtain the PG metric.

2.3 General vector multiplication in the Weyl–Dirac environment

If we multiply two vectors in the Pauli environment we obtain the familiar scalar quantity $C = A^T B$. In the Weyl– and Dirac environments, however, such a multiplication is already implicit in the matrix form of the vectors. When \mathcal{A} and \mathcal{B} are multiplied to form \mathcal{C} , the resulting object decomposes naturally into norm, spin, and mixed components. Because this product appears frequently in later sections—in particular in the construction of the biquaternionic line element and in the derivation of the PG coframe—we work it out here explicitly, starting from the Weyl representation.

We begin by writing the general Weyl multiplication rule

$$\mathcal{C} = \mathcal{A}\mathcal{B} = -(c \mathbb{1} + i \mathbf{d} \cdot \boldsymbol{\Sigma} + \mathbf{e} \cdot \boldsymbol{\alpha}), \quad (2)$$

where the coefficients are the standard Minkowski bilinear combinations

$$c = a_0 b_0 - \mathbf{a} \cdot \mathbf{b}, \quad \mathbf{d} = \mathbf{a} \times \mathbf{b}, \quad \mathbf{e} = a_0 \mathbf{b} - a_0 \mathbf{b}.$$

These expressions are identical in both the Weyl and Dirac representations.

It is useful to make the product explicit in matrix form:

$$\begin{aligned} \mathcal{C} = \mathbb{A}\mathbb{B} &= \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix} \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix} = \begin{bmatrix} -AB^T & 0 \\ 0 & -A^T B \end{bmatrix} \\ &= - \begin{bmatrix} (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) \hat{1} & 0 \\ 0 & (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) \hat{1} \end{bmatrix} - \begin{bmatrix} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{K} & 0 \\ 0 & (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{K} \end{bmatrix} \end{aligned} \quad (3)$$

$$- \begin{bmatrix} -(a_0 \mathbf{b} - \mathbf{a} b_0) \cdot \boldsymbol{\sigma} & 0 \\ 0 & (a_0 \mathbf{b} - \mathbf{a} b_0) \cdot \boldsymbol{\sigma} \end{bmatrix}. \quad (4)$$

Collecting these contributions yields

$$\mathcal{C} = - (c \mathbb{1} + \mathbf{d} \cdot \boldsymbol{\Sigma} + \mathbf{e} \cdot \boldsymbol{\alpha}). \quad (5)$$

The difference between $A^T B$ and AB^T corresponds to exchanging the signs of a_0 and b_0 , which affects only the \mathbf{e} -vector contribution appearing in $\boldsymbol{\alpha}$. This change is absorbed into the definition of $\boldsymbol{\alpha}$, allowing \mathbf{e} to factor out cleanly. On the Dirac level the multiplication requires somewhat more work, but the similarity transformation with S always converts the Weyl results into the Dirac representation. This transformation affects only the β 's (or the norm block), $\boldsymbol{\Sigma}$, and $\boldsymbol{\alpha}$, and never the real coordinate values c , \mathbf{d} , and \mathbf{e} .

An advantage of performing these products in the Weyl environment is the diagonality on the 2×2 spin-matrix level.

Norm as a special case. Taking $\mathbb{B} = \mathbb{A}$ gives the BQ norm:

$$\begin{aligned} \mathbb{A}\mathbb{A} &= \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix} \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix} = \begin{bmatrix} -AA^T & 0 \\ 0 & -A^T A \end{bmatrix} \\ &= \begin{bmatrix} -(a_0^2 - \mathbf{a}^2) \hat{1} & 0 \\ 0 & -(a_0^2 - \mathbf{a}^2) \hat{1} \end{bmatrix} = -(a_0^2 - \mathbf{a}^2) \mathbb{1}. \end{aligned} \quad (6)$$

Applied to the Minkowski line element in flat spacetime, this yields

$$d\mathbb{R} d\mathbb{R} = -(c^2 dt^2 - d\mathbf{r}^2) \mathbb{1} = (-c^2 dt^2 + d\mathbf{r}^2) \mathbb{1} = -c^2 dt_0^2 \mathbb{1}, \quad (7)$$

where dt_0 is the proper time increment.

3 Introducing gravity as a rapidity boost of the basis

3.1 Inserting ds^2 in a gravitational rapidity field ψ_g

We insert a vector \mathbb{A} in a homogenous field of gravity in the rotated direction $\beta_r = R\beta_l R^{-1}$, by using the gravitational rapidity boost Q_g as

$$\mathbb{A}^G \equiv Q_g \mathbb{A} Q_g^{-1} \quad (8)$$

with

$$Q_g(x) \equiv \exp\left[\frac{1}{2}\psi_r(x)\beta_r\beta_0\right] \quad (9)$$

and

$$\frac{v_r}{c} = \frac{v_r(x)}{c} \equiv \tanh\psi_r(x), \quad \gamma = \gamma(x) \equiv \cosh\psi_r = \frac{1}{\sqrt{1 - \frac{v_r^2}{c^2}}}. \quad (10)$$

For the gravitational boost we have

$$\mathbb{A}^G = Q_g \mathbb{A} Q_g^{-1} = Q_g A_\mu \beta^\mu Q_g^{-1} = A_\mu Q_g \beta^\mu Q_g^{-1} = A_\mu \hat{\beta}^\mu \quad (11)$$

We can then immediately use the already proven relationship

$$\Lambda_W \beta^\mu \Lambda_W^{-1} = \Lambda_\nu{}^\mu \beta^\nu. \quad (12)$$

to write it as a Lorentz matrix acting on the base vector ($\mu, \nu = t, r, \theta, \phi$):

$$\hat{\beta}_a = \Lambda_a{}^b(\psi_r) \beta_b, \quad \Lambda(\psi_r) = \begin{pmatrix} \gamma & \gamma \frac{v_r}{c} & 0 & 0 \\ \gamma \frac{v_r}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For a pure boost in the 0– r plane we have the standard relations

$$\hat{\beta}_0 = Q_g \beta_0 Q_g^{-1} = \gamma \beta_0 + \gamma \frac{v_r}{c} \beta_r, \quad (13)$$

$$\hat{\beta}_r = Q_g \beta_r Q_g^{-1} = \gamma \frac{v_r}{c} \beta_0 + \gamma \beta_r, \quad (14)$$

$$\hat{\beta}_\theta = Q_g \beta_\theta Q_g^{-1} = \beta_\theta, \quad \hat{\beta}_\phi = Q_g \beta_\phi Q_g^{-1} = \beta_\phi, \quad (15)$$

But as we have seen, this Gravity transformation can also work on the coordinates, leaving the basis untouched, giving:

$$\mathbb{A}^G = A_\mu \hat{\beta}^\mu = \hat{A}_\mu \beta^\mu = \Lambda_\mu{}^a A_a \beta^\mu \quad (16)$$

3.2 The BQ derivation of the Painlevé–Gullstrand metric

The norm of the metric in Minkowski, flat space-time is

$$d\mathcal{R}d\mathcal{R} = -(c^2 dt^2 - d\mathbf{r}^2)\mathbb{1} = (-c^2 dt^2 + d\mathbf{r}^2)\mathbb{1} = -c^2 dt_0^2 \mathbb{1} \quad (17)$$

which, for the G-boosted instead of the L-boosted frame becomes $d\mathcal{R}^G d\mathcal{R}^G$. For obtaining the same form as the PG-metric, we take

$$\mathbb{A}^{-G} \equiv Q_g^{-1} \mathbb{A} Q_g \quad (18)$$

This gives us the G-boosted Dirac basis

$$\tilde{\beta}_0 = Q_g^{-1} \beta_0 Q_g = \cosh \psi \beta_0 - \sinh \psi \beta_r = \gamma \left(\beta_0 - \frac{v}{c} \beta_r \right), \quad (19)$$

$$\tilde{\beta}_r = Q_g^{-1} \beta_r Q_g = -\sinh \psi \beta_0 + \cosh \psi \beta_r = \gamma \left(-\frac{v}{c} \beta_0 + \beta_r \right), \quad (20)$$

with the usual orthonormal relations $\tilde{\beta}_0^2 = -1$, $\tilde{\beta}_r^2 = +1$, $\tilde{\beta}_0 \tilde{\beta}_r + \tilde{\beta}_r \tilde{\beta}_0 = 0$.

The full gravitational line element in this boosted frame is

$$d\mathcal{R}^G = dR_\mu \hat{\beta}_\mu = c dt \hat{\beta}_0 + dr \hat{\beta}_r + r d\theta \hat{\beta}_\theta + r \sin \theta d\phi \hat{\beta}_\phi,$$

and substituting the original β 's in $d\mathcal{R}^G$ gives:

$$\begin{aligned} d\mathcal{R}^G &= dR_\mu \hat{\beta}^\mu = dR_\mu^G \beta^\mu \\ &= c dt \gamma \left(\beta_0 - \frac{v}{c} \beta_r \right) + dr \gamma \left(-\frac{v}{c} \beta_0 + \beta_r \right) + r d\theta \beta_\theta + r \sin \theta d\phi \beta_\phi \\ &= \gamma c dt \left(1 - \frac{v dr}{c^2 dt} \right) \beta_0 + \gamma (dr - v dt) \beta_r + r d\theta \beta_\theta + r \sin \theta d\phi \beta_\phi \\ &= \gamma_f c dt \left(1 - \frac{v_f v_s}{c^2} \right) \beta_0 + \gamma_f (dr - v_f dt) \beta_r + r d\theta \beta_\theta + r \sin \theta d\phi \beta_\phi, \end{aligned} \quad (21)$$

with v_f as the flow velocity of the external field and $v_s = \frac{dr}{dt}$ as the ‘free’ velocity of space relative to that field, it’s internal reaction to the external condition as it’s self-organisation. If their relative velocities are zero, the local space is perfectly riding the enforced flow.

If we take the non-relativistic limit of both field and space velocities, we set $\gamma_f \approx 1$, and $1 - \frac{v_f v_s}{c^2} \approx 1$, and write $v_f = w$ we get

$$d\mathcal{R}^G = dR_\mu^G \beta^\mu = \theta_\mu \beta^\mu = c dt \beta_0 + (dr - w dt) \beta_r + r d\theta \beta_\theta + r \sin \theta d\phi \beta_\phi, \quad (22)$$

where we have encoded the ‘‘river’’ flow by choosing the coframe $\theta_0 = c dt$, $\theta_r = dr - w dt$ for the radial sector. By identifying $\theta_\mu \equiv dR_\mu^G$ we effectively bridged BQ and PG and thereby made the connection with GR.

The spacetime metric is simply the square:

$$ds^2 \mathbb{1} = d\mathcal{R}^G d\mathcal{R}^G = \left(-(\theta_0)^2 + (\theta_r)^2 + r^2 d\Omega^2 \right) \mathbb{1}. \quad (23)$$

Substituting the explicit coframe,

$$\begin{aligned} ds^2 &= -c^2 dt^2 + (dr - w dt)^2 + r^2 d\Omega^2 \\ &= -(c^2 - w(r)^2) dt^2 - 2w(r) dt dr + dr^2 + r^2 d\Omega^2. \end{aligned} \quad (24)$$

For the Schwarzschild case we take the free-fall river velocity $w(r) = \sqrt{2GM/r}$, which yields the standard Painlevé–Gullstrand form of the metric,

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 - 2\sqrt{\frac{2GM}{r}} dt dr + dr^2 + r^2 d\Omega^2. \quad (25)$$

The derivation reveals that the usual PG metric is, from a flow perspective, a non-relativistic approximation of the fully relativistic flow Eqn.(21), with

$$\begin{aligned} \theta_0 &= \gamma_f c dt \left(1 - \frac{v_f v_s}{c^2}\right) \\ \theta_r &= \gamma_f (dr - v_f dt) = \gamma_f \left(1 - \frac{v_f}{v_s}\right) dr. \end{aligned} \quad (26)$$

This needs further examination, but we pause that for a later time.

4 From the BQ Line Element to the Tetrad and the Painlevé–Gullstrand Metric

In this section we give a complete and transparent derivation of the tetrad $e^a{}_\mu$ and the corresponding metric $g_{\mu\nu}$ starting from the biquaternionic (BQ) gravitationally boosted displacement $d\mathcal{R}^G$ introduced in Section 4. All steps are made explicit so that the connection between the BQ representation, the tetrad formalism of general relativity, and the Painlevé–Gullstrand (PG) metric remains transparent.

Throughout we work in spherical coordinates $x^\mu = (t, r, \theta, \phi)$ and use internal Lorentz indices $a = (0, r, \theta, \phi)$ for the local Minkowski frame with $\eta_{ab} = \text{diag}(-1, +1, +1, +1)$. The BQ basis β^a realises the local Minkowski basis in the Dirac/BQ algebra.

4.1 Identifying the boosted 1-forms θ^a from the BQ line element

From the gravitationally boosted line element in the BQ/Dirac algebra,

$$d\mathcal{R}^G = dR_a^G \beta^a = c dt \beta_0 + (dr - w(r) dt) \beta_r + r d\theta \beta_\theta + r \sin \theta d\phi \beta_\phi, \quad (27)$$

we read off the gravitationally boosted 1-forms

$$\theta^0 \equiv dR_0^G = c dt, \quad \theta^r \equiv dR_r^G = dr - w(r) dt, \quad (28)$$

$$\theta^\theta \equiv dR_\theta^G = r d\theta, \quad \theta^\phi \equiv dR_\phi^G = r \sin \theta d\phi. \quad (29)$$

These 1-forms constitute the *coframe* associated with the boosted PQ-type geometry. In the local Lorentz frame the BQ scalar product is exactly the Minkowski norm, hence

$$ds^2 = \eta_{ab} \theta^a \theta^b = -(\theta^0)^2 + (\theta^r)^2 + (\theta^\theta)^2 + (\theta^\phi)^2. \quad (30)$$

Substituting the explicit expressions gives

$$ds^2 = -c^2 dt^2 + (dr - w(r)dt)^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (31)$$

$$= -(c^2 - w^2(r)) dt^2 - 2w(r) dt dr + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (32)$$

This is the standard PG form when $w(r) = \sqrt{2GM/r}$.

4.2 Reading off the tetrad $e^a{}_\mu$ from θ^a

By definition the coframe θ^a is written as

$$\theta^a = e^a{}_\mu dx^\mu, \quad x^\mu = (t, r, \theta, \phi). \quad (33)$$

Comparing with the expressions above we obtain the tetrad components:

$$\theta^0 = c dt \Rightarrow e^0{}_t = c, \quad e^0{}_r = e^0{}_\theta = e^0{}_\phi = 0, \quad (34)$$

$$\theta^r = dr - w(r) dt \Rightarrow e^r{}_t = -w(r), \quad e^r{}_r = 1, \quad e^r{}_\theta = e^r{}_\phi = 0, \quad (35)$$

$$\theta^\theta = r d\theta \Rightarrow e^\theta{}_\theta = r, \quad e^\theta{}_t = e^\theta{}_r = e^\theta{}_\phi = 0, \quad (36)$$

$$\theta^\phi = r \sin \theta d\phi \Rightarrow e^\phi{}_\phi = r \sin \theta, \quad e^\phi{}_t = e^\phi{}_r = e^\phi{}_\theta = 0. \quad (37)$$

Collecting all components, the tetrad matrix is

$$e^a{}_\mu = \begin{pmatrix} e^0{}_t & e^0{}_r & e^0{}_\theta & e^0{}_\phi \\ e^r{}_t & e^r{}_r & e^r{}_\theta & e^r{}_\phi \\ e^\theta{}_t & e^\theta{}_r & e^\theta{}_\theta & e^\theta{}_\phi \\ e^\phi{}_t & e^\phi{}_r & e^\phi{}_\theta & e^\phi{}_\phi \end{pmatrix} = \begin{pmatrix} c & 0 & 0 & 0 \\ -w(r) & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \sin \theta \end{pmatrix}. \quad (38)$$

4.3 Constructing $g_{\mu\nu}$ from the tetrad

The coordinate-basis metric follows from the standard tetrad relation

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu, \quad \eta_{ab} = \text{diag}(-1, +1, +1, +1). \quad (39)$$

We compute the non-vanishing components explicitly.

Time-time component.

$$g_{tt} = \eta_{00}(e^0{}_t)^2 + \eta_{rr}(e^r{}_t)^2 \quad (40)$$

$$= (-1)c^2 + (+1)w^2(r) = -c^2 + w^2(r). \quad (41)$$

Mixed components.

Since only $e^r{}_t$ and $e^r{}_r$ are non-zero among mixed terms,

$$g_{tr} = g_{rt} = \eta_{rr} e^r{}_t e^r{}_r = 1 \cdot (-w(r)) \cdot 1 = -w(r). \quad (42)$$

Radial component.

$$g_{rr} = \eta_{rr} (e^r_r)^2 = 1. \quad (43)$$

Angular components.

$$g_{\theta\theta} = \eta_{\theta\theta} (e^\theta_\theta)^2 = r^2, \quad (44)$$

$$g_{\phi\phi} = \eta_{\phi\phi} (e^\phi_\phi)^2 = (r \sin \theta)^2 = r^2 \sin^2 \theta. \quad (45)$$

All off-diagonal angular terms vanish. Thus the metric in the coordinate basis is

$$g_{\mu\nu} = \begin{pmatrix} -c^2 + w^2(r) & -w(r) & 0 & 0 \\ -w(r) & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}, \quad (46)$$

and hence

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (47)$$

$$= -(c^2 - w^2(r)) dt^2 - 2w(r) dt dr + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (48)$$

which agrees exactly with the expression obtained from the BQ norm (32). For the special choice $w(r) = \sqrt{2GM/r}$ this reduces to the Painlevé–Gullstrand form of the Schwarzschild solution.

5 PG metric from a single BQ rapidity field with radial and azimuthal flow

In Sec. 3 we showed how a single radial rapidity field $\psi_r(r)$, encoded in the gravitational rotor Q_g , produces the Painlevé–Gullstrand (PG) coframe and metric once the nonrelativistic field limit is taken at the level of the boosted BQ displacement, cf. the transition from Eq. (21) to Eq. (22). In many applications, however, the gravitational flow is not purely radial but has both a radial and an azimuthal component, as in rotating or shearing configurations. In this section we treat such a situation as a *single physical flow* described by one 3–velocity field and one rapidity, and we show that the same PG construction goes through with hardly any modification.

5.1 Single-flow rotor in the (r, ϕ) -plane

We assume that the gravitational flow at a given point is described by a single 3–velocity

$$\mathbf{v}_f(r, \theta, \phi) = (w(r), 0, v_\phi(r)) \quad (49)$$

in the (r, ϕ) -plane, written in the local orthonormal basis aligned with $(\beta_r, \beta_\theta, \beta_\phi)$. Its magnitude and direction are

$$v_f(r) = \sqrt{w(r)^2 + v_\phi(r)^2}, \quad \hat{\mathbf{n}}(r, \theta, \phi) = \frac{1}{v_f(r)} (w(r) \hat{\mathbf{e}}_r + v_\phi(r) \hat{\mathbf{e}}_\phi), \quad (50)$$

where $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\phi$ denote the local spatial unit vectors associated with β_r and β_ϕ . We parameterize the flow by a single rapidity $\psi_f(r)$ defined by

$$v_f(r) = c \tanh \psi_f(r), \quad \gamma_f(r) = \cosh \psi_f(r) = \frac{1}{\sqrt{1 - \frac{v_f(r)^2}{c^2}}}. \quad (51)$$

The corresponding boost bivector is

$$K_f(r, \theta, \phi) = (\hat{\mathbf{n}} \cdot \boldsymbol{\beta}) \beta_0 = n_r(r) \beta_r \beta_0 + n_\phi(r) \beta_\phi \beta_0, \quad n_r = \frac{w}{v_f}, \quad n_\phi = \frac{v_\phi}{v_f}, \quad (52)$$

and the gravitational rotor is taken to be the single-flow rotor

$$Q_g(x) = \exp \left[\frac{1}{2} \psi_f(r) K_f(r, \theta, \phi) \right]. \quad (53)$$

By construction this is a *pure boost* in the direction $\hat{\mathbf{n}}$ with rapidity ψ_f , and it carries all information about the gravitational flow through (w, v_ϕ) .

5.2 Relativistic boosted displacement

We start again from the flat-space BQ line element

$$d\mathcal{R} = c dt \beta_0 + dr \beta_r + r d\theta \beta_\theta + r \sin \theta d\phi \beta_\phi. \quad (54)$$

The gravitationally boosted displacement is defined as

$$d\mathcal{R}^G = Q_g d\mathcal{R} Q_g^{-1}, \quad (55)$$

with Q_g given by (53). Let the “free” spatial velocity of space relative to the chosen coordinates be written as

$$\mathbf{v}_s = \left(\frac{dr}{dt}, r \frac{d\theta}{dt}, r \sin \theta \frac{d\phi}{dt} \right), \quad (56)$$

and denote its radial and azimuthal components by

$$v_r = \frac{dr}{dt}, \quad v_\phi^{(s)} = r \sin \theta \frac{d\phi}{dt}. \quad (57)$$

The scalar product that enters the time component of the boost is then

$$\mathbf{v}_f \cdot \mathbf{v}_s = w(r) v_r + v_\phi(r) v_\phi^{(s)}. \quad (58)$$

By applying the lemma of Sec. 2.1 to the single-flow rotor (53), and using the standard Lorentz boost relations for a velocity \mathbf{v}_f relative to \mathbf{v}_s , one finds that the boosted line element takes the compact relativistic form

$$\begin{aligned} d\mathcal{R}^G &= dR^{G,\mu} \beta_\mu \\ &= \gamma_f c dt \left(1 - \frac{\mathbf{v}_f \cdot \mathbf{v}_s}{c^2} \right) \beta_0 + \gamma_f (dr - w(r) dt) \beta_r + r d\theta \beta_\theta \\ &\quad + \gamma_f (r \sin \theta d\phi - v_\phi(r) dt) \beta_\phi. \end{aligned} \quad (59)$$

This is the direct analogue of Eq. (21), now for a single 3-velocity field with both radial and azimuthal components. The entire gravitational flow is carried by the pair (\mathbf{v}_f, γ_f) , and the dependence on w and v_ϕ appears only through their projections on the (r, ϕ) axes.

5.3 Nonrelativistic field limit and PG-type coframe

To recover the PG “river” form, we now take the same nonrelativistic field limit as in the step from Eq. (21) to Eq. (22). We assume that the gravitational flow is slow compared to c ,

$$\frac{v_f}{c} = \frac{\sqrt{w^2 + v_\phi^2}}{c} \ll 1, \quad (60)$$

so that

$$\gamma_f \simeq 1, \quad 1 - \frac{\mathbf{v}_f \cdot \mathbf{v}_s}{c^2} \simeq 1. \quad (61)$$

Under this approximation the boosted line element (59) reduces to

$$d\mathcal{R}^G = c dt \beta_0 + (dr - w(r) dt) \beta_r + r d\theta \beta_\theta + (r \sin \theta d\phi - v_\phi(r) dt) \beta_\phi. \quad (62)$$

This is the exact analogue of Eq. (22), now for a combined radial and azimuthal flow.

Comparing (62) with the general expansion $d\mathcal{R}^G = \theta_\mu \beta^\mu$ we immediately identify the coframe components

$$\theta_0 = c dt, \quad \theta_r = dr - w(r) dt, \quad \theta_\theta = r d\theta, \quad \theta_\phi = r \sin \theta d\phi - v_\phi(r) dt. \quad (63)$$

Thus, exactly as in the purely radial case, the BQ algebra provides the PG-type river coframe directly from the linear boosted displacement, now including an azimuthal flow component, once the nonrelativistic field limit is taken at the level of $d\mathcal{R}^G$.

5.4 Reading off the tetrad $e^a{}_\mu$ from the coframe θ^a

The PG-type coframe obtained in Eq. (63) is

$$\theta^0 = c dt, \quad \theta^r = dr - w(r) dt, \quad \theta^\theta = r d\theta, \quad \theta^\phi = r \sin \theta d\phi - v_\phi(r) dt. \quad (64)$$

By definition,

$$\theta^a = e^a{}_\mu dx^\mu, \quad dx^\mu = (dt, dr, d\theta, d\phi), \quad (65)$$

so the tetrad components $e^a{}_\mu$ are obtained simply by identifying the coefficients of the coordinate one-forms in (64). Reading them off component-by-component gives:

- For $\theta^0 = c dt$:

$$e^0{}_t = c, \quad e^0{}_r = 0, \quad e^0{}_\theta = 0, \quad e^0{}_\phi = 0.$$

- For $\theta^r = dr - w(r) dt$:

$$e^r{}_t = -w(r), \quad e^r{}_r = 1, \quad e^r{}_\theta = 0, \quad e^r{}_\phi = 0.$$

- For $\theta^\theta = r d\theta$:

$$e^\theta{}_t = 0, \quad e^\theta{}_r = 0, \quad e^\theta{}_\theta = r, \quad e^\theta{}_\phi = 0.$$

- For $\theta^\phi = r \sin \theta d\phi - v_\phi(r) dt$:

$$e^\phi{}_t = -v_\phi(r), \quad e^\phi{}_r = 0, \quad e^\phi{}_\theta = 0, \quad e^\phi{}_\phi = r \sin \theta.$$

Collecting all components, the tetrad matrix in the coordinate basis (t, r, θ, ϕ) is

$$e^a{}_\mu = \begin{pmatrix} c & 0 & 0 & 0 \\ -w(r) & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ -v_\phi(r) & 0 & 0 & r \sin \theta \end{pmatrix}, \quad (66)$$

where the rows correspond to the orthonormal frame indices $a = (0, r, \theta, \phi)$ and the columns to the coordinate indices $\mu = (t, r, \theta, \phi)$.

This tetrad encodes all gravitational information contained in the linear BQ displacement dR^G , and squaring the coframe via $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$ reproduces the PG-type metric derived in the next subsection.

5.5 Constructing $g_{\mu\nu}$ from the tetrad

Given the tetrad $e^a{}_\mu$ in Eq. (66), the spacetime metric is obtained in the standard way from

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu, \quad \eta_{ab} = \text{diag}(-1, +1, +1, +1), \quad (67)$$

where the orthonormal frame indices are $a = (0, r, \theta, \phi)$ and the coordinate indices are $\mu = (t, r, \theta, \phi)$. Using the explicit matrix (66),

$$e^a{}_\mu = \begin{pmatrix} c & 0 & 0 & 0 \\ -w(r) & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ -v_\phi(r) & 0 & 0 & r \sin \theta \end{pmatrix},$$

we now compute the metric components column by column.

Time–time component g_{tt} .

$$g_{tt} = \eta_{00}(e^0{}_t)^2 + \eta_{rr}(e^r{}_t)^2 + \eta_{\phi\phi}(e^\phi{}_t)^2 = -c^2 + w(r)^2 + v_\phi(r)^2.$$

Time–radial component $g_{tr} = g_{rt}$.

$$g_{tr} = \eta_{rr} e^r{}_t e^r{}_r = -w(r).$$

Time–azimuthal component $g_{t\phi} = g_{\phi t}$.

$$g_{t\phi} = \eta_{\phi\phi} e^\phi{}_t e^\phi{}_\phi = -r \sin \theta v_\phi(r).$$

Purely spatial components.

$$\begin{aligned} g_{rr} &= \eta_{rr}(e^r{}_r)^2 = 1, & g_{\theta\theta} &= \eta_{\theta\theta}(e^\theta{}_\theta)^2 = r^2, \\ g_{\phi\phi} &= \eta_{\phi\phi}(e^\phi{}_\phi)^2 = r^2 \sin^2 \theta, & g_{r\phi} &= g_{\theta r} = g_{\theta\phi} = 0. \end{aligned}$$

Collecting all components, the metric in the coordinate basis (t, r, θ, ϕ) takes the Painlevé–Gullstrand–type form

$$g_{\mu\nu} = \begin{pmatrix} -(c^2 - w^2 - v_\phi^2) & -w & 0 & -r \sin \theta v_\phi \\ -w & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ -r \sin \theta v_\phi & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}. \quad (68)$$

This is precisely the PG river metric for a single physical flow with components $(w(r), 0, v_\phi(r))$. Setting $v_\phi(r) = 0$ recovers the standard radial PG form, and inserting $w(r) = \sqrt{2GM/r}$ gives the Schwarzschild metric in PG coordinates.

6 Conclusion

In this work we have provided a compact and fully transparent derivation of the complete algebraic chain that links biquaternionic rapidity fields to the Painlevé–Gullstrand (PG) coframe and the corresponding spacetime metric $g_{\mu\nu}$. By making each step of the sequence

$$\psi(x) \longrightarrow Q_g(x) \longrightarrow d\mathbb{R}^G \longrightarrow \theta^a \longrightarrow g_{\mu\nu}$$

explicit and traceable, we have shown that gravitational geometry emerges directly and naturally from the BQ calculus itself.

A central result of this paper is that the PG coframe θ^a can be read off *directly from the linear boosted displacement* $d\mathbb{K}^G$ produced by the gravitational rotor Q_g . No intermediate metric assumptions, no ansatz for the tetrad, and no additional geometric input are required: the coframe arises as the coefficient array of the BQ–gravitational line element. This identifies the PG structure not as a coordinate choice, but as the natural interface between the linear BQ description of spacetime flow and the quadratic metric form obtained from the BQ norm.

This closes the remaining algebraic gap in the rotor-based construction of gravity and places the entire framework on a firm and reproducible foundation. The derivation clarifies precisely how the gravitational rotor encodes the rapidity field, how it acts on the Minkowski basis, and how the standard GR tetrad and metric emerge as algebraic consequences of this action.

The resulting framework now allows rapidity-based gravitational fields to be translated *directly and uniquely* into geometric structure. It provides a clean and constructive interface between the algebraic formalism developed in [1, 2, 3] and recent applications to astrophysical systems such as relativistic jets [4]. With the linear–to–quadratic map now fully understood, the method is ready for extension to multi-rapidity configurations, to the Dirac and Klein–Gordon equations on curved backgrounds, and to the systematic exploration of PG-type metrics generated by physically motivated flow fields.

In short, the gravitational geometry carried by the PG tetrad is now seen to be the direct quadratic image of a single BQ rotor acting on the linear displacement field. This completes the algebraic fusion of rapidity, rotor, and geometry, and provides a transparent route from BQ kinematics to GR spacetime structure.

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