

# Gaussian Ensembles of Hopf Spinors and Internal Directions of Light

Axel G. Schubert

November 2025

## Abstract

In four-dimensional Lorentzian spacetime, the kinematics of photon rays can be described in a spinor language in which each future-directed null covector  $k_\mu$  and timelike observer  $u^\mu$  determines a normalized spinor  $z_A \in S^3$  and an associated internal orthonormal triad of directions  $\{\vec{n}^{(1)}(z), \vec{n}^{(2)}(z), \vec{n}^{(3)}(z)\}$  via the Hopf fibration. [1–3] This Hopf structure packages propagation direction, phase and polarization into a single spinor bundle over null congruences, entirely within the standard Einstein–Maxwell framework. In many physical situations, however, one is interested not in individual null rays but in ensembles of photons or null geodesics, for which a statistical description of internal directions is more appropriate than a strictly deterministic one.

In this paper we develop a minimal Gaussian framework for such ensembles. We introduce random normalized Hopf spinors  $Z_A$  on the spinor sphere and the induced random internal triads  $\{\vec{N}^{(1)}, \vec{N}^{(2)}, \vec{N}^{(3)}\}$  in  $\mathbb{R}^3$ . Treating the internal directions as  $\mathbb{R}^3$ -valued random variables constrained to lie on the unit sphere, we model their statistics by multivariate normal distributions in regimes where the ensembles are narrowly concentrated around a preferred frame. Gaussian models are singled out by their well-known maximum-entropy property under fixed mean and covariance and thus provide a natural, least-biased reference class once only low-order moments are specified. [4, 5]

The second-order statistics of the internal triad are organised into a correlation tensor  $C^{ab}{}_{ij}$ , where  $a, b$  label the legs of the triad and  $i, j$  are spatial indices. We decompose the spatial part of this tensor into a scalar and a symmetric traceless tensor channel under  $\text{SO}(3)$ , providing a systematic classification of isotropic ensembles and weakly anisotropic deformations. In parallel, we consider ensemble averages of spinor bilinears  $M_{AA} = \mathbb{E}[Z_A \bar{Z}_{\dot{A}}]$ , which transform naturally under the spinor representation and may be regarded as  $2 \times 2$  density matrices of unit trace. The mixed bilinear  $M_{A\dot{A}}$  is related to the mean internal direction and can be mapped to an effective four-vector by the Infeld–van der Waerden symbols.

The constructions presented here are purely kinematical and statistical: they do not introduce any new field equations beyond those of the underlying spacetime geometry and null kinematics. Instead, they establish a Gaussian ensemble description of Hopf spinors and internal directions, together with a channel decomposition of their second-order statistics. This provides a technical backbone for subsequent work in which specific correlation patterns in such ensembles will be related to effective scalar, vector and tensor quantities on spacetime.

## 1 Introduction

The description of massless fields and null directions in four-dimensional Lorentzian spacetime admits a particularly natural formulation in terms of two-component spinors and twistors. [1, 2] Any real null vector can be written as a rank-one Hermitian spinor,  $k_{AA} = \lambda_A \tilde{\lambda}_{\dot{A}}$ , so that a photon of definite helicity appears kinematically as a correlated pair of Weyl spinors. This observation underlies both the Newman–Penrose formalism and twistor descriptions of massless

fields, in which null directions and their incidence relations are taken as primary geometric objects.

At the same time, normalized two–component spinors carry a well–known internal geometry: they live on a three–sphere and map, via the Pauli matrices, to unit vectors on a two–sphere. The resulting map is the Hopf fibration. In quantum mechanics this structure is familiar from the Bloch–sphere description of spin–1/2 systems; in polarization optics it appears in the representation of pure polarization states on the Poincaré sphere. [6] In both cases a normalized spinor encodes a point on  $S^2$  together with an internal  $U(1)$  phase living on the  $S^1$  fibres of  $S^3$ .

In recent work, this Hopf geometry has been made explicit in the context of null kinematics and photon propagation. [3] A future–directed null covector  $k_\mu$  together with a timelike observer field  $u^\mu$  determines an observer–dependent spatial propagation direction  $\vec{n} \in S^2$ , a normalized spinor  $z_A$  whose Hopf image reproduces  $\vec{n}$ , and an associated internal orthonormal triad  $\{\vec{n}^{(1)}(z), \vec{n}^{(2)}(z), \vec{n}^{(3)}(z) = \vec{n}(z)\}$  attached to each photon ray. In this way direction, phase and polarization of light are encoded in a single spinor bundle over null congruences, entirely within the standard Einstein–Maxwell framework.

The present paper takes this kinematical Hopf structure as a starting point and asks a different kind of question: how should one describe, in a minimal and coordinate–invariant way, ensembles of such Hopf spinors and internal directions? In many physical situations one is not interested in a single null ray with fixed internal data, but in large collections of photons or null geodesics whose directions and polarizations fluctuate. Examples include partially polarised light in classical optics, random wave and speckle patterns in complex media, and stochastic treatments of radiation fields in astrophysical or cosmological settings. [6, 7] In these contexts it is natural to replace a deterministic specification of  $z_A$  by a statistical description of a random normalized spinor  $Z_A$  and the associated random internal triad.

Once this step is taken, the question arises which classes of probability distributions on the spinor sphere and on the induced internal directions in  $\mathbb{R}^3$  provide a useful and mathematically controlled starting point. In this work we focus on Gaussian ensembles, motivated by three considerations. Throughout, the term “Gaussian ensemble” is used in the elementary sense of a multivariate normal distribution on the internal degrees of freedom (or on an appropriate linear subspace), rather than in the sense of random matrix theory.

First, multivariate normal distributions are characterised by the property that they maximise the differential entropy among all continuous distributions with the same mean and covariance. [4, 5] They provide the least biased, or maximal–uncertainty, models consistent with specified first and second moments. Second, Gaussian laws arise naturally as effective descriptions of sums of many weakly correlated contributions by central–limit mechanisms, making them robust and widely applicable. Third, in regimes where an ensemble of internal directions is tightly concentrated around a preferred frame, the curvature of the unit sphere can be neglected locally and the distribution can be well approximated by a Gaussian density on  $\mathbb{R}^3$ , fully described by its mean vector and covariance matrix.

The aim of this paper is not to argue that internal directions of light must be Gaussian distributed in any fundamental sense, but rather to establish a Gaussian framework as a technically simple and conceptually transparent reference model. Within this framework, the statistical properties of the random internal triad  $\{\vec{N}^{(1)}, \vec{N}^{(2)}, \vec{N}^{(3)}\}$  induced by a random Hopf spinor  $Z_A$  can be encoded in a small number of objects: ensemble means, covariance matrices and correlation tensors. These can then be organised into scalar, vector and tensor channels under spatial rotations, providing a systematic way to classify isotropic ensembles, weakly anisotropic deformations and possible correlation patterns.

The structure of the paper is as follows. In Sec. 2 we briefly recall the Hopf description of null kinematics and explain how individual photon rays give rise to internal orthonormal triads. In Sec. 3 we introduce random normalized Hopf spinors  $Z_A$  and define Gaussian models for the statistics of the associated internal directions in  $\mathbb{R}^3$ , focusing on the regime where

ensembles are narrowly concentrated and second moments dominate. In Sec. 4 we collect the second-order statistics of the internal triad into a correlation tensor  $C^{ab}{}_{ij}$  and decompose its spatial indices into scalar, vector and a symmetric traceless tensor channels under  $\text{SO}(3)$ . In Sec. 5 we complement this description by considering ensemble averages of spinor bilinears  $M_{A\dot{A}} = \mathbb{E}[Z_A \bar{Z}_{\dot{A}}]$  and briefly discuss their interpretation as  $2 \times 2$  density matrices and effective four-vectors. Sec. 6 summarises the main points and outlines how the Gaussian framework developed here can serve as a technical backbone for subsequent work on correlation structures of Hopf spinor ensembles and their possible effective descriptions on spacetime.

**Relation to previous work.** The present construction combines three standard ingredients in a way that, to the author's knowledge, has not been made explicit in the existing literature. First, the two-spinor and twistor description of null directions and photon kinematics is by now classical. [1, 2, 8, 9] Second, the Hopf fibration  $S^3 \rightarrow S^2$  and its realization on normalized spinors underlie both the Bloch-sphere picture in quantum mechanics and the Poincaré-sphere description of polarization optics. [6, 10] Third, probabilistic models on spaces of directions and rotations are well developed in directional statistics, where distributions on  $S^2$  and  $\text{SO}(3)$  such as the Fisher or Kent distributions play a central role. [11]

What appears to be new in the present work is the specific combination of these ingredients: null congruences are equipped with Hopf spinors and their internal triads as additional kinematical data, and Gaussian ensembles are introduced directly on the internal degrees of freedom associated with these spinors. Correlation functions of these ensembles are then organized into a scalar and a symmetric traceless tensor channel under spatial rotations, providing a purely kinematical framework in which effective field-like quantities can be discussed without modifying the underlying Einstein-Maxwell dynamics. In this sense, the paper does not propose an alternative to the standard field equations, but a reorganization of the same kinematics in terms of Hopf spinors and their internal directions.

## 2 From Hopf kinematics to statistical ensembles

In order to introduce Gaussian ensembles of internal directions it is useful to summarise, in a compact form, the Hopf description of photon kinematics that we take as background. We then explain how this kinematical structure is lifted from individual null rays to random ensembles.

### 2.1 Hopf description of internal directions

In the two-spinor formalism any real vector  $v^\mu$  in a four-dimensional Lorentzian spacetime  $(M, g_{\mu\nu})$  can be represented as a Hermitian spinor  $v_{A\dot{A}} = v^\mu \sigma_{\mu A\dot{A}}$ , where  $\sigma_{\mu A\dot{A}}$  are the Infeld-van der Waerden symbols. [1] The vector is null if and only if  $v_{A\dot{A}}$  has vanishing determinant, in which case it factorises as a rank-one outer product

$$k_{A\dot{A}} = \lambda_A \tilde{\lambda}_{\dot{A}}, \quad (1)$$

for some nonzero spinors  $\lambda_A$  and  $\tilde{\lambda}_{\dot{A}}$ , unique up to a complex rescaling. For a massless field this  $k^\mu$  is the wave covector or momentum of a photon, and the dyad  $(\lambda_A, \tilde{\lambda}_{\dot{A}})$  encodes its helicity structure. [2]

Given a timelike observer field  $u_\mu$  with  $u_\mu u^\mu = -1$  and a future-directed null covector  $k_\mu$ , we define the measured frequency

$$\omega = -u_\mu k^\mu > 0, \quad (2)$$

and the corresponding spatial propagation direction in the rest frame of  $u^\mu$  by

$$n^\mu = \frac{1}{\omega} k^\mu - u^\mu. \quad (3)$$

A short calculation shows that  $u_\mu n^\mu = 0$  und  $n_\mu n^\mu = +1$ , so that  $n^\mu$  is a unit spacelike vector orthogonal to  $u^\mu$ , and

$$k^\mu = \omega (u^\mu + n^\mu). \quad (4)$$

A short computation shows that  $n^\mu$  is unit spacelike and orthogonal to  $u^\mu$ , so that  $n^\mu$  represents a point  $\vec{n}$  on the unit sphere of spatial directions in the observer frame.

At the level of spinors, an orthonormal spatial triad in the rest frame of  $u^\mu$  is represented by the Pauli matrices  $\sigma_i$ ,  $i = 1, 2, 3$ . A normalized two-component spinor  $z_A$  with  $z^\dagger z = 1$  defines a unit vector

$$n_i(z) = z^\dagger \sigma_i z, \quad i = 1, 2, 3, \quad (5)$$

which lies on the two-sphere  $S^2 \subset \mathbb{R}^3$ . The map  $z \mapsto \vec{n}(z)$  is the Hopf fibration  $S^3 \rightarrow S^2$ , and the preimages of points on  $S^2$  are circles  $S^1$  corresponding to the  $U(1)$  phase freedom of  $z_A$ . [1, 2]

In the construction developed in Ref. [3], each photon ray, specified by a pair  $(u^\mu, k^\mu)$  at a spacetime point, is assigned a normalized spinor  $z_A$  whose Hopf image reproduces the spatial direction  $\vec{n}$  determined by  $(u^\mu, k^\mu)$ . Moreover,  $z_A$  determines an  $SU(2)$  matrix  $U(z)$  which maps a fixed reference spinor to  $z_A$ , and conjugation with  $U(z)$  rotates the Pauli matrices into a new set

$$\Sigma_i(z) = U(z) \sigma_i U(z)^\dagger, \quad i = 1, 2, 3. \quad (6)$$

The three vectors obtained from  $\Sigma_i(z)$  via the standard  $SO(3)$ – $SU(2)$  correspondence form an orthonormal triad  $\{\vec{n}^{(1)}(z), \vec{n}^{(2)}(z), \vec{n}^{(3)}(z)\}$  in  $\mathbb{R}^3$ , with  $\vec{n}^{(3)}(z) = \vec{n}(z)$ . We refer to this triad as the internal directions of light associated with the null ray  $(u^\mu, k^\mu)$ .

## 2.2 From single rays to ensembles

The Hopf description reviewed above is purely kinematical: it assigns to each individual null ray a normalized spinor  $z_A$  and an internal triad of directions, without making any probabilistic assumptions. In many applications, however, one is interested in ensembles of photon rays or null geodesics rather than in single trajectories. Examples include partially polarised beams, random wave fields and stochastic descriptions of radiation in complex media. In such situations it is natural to replace the deterministic spinor  $z_A$  by a random normalized spinor and to consider the induced randomness in the internal directions.

To formalise this, let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and define a random normalized spinor

$$Z_A : \Omega \rightarrow S^3, \quad Z^\dagger Z = 1 \text{ almost surely.} \quad (7)$$

Each outcome  $\omega \in \Omega$  corresponds to a particular spinor  $Z_A(\omega)$  and hence to a particular internal triad

$$\vec{N}^{(a)}(\omega) = \vec{n}^{(a)}(Z(\omega)), \quad a = 1, 2, 3, \quad (8)$$

where  $\vec{n}^{(a)}(z)$  denotes the internal directions constructed from  $z_A$  as above. The three random vectors  $\vec{N}^{(1)}, \vec{N}^{(2)}, \vec{N}^{(3)}$  are  $\mathbb{R}^3$ -valued random variables satisfying, for each  $\omega$ ,

$$\vec{N}^{(a)}(\omega) \cdot \vec{N}^{(b)}(\omega) = \delta^{ab}, \quad a, b = 1, 2, 3. \quad (9)$$

In other words, an ensemble of Hopf spinors  $\{Z_A(\omega)\}$  induces an ensemble of internal orthonormal triads in  $\mathbb{R}^3$ .

In this paper we do not attempt to model the full spacetime dependence of such ensembles. Instead, we work at a fixed spacetime point and suppress the dependence on  $(u^\mu, k^\mu)$ , regarding  $Z_A$  and  $\vec{N}^{(a)}$  as purely internal random variables. This is sufficient for our purposes, which are to develop a Gaussian description of the statistics of the internal directions and to organise their second-order correlations into scalar, vector and tensor channels.

### 2.3 Local concentration and Gaussian approximation

The space of unit vectors in  $\mathbb{R}^3$  is the two-sphere  $S^2$ , while a full orthonormal triad lives in the rotation group  $\text{SO}(3)$ . In principle, probability distributions for the internal directions should be defined directly on these curved manifolds. In many situations of interest, however, the ensemble is tightly concentrated around a preferred internal frame or a narrow cone of directions. In such regimes one may choose coordinates in which the fluctuations of the internal directions are small, and approximate the relevant distributions by Gaussian densities on the tangent spaces.

Concretely, let  $\bar{n}^{(a)}$ ,  $a = 1, 2, 3$ , be a reference orthonormal triad and write the random internal directions as

$$\vec{N}^{(a)} = \bar{n}^{(a)} + \delta\vec{N}^{(a)}, \quad (10)$$

with  $\delta\vec{N}^{(a)}$  small in the sense that their typical magnitude is much less than unity. To leading order in  $\delta\vec{N}^{(a)}$  the orthonormality constraints can be linearised, and the joint distribution of the  $\delta\vec{N}^{(a)}$  can be modelled by a multivariate normal law on  $\mathbb{R}^3$  (or on the appropriate subspace defined by the linearised constraints). In this approximation the ensemble is completely characterised by the means and covariances of the components of  $\vec{N}^{(a)}$ .

The next sections make this statement precise. In Sec. 3 we set up Gaussian models for the internal directions and introduce the corresponding mean vectors and covariance matrices. In Sec. 4 we organise the second-order statistics of the internal triad into a correlation tensor and decompose it into scalar, vector and a symmetric traceless tensor channels under spatial rotations. Throughout, we remain within the kinematical Hopf framework summarised above and use Gaussian ensembles only as a convenient and well-understood reference class once only low-order moments are specified.

## 3 Gaussian ensembles of Hopf spinors and internal directions

Having set up the passage from individual Hopf spinors to random internal triads, we now introduce Gaussian models for the statistics of the internal directions. Throughout this section we work at a fixed spacetime point and keep the underlying null kinematics implicit; all random variables live in the internal space of directions attached to a given observer.

### 3.1 Random internal directions and basic moments

Let  $\vec{N}^{(a)}$ ,  $a = 1, 2, 3$ , denote the three random internal directions in  $\mathbb{R}^3$  induced by a random normalized Hopf spinor  $Z_A$  as described in Sec. 2. For each outcome  $\omega \in \Omega$  the three vectors  $\vec{N}^{(a)}(\omega)$  form an orthonormal triad. We write components with respect to a fixed spatial basis as

$$\vec{N}^{(a)} = (N_1^{(a)}, N_2^{(a)}, N_3^{(a)}), \quad a = 1, 2, 3, \quad (11)$$

and use angular brackets or  $\mathbb{E}[\cdot]$  to denote ensemble averages.

The first moments of the internal directions are the mean vectors

$$\boldsymbol{\mu}^{(a)} = \mathbb{E}[\vec{N}^{(a)}], \quad \mu_i^{(a)} = \mathbb{E}[N_i^{(a)}], \quad a = 1, 2, 3, \quad i = 1, 2, 3, \quad (12)$$

which encode any preferred orientation of the internal triad in the ensemble. The second moments are collected in the covariance matrices

$$\Sigma_{ij}^{(ab)} = \mathbb{E}[(N_i^{(a)} - \mu_i^{(a)})(N_j^{(b)} - \mu_j^{(b)})], \quad a, b = 1, 2, 3, \quad i, j = 1, 2, 3. \quad (13)$$

For fixed  $a, b$  the  $3 \times 3$  matrix  $\Sigma_{ij}^{(ab)}$  is symmetric in  $i, j$ , and the full collection  $\Sigma_{ij}^{(ab)}$  describes both the fluctuations of each leg of the internal triad and their mutual correlations.

Because the  $\vec{N}^{(a)}$  are constrained to form an orthonormal triad for each  $\omega$ , their components are not independent random variables. In this paper we will not attempt to implement the orthonormality constraints exactly at the level of probability densities. Instead, we focus on regimes in which the ensemble is tightly concentrated around a preferred frame and treat the fluctuations as small, so that the constraints can be linearised and the statistics approximated by Gaussian laws on the corresponding tangent space.

### 3.2 Gaussian models for internal directions

To motivate the Gaussian approximation, recall that multivariate normal distributions maximise the differential entropy subject to fixed mean and covariance. [4,5] They provide the least biased continuous distributions compatible with specified first and second moments, and arise naturally as effective descriptions of sums of many weakly correlated contributions. In the present context, we consider ensembles of internal triads whose fluctuations around a preferred frame are small. In such a regime the curvature of the unit sphere and of  $\text{SO}(3)$  can be neglected locally and the distributions of the components  $N_i^{(a)}$  can be well approximated by Gaussian densities on  $\mathbb{R}^3$  (or on an appropriate linear subspace).

Concretely, fix a reference orthonormal triad  $\{\bar{n}^{(1)}, \bar{n}^{(2)}, \bar{n}^{(3)}\}$  and write

$$\vec{N}^{(a)} = \bar{n}^{(a)} + \delta\vec{N}^{(a)}, \quad a = 1, 2, 3, \quad (14)$$

with  $\delta\vec{N}^{(a)}$  small. To leading order in the fluctuations the orthonormality conditions become linear relations among the components of  $\delta\vec{N}^{(a)}$ , which define a linear subspace  $V \subset \mathbb{R}^9$  in which the fluctuations live. A Gaussian model is then specified by choosing a mean vector and a positive semidefinite covariance matrix on  $V$  and restricting the associated multivariate normal distribution on  $\mathbb{R}^9$  to  $V$ .

In practice, it is often convenient to work directly with the components  $N_i^{(a)}$  and to treat the Gaussian density on  $\mathbb{R}^9$  as an approximation which is valid as long as the probability mass is concentrated in a region where the orthonormality constraints are well approximated by their linearisation. In this spirit we consider triplets of jointly Gaussian random vectors  $(\vec{N}^{(1)}, \vec{N}^{(2)}, \vec{N}^{(3)})$  with given means (12) and covariances (13), and use these low-order moments as the primary statistical characteristics of the ensemble.

### 3.3 Isotropic reference ensemble and anisotropic deformations

A particularly simple example is provided by isotropic ensembles in which there is no preferred internal frame. In such a case all three mean vectors vanish,

$$\boldsymbol{\mu}^{(a)} = \mathbf{0}, \quad a = 1, 2, 3, \quad (15)$$

and the covariance matrices are proportional to the identity in the spatial indices,

$$\Sigma_{ij}^{(ab)} = \sigma^2 \delta^{ab} \delta_{ij}, \quad (16)$$

for some nonnegative variance parameter  $\sigma^2$ . The proportionality to  $\delta^{ab}$  expresses that all three legs of the internal triad fluctuate in the same way, while the factor  $\delta_{ij}$  ensures that there is no preferred spatial direction. In the limit  $\sigma^2 \rightarrow 0$  the ensemble collapses to a single deterministic internal triad, while for small but nonzero  $\sigma^2$  the variables  $\vec{N}^{(a)}$  perform small, isotropic fluctuations around  $\bar{n}^{(a)}$ .

More general ensembles break this isotropy. A nonzero mean  $\boldsymbol{\mu}^{(a)}$  singles out a preferred orientation of the  $a$ -th leg of the triad, while deviations of  $\Sigma_{ij}^{(ab)}$  from the isotropic form (16) encode anisotropic correlations among the components of the internal directions. In Sec. 4 we will reorganise the information contained in  $\Sigma_{ij}^{(ab)}$  into a correlation tensor and decompose its

spatial indices into scalar, vector and a symmetric traceless tensor channels under  $\text{SO}(3)$ . This channel decomposition provides a convenient way to classify isotropic ensembles and weakly anisotropic deformations.

The Gaussian ensembles considered here should be viewed as local reference models: they capture the leading behaviour of internal fluctuations in regimes where the ensemble is narrowly concentrated and only first and second moments are controlled. In regimes with large fluctuations or strong non-Gaussian features, higher cumulants of the internal directions or of the underlying spinors  $Z_A$  may become important; such extensions lie beyond the scope of the present paper.

## 4 Correlation tensors of internal directions

The covariance matrices (13) capture the fluctuations and cross-correlations of the components of the internal triad. For later use it is convenient to package this information into a single correlation tensor and to decompose its spatial indices into irreducible pieces under  $\text{SO}(3)$ .

### 4.1 Definition of the correlation tensor

We define the (non-centred) second moments of the internal directions by

$$C^{ab}_{ij} = \mathbb{E}[N_i^{(a)} N_j^{(b)}], \quad a, b = 1, 2, 3, \quad i, j = 1, 2, 3, \quad (17)$$

so that

$$C^{ab}_{ij} = \Sigma_{ij}^{(ab)} + \mu_i^{(a)} \mu_j^{(b)}. \quad (18)$$

Since  $N_i^{(a)} N_j^{(b)}$  is real and symmetric under interchange of  $i$  and  $j$ , the tensor  $C^{ab}_{ij}$  is symmetric in its spatial indices,

$$C^{ab}_{ij} = C^{ab}_{ji}. \quad (19)$$

The indices  $a, b$  label the legs of the internal triad, while  $i, j$  transform as spatial vector indices under rotations of the observer frame.

The full tensor  $C^{ab}_{ij}$  contains more information than will typically be needed. In many applications it is sufficient to consider ensembles that are close to isotropy and to characterise deviations from isotropy in terms of a small number of scalar and tensorial quantities. To this end we focus on the spatial indices and decompose  $C^{ab}_{ij}$  into irreducible pieces under  $\text{SO}(3)$ .

### 4.2 Scalar and tensor channels in the spatial indices

For fixed  $a, b$  the object  $C^{ab}_{ij}$  is a real symmetric  $3 \times 3$  matrix in the indices  $i, j$ . Under spatial rotations it transforms as a second-rank tensor, which decomposes into a scalar (trace) part and a symmetric traceless tensor part. Since  $C^{ab}_{ij}$  is symmetric in  $i, j$ , there is no antisymmetric (vector) part in these indices.

We define the scalar channel by the trace

$$C_{\text{scal}}^{ab} = \frac{1}{3} \delta^{ij} C^{ab}_{ij}, \quad (20)$$

and the traceless tensor channel by

$$T^{ab}_{ij} = C^{ab}_{ij} - C_{\text{scal}}^{ab} \delta_{ij}, \quad \delta^{ij} T^{ab}_{ij} = 0. \quad (21)$$

The quantities  $C_{\text{scal}}^{ab}$  are invariant under spatial rotations, while  $T^{ab}_{ij}$  transforms as a symmetric traceless rank-two tensor. For each pair  $(a, b)$  this yields one scalar and one tensor channel in the spatial indices.

The isotropic reference ensemble introduced in Sec. 3 is characterised by

$$\boldsymbol{\mu}^{(a)} = \mathbf{0}, \quad C^{ab}{}_{ij} = \sigma^2 \delta^{ab} \delta_{ij}, \quad (22)$$

so that

$$C_{\text{scal}}^{ab} = \sigma^2 \delta^{ab}, \quad T^{ab}{}_{ij} = 0. \quad (23)$$

In this case the only nontrivial information in the correlation tensor is the single variance parameter  $\sigma^2$ , and there are no anisotropic tensor contributions. Weak deviations from isotropy can be described by keeping  $C_{\text{scal}}^{ab}$  close to  $\sigma^2 \delta^{ab}$  and treating the traceless tensors  $T^{ab}{}_{ij}$  as small.

### 4.3 Organisation of information and possible extensions

The main purpose of the correlation tensor  $C^{ab}{}_{ij}$  and its decomposition (20)–(21) is organisational: it provides a compact way to store the second-order statistics of the internal triad and to separate rotationally invariant information from anisotropic deformations. In addition to the scalar channel, which controls the overall strength of fluctuations, the tensor channel encodes directional patterns in the correlations of the internal directions.

In more elaborate settings one may wish to consider additional structures. For instance, if the internal triad is coupled to a preferred spatial direction or to a background vector field, mixed scalar–vector channels can arise when the indices  $a, b$  are also decomposed into symmetric and antisymmetric parts. Such extensions are not needed for the present work and will not be pursued here. For our purposes it suffices that the correlation tensor provides a natural home for the scalar and tensor channels that appear in weakly anisotropic Gaussian ensembles of internal directions.

## 5 Ensemble averages of spinor bilinears

So far we have described the statistics of internal directions in terms of random vectors  $\vec{N}^{(a)}$  in  $\mathbb{R}^3$ . In this section we return to the underlying Hopf spinors and introduce ensemble averages of spinor bilinears as complementary objects. These bilinears transform naturally under the spinor representation and provide an efficient way to summarise one–point information about the ensemble.

### 5.1 Density matrix and effective four–vector

Let  $Z_A$  be a random normalized spinor on the spinor sphere,

$$Z^\dagger Z = 1 \quad \text{almost surely}, \quad (24)$$

and define the mixed spinor bilinear

$$M_{A\dot{A}} = \mathbb{E}[Z_A \bar{Z}_{\dot{A}}], \quad (25)$$

where overline denotes complex conjugation in a fixed spin frame. The object  $M_{A\dot{A}}$  is a  $2 \times 2$  Hermitian matrix with unit trace,

$$M_{A\dot{A}} = \overline{M_{A\dot{A}}}, \quad M^A{}_A = \mathbb{E}[Z^\dagger Z] = 1. \quad (26)$$

It may be viewed as a density matrix on the two–dimensional spinor space, familiar from the description of mixed polarization or spin states in quantum mechanics. [6]

Using the Infeld–van der Waerden symbols  $\sigma^{\mu A\dot{A}}$  we can map  $M_{A\dot{A}}$  to an effective four–vector  $V^\mu$  by

$$V^\mu = \sigma^{\mu A\dot{A}} M_{A\dot{A}}. \quad (27)$$

Conversely, any Hermitian  $M_{A\dot{A}}$  can be reconstructed from  $V^\mu$  by

$$M_{A\dot{A}} = \frac{1}{2} V^\mu \sigma_{\mu A\dot{A}}, \quad (28)$$

up to convention-dependent factors. For a pure ensemble in which  $Z_A$  takes a single value almost surely,  $M_{A\dot{A}}$  is a rank-one projector and  $V^\mu$  corresponds to a null or timelike vector lying on or inside the forward light cone, depending on the conventions chosen. For mixed ensembles  $V^\mu$  lies strictly inside the cone.

## 5.2 Relation to mean internal directions

The internal directions are related to the Hopf spinor by

$$n_i(z) = z^\dagger \sigma_i z, \quad i = 1, 2, 3, \quad (29)$$

where  $\sigma_i$  are the Pauli matrices representing the spatial directions in the rest frame of the observer. For the random spinor  $Z_A$  the mean internal direction is therefore

$$\mu_i = \mathbb{E}[n_i(Z)] = \mathbb{E}[Z^\dagger \sigma_i Z] = \text{tr}(\sigma_i M), \quad (30)$$

where  $M$  is the  $2 \times 2$  matrix with entries  $M_{A\dot{A}}$ . Thus the mean internal direction is nothing but the Bloch vector associated with the density matrix  $M_{A\dot{A}}$ .

In terms of the effective four-vector  $V^\mu$ , and with conventions in which  $\sigma^0$  is the identity and  $\sigma^i$  are the Pauli matrices, the relation (27) reads

$$V^0 = \text{tr}(M) = 1, \quad V^i = \text{tr}(\sigma_i M) = \mu_i, \quad (31)$$

so that  $V^\mu$  encodes both the normalisation of the ensemble and the mean internal direction. In this sense the mixed bilinear  $M_{A\dot{A}}$  and the vector  $V^\mu$  provide compact ways to store the one-point information about the distribution of internal directions.

## 5.3 Higher-order spinor correlators

While  $M_{A\dot{A}}$  summarises one-point information, higher-order spinor correlators such as

$$\mathbb{E}[Z_A Z_B \bar{Z}_{\dot{C}} \bar{Z}_{\dot{D}}] \quad (32)$$

contain the same information as the second moments of the internal directions and their correlation tensor. In the Gaussian models discussed in Sec. 3, the components of the internal directions  $N_i^{(a)}$  are jointly Gaussian random variables, so that higher-order moments of  $N_i^{(a)}$  are determined by their covariances via Wick's theorem. The corresponding spinor correlators can be expressed in terms of these Gaussian moments, although we will not carry out this translation explicitly here. In practice it is often simpler to work directly with the correlation tensor  $C_{ij}^{ab}$  introduced in Sec. 4, but in situations where spinor covariance is manifest it can be useful to express the statistics at the spinor level.

In the present paper we will not attempt to develop the spinor correlator language in detail. The main point is that there are two equivalent but complementary viewpoints on the statistics of internal directions:

- a vectorial viewpoint, in which the basic objects are the random internal directions  $\vec{N}^{(a)}$  and their correlation tensor  $C^{ab}_{ij}$ ,
- and a spinorial viewpoint, in which the basic objects are the random Hopf spinor  $Z_A$ , the mixed bilinear  $M_{A\dot{A}}$  and, in Gaussian approximations, higher-order correlators built from  $Z_A$  and its conjugate.

Both viewpoints are compatible and can be translated into one another. The choice between them is largely a matter of convenience and of which symmetries one wishes to keep manifest.

## 6 Conclusion and outlook

In this paper we have developed a Gaussian framework for ensembles of Hopf spinors and their associated internal directions of light. Starting from the Hopf description of null kinematics, in which each future-directed null covector and timelike observer determine a normalized spinor  $z_A$  and an internal orthonormal triad of directions, we promoted the Hopf spinor to a random normalized spinor  $Z_A$  and analysed the induced statistics of the internal triad.

On the vector side, we described the internal directions by random vectors  $\vec{N}^{(a)}$  in  $\mathbb{R}^3$ , introduced their mean vectors  $\boldsymbol{\mu}^{(a)}$  and covariance matrices  $\Sigma_{ij}^{(ab)}$ , and collected the second moments into a correlation tensor  $C^{ab}_{ij}$ . For tightly concentrated ensembles we motivated Gaussian models on the tangent space of the space of internal triads, justified by the maximum-entropy property of multivariate normal distributions under fixed mean and covariance. The spatial indices of the correlation tensor were decomposed into a scalar channel and a symmetric traceless tensor channel under  $\text{SO}(3)$ , providing a systematic organisation of isotropic ensembles and weakly anisotropic deformations.

On the spinor side, we introduced the mixed bilinear  $M_{A\dot{A}} = \mathbb{E}[Z_A \bar{Z}_{\dot{A}}]$  as a  $2 \times 2$  density matrix of unit trace, and mapped it to an effective four-vector  $V^\mu$  using the Infeld-van der Waerden symbols. The mean internal direction was shown to coincide with the Bloch vector associated with  $M_{A\dot{A}}$ , so that  $M_{A\dot{A}}$  and  $V^\mu$  provide compact representations of the one-point statistics of the internal directions. Higher-order spinor correlators encode the same information as the correlation tensor  $C^{ab}_{ij}$  and can be used to formulate the Gaussian ensemble in a spinor-covariant language if desired.

### 6.1 Interpretation and geometric perspective

The constructions developed here are purely kinematical and statistical. The background spacetime is fixed, no stress-energy tensor is introduced, and all structures live in the null sector: future-directed null covectors, their observer-dependent spatial directions, the associated Hopf spinors, and the internal orthonormal triads. In this sense the framework describes a “bare” network of null geodesics endowed with internal directions, without kinetic or potential energies and without massive degrees of freedom.

From a geometric point of view, the Gaussian description organises fluctuations of internal directions around a preferred frame in a way that is structurally similar to familiar null structures in Lorentzian geometry. A multivariate normal distribution with mean  $\boldsymbol{\mu}$  and covariance  $\Sigma$  splits the configuration space into two open sectors separated by a measure-zero set, the mean. The mean is an extremal point of the density but carries no probability mass itself; typical realisations lie in a thin shell around it. Likewise, null hypersurfaces such as light cones or event horizons separate causally distinct regions in spacetime and themselves form null sets in the measure-theoretic sense. In both cases a distinguished “null” structure organises the sectors around it, without being itself a typical trajectory or sample.

In the present work this analogy is used only at a structural level. The mean of a Gaussian ensemble of internal directions plays the role of a statistical reference frame, and the correlation tensor encodes how fluctuations are distributed around it. The underlying null geodesics and their Hopf spinors provide the geometric backbone, but no additional dynamics is attributed to the Gaussian structure itself. In particular, no claim is made here that probabilistic null sets are physically equivalent to geometric null surfaces, or that Gaussian extremality replaces the role of field equations. The point is rather that Gaussian ensembles and null kinematics share a common pattern in which a distinguished “middle” structure of measure zero organises two open sectors, and that this pattern can be exploited to formulate a controlled statistical description of internal directions of light.

## 6.2 Outlook

The Gaussian ensembles considered here are deliberately conservative: they are local in spacetime, purely kinematical and statistical, and do not introduce any new dynamical equations beyond those of the underlying spacetime geometry and null kinematics. Nevertheless, they provide a technically simple and conceptually transparent reference model for the statistics of internal directions of light. Several possible extensions suggest themselves.

1. *Spatial transport and curvature effects.* In curved spacetimes the Hopf spinors and their internal triads are transported along null geodesics by the spin connection. It would be natural to study how Gaussian ensembles of internal directions evolve along such congruences, and how curvature-induced holonomies in the spinor bundle manifest themselves in the correlation tensor  $C^{ab}{}_{ij}$ .
2. *Correlation channels and effective quantities.* The scalar and tensor channels of the correlation tensor are natural candidates for constructing effective scalar and tensor quantities on spacetime. One may ask, for instance, how specific patterns in  $C^{ab}{}_{ij}$  could be related to effective measures of anisotropy in ensembles of null rays or to coarse-grained observables built from internal directions.
3. *Beyond Gaussian ensembles.* In regimes where fluctuations of the internal directions are large or strongly non-Gaussian, higher cumulants of the internal triad or of the underlying spinors  $Z_A$  become relevant. The Gaussian framework developed here can then serve as a baseline, with deviations from Gaussianity organised in terms of higher-order correlation functions.
4. *Multi-spinor and multi-twistor configurations.* Finally, the present work has focussed on ensembles of single Hopf spinors in the null sector. In many contexts of interest, massive sectors or more complex internal structures are expected to involve correlations between several independent null directions or spinors. Extending the ensemble picture to multi-spinor or two-twistor configurations is a natural next step and may provide a bridge from the purely null, massless kinematics studied here to effective descriptions in which massive degrees of freedom and curvature play a role. Such questions lie beyond the scope of the present paper and are left for future work.

In summary, the Gaussian ensembles of Hopf spinors and internal directions introduced here establish a simple but flexible statistical layer on top of the Hopf description of null kinematics. They offer a controlled way to encode first and second moments of internal directions in both vectorial and spinorial languages, and they can serve as a technical backbone for future work on correlation structures in twistor-based descriptions of spacetime and fields. In this sense, the present work does not propose new dynamics, but provides an explicit Gaussian ensemble formulation of the internal Hopf geometry of null directions, organised in terms of scalar and tensor correlation channels, which can be used as a building block in more elaborate twistor-based frameworks.

## References

- [1] Roger Penrose and Wolfgang Rindler. *Spinors and Space-Time. Volume 1: Two-Spinor Calculus and Relativistic Fields.* Cambridge University Press, Cambridge, 1984.
- [2] Roger Penrose and Wolfgang Rindler. *Spinors and Space-Time. Volume 2: Spinor and Twistor Methods in Space-Time Geometry.* Cambridge University Press, Cambridge, 1986.

- [3] Axel Schubert. Hopf fibration and twistor null geometry: Internal directions of light, 2025. Preprint.
- [4] Edwin T. Jaynes. Information theory and statistical mechanics. *Physical Review*, 106(4):620–630, 1957.
- [5] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory*. Wiley, Hoboken, 2 edition, 2006.
- [6] Max Born and Emil Wolf. *Principles of Optics*. Cambridge University Press, Cambridge, 7 edition, 1999.
- [7] Subrahmanyan Chandrasekhar. *Radiative Transfer*. Dover Publications, New York, 1960.
- [8] Ezra T. Newman and Roger Penrose. An approach to gravitational radiation by a method of spin coefficients. *Journal of Mathematical Physics*, 3(3):566–578, 1962.
- [9] Robert M. Wald. *General Relativity*. University of Chicago Press, Chicago, 1984.
- [10] Mikio Nakahara. *Geometry, Topology and Physics*. Taylor & Francis, Bristol, 2 edition, 2003.
- [11] Kanti V. Mardia and Peter E. Jupp. *Directional Statistics*. John Wiley & Sons, Chichester, 2000.