Hopf Fibration and Twistor Null Geometry: Internal Directions of Light

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Abstract

We make explicit the role of the Hopf fibration in the two–spinor description of null directions and photon kinematics in four–dimensional spacetime. Working in the standard two–spinor formalism, [1, 2] any future–directed null vector k^μ is represented by a bispinor $k_{A\dot{A}} = \lambda_A \tilde{\lambda}_{\dot{A}}$, and a timelike observer field u^μ singles out an observer–dependent unit spatial propagation direction $\vec{n} \in S^2$ via $k^\mu = \omega(u^\mu + n^\mu)$ with $\omega = -u_\mu k^\mu > 0$. We show that this $\vec{n}$ can be written as the image of a normalized spinor z_A on the unit spinor sphere S^3 under the Hopf map $S^3 \to S^2$, so that z_A acts as a spinorial "square root" of the photon direction in the observer frame.

Using the SU(2) action on spinors, we associate with z_A a rotated set of Pauli matrices and thereby define an orthonormal internal triad $\{\vec{n}^{(1)}, \vec{n}^{(2)}, \vec{n}^{(3)} = \vec{n}\}$ attached to each null ray. The Hopf fibration thus provides a natural fiber structure over the space of spatial directions, in which propagation direction, phase and polarization of light are encoded geometrically in a single spinor bundle. We illustrate the construction for plane waves in Minkowski spacetime and for radial null rays in the Schwarzschild geometry. [3] The framework remains purely kinematical and within classical Einstein–Maxwell theory: we do not introduce new dynamical fields, but reorganize familiar two–spinor geometry so that the internal directions of light are explicit. This kinematical Hopf picture is intended as a geometric starting point for subsequent work on statistical ensembles of spinors, correlation–based effective channels and multi–twistor configurations.

1 Introduction

In four–dimensional Lorentzian spacetime, massless fields and null directions admit a particularly natural description in terms of two–component spinors and twistors. Any real null vector can be written as a bispinor $k_{A\dot{A}} = \lambda_A \tilde{\lambda}_{\dot{A}}$, so that a photon of definite helicity appears kinematically as a correlated pair of Weyl spinors. [1, 2] This observation underlies both the Newman–Penrose formalism [4] and twistor descriptions of massless fields, [1, 5] in which null directions and their incidence relations are taken as primary geometric objects.

At the same time, normalized two–component spinors carry a well–known internal geometry: they live on a three–sphere S^3 and map, via the Pauli matrices, to unit vectors on a two–sphere S^2. The resulting map $\pi_{\text{Hopf}} : S^3 \to S^2$ is the Hopf fibration. [6] In quantum mechanics this structure is familiar from the Bloch–sphere description of spin–$\frac{1}{2}$ systems, [7] and in polarization optics it appears in the representation of pure polarization states on the Poincaré sphere. [8] In both cases a normalized spinor encodes a point on S^2 together with an internal U(1) phase living on the S^1 fibers of S^3.

The purpose of this paper is to make this Hopf geometry explicit in the context of null kinematics and photon propagation. We consider null covectors k_μ together with a timelike observer field u^μ and show how, in the standard two–spinor formalism, each photon ray naturally carries: (i) an observer–dependent spatial propagation direction $\vec{n} \in S^2$, (ii) a normalized spinor $z_A \in S^3$ serving as a "square root" of $\vec{n}$, and (iii) a canonically defined orthonormal triad of internal directions obtained by rotating the Pauli matrices with an SU(2) element associated with z. The Hopf fibration $S^3 \to S^2$ thus provides a precise geometric framework in which direction, phase and polarization of light are treated as different facets of a single spinor bundle attached to null congruences.

While each of these ingredients is standard on its own, their combination in the present work is somewhat different in emphasis from common treatments. We do not introduce additional dynamics or new field equations; instead, we make explicit a purely kinematical bundle structure in which, for every pair (u^μ, k^μ), the observer split yields a unit spatial direction $\vec{n} \in S^2$, a corresponding Hopf spinor $z_A \in S^3$ defined up to phase, and an internal orthonormal triad $\{\vec{n}^{(1)}, \vec{n}^{(2)}, \vec{n}^{(3)} = \vec{n}\}$ attached to the null ray. In this sense, the paper isolates and organizes the internal Hopf geometry of null directions in a form that is intended to be directly usable as a geometric building block in subsequent work on statistical ensembles of spinors, correlation channels and multi–twistor configurations.

Throughout we work within the usual Einstein–Maxwell setting and focus on kinematical aspects. [3] No modification of the field equations is assumed; the Hopf structure is already implicit in the spinor description of null vectors and is here brought to the foreground. The resulting picture is closely related to the standard Poincaré–sphere description of polarization, but emphasizes the role of the spinor sphere S^3 and the SU(2) action on internal directions of light. In this sense, the present work is intended as a geometric building block that isolates the internal Hopf geometry of null directions for use in subsequent, more elaborate constructions.

The paper is organized as follows. In Sec. 2 we review the basic spinor geometry of the unit spinor sphere and the Hopf fibration $S^3 \to S^2$, emphasizing the interpretation of normalized spinors as square roots of spatial directions. In Sec. 3 we relate this structure to null kinematics by introducing an observer split of null vectors and constructing, for each pair (u^μ, k^μ), a corresponding Hopf spinor. Sec. 4 defines the internal directions of light as an orthonormal triad attached to each photon ray via the rotated Pauli matrices. In Sec. 5 we illustrate the construction in simple examples, including plane waves in Minkowski spacetime and radial null rays in the Schwarzschild geometry. Sec. 6 summarizes the main points and outlines how this kinematical Hopf framework can serve as a starting point for more elaborate descriptions in which statistical ensembles of spinors, correlation tensors and multi–twistor configurations play a role.

2 Spinor sphere and Hopf fibration

In this section we collect the minimal spinorial and geometric facts about the Hopf fibration that will be used in the sequel. [6] The central idea is that a normalized two–component spinor naturally lives on a three–sphere S^3, and that the Pauli matrices define a map from this S^3 to the two–sphere S^2 of unit spatial directions. This is the Hopf map. In the context of null kinematics it will provide the internal "direction space" associated with a single photon ray.

2.1 Two–spinor space and the unit spinor sphere

Let $\mathbb{S}$ be a two–dimensional complex spinor space with basis $\{e_A\}$, $A = 0, 1$, equipped with the antisymmetric spinor $\epsilon_{AB} = -\epsilon_{BA}$ and its inverse ϵ^{AB}, satisfying $\epsilon_{AC}\epsilon^{BC} = \delta_A^B$. We write spinor components as $\psi_A \in \mathbb{S}$ and raise and lower indices according to

$$\psi^A = \epsilon^{AB}\psi_B, \qquad \psi_A = \epsilon_{AB}\psi^B. \tag{1}$$

To discuss normalized spinors it is convenient to equip $\mathbb{S}$ with a Hermitian inner product $(\cdot, \cdot)$, anti–linear in the first argument. Relative to an orthonormal spin frame $\{o_A, \iota_A\}$ this can be written as

$$(\psi, \phi) = \overline{\psi^A}\phi_A, \tag{2}$$

where the bar denotes complex conjugation in the chosen frame. A spinor $z_A \in \mathbb{S}$ is called *normalized* if

$$(z, z) = \overline{z^A} z_A = 1. \tag{3}$$

The space of normalized spinors

$$S^3 \;\cong\; \{z_A \in \mathbb{S} \mid (z, z) = 1\} \tag{4}$$

is a three–sphere. Concretely, if we write

$$z_A = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \in \mathbb{C}^2, \qquad (z, z) = |z_0|^2 + |z_1|^2, \tag{5}$$

then the condition $(z, z) = 1$ is simply $|z_0|^2 + |z_1|^2 = 1$, the standard embedding $S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$.

The group SU(2) acts transitively on S^3 by

$$z_A \mapsto U_A^{\;B} z_B, \qquad U \in \mathrm{SU}(2), \tag{6}$$

preserving both ϵ_{AB} and the Hermitian norm (z, z). In this sense the unit spinor sphere can be identified with SU(2) itself.

2.2 The Hopf map $S^3 \to S^2$

Let σ^i, $i = 1, 2, 3$, be the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7}$$

regarded as Hermitian endomorphisms of $\mathbb{S}$. For a normalized spinor z_A with $(z, z) = 1$ we define a real three–vector $n^i(z)$ by

$$n^i(z) = z^\dagger \sigma^i z, \qquad z^\dagger = (\overline{z_0}, \overline{z_1}), \tag{8}$$

where matrix multiplication is understood. In components this reads

$$n^1 = 2\,\mathrm{Re}(\overline{z_0} z_1), \quad n^2 = 2\,\mathrm{Im}(\overline{z_0} z_1), \quad n^3 = |z_0|^2 - |z_1|^2. \tag{9}$$

A short computation using $(z, z) = 1$ shows that

$$\sum_{i=1}^{3} (n^i)^2 = 1, \tag{10}$$

so $n^i(z)$ defines a point on the two–sphere $S^2 \subset \mathbb{R}^3$. The assignment

$$\pi_{\mathrm{Hopf}} : S^3 \to S^2, \qquad z \mapsto \vec{n}(z) = (n^1(z), n^2(z), n^3(z)) \tag{11}$$

is the *Hopf map.*

Two immediate properties of π_{Hopf} are crucial for our purposes:

(i) **Phase invariance.** For any z_A and any $e^{i\chi} \in \mathrm{U}(1)$,

$$n^i(e^{i\chi} z) = n^i(z). \tag{12}$$

Thus the Hopf map depends only on the projective class of z, and the orbits of the U(1) phase action $z \mapsto e^{i\chi} z$ are precisely the fibers of π_{Hopf}.

(ii) **Surjectivity and fiber structure.** Every point $\vec{n} \in S^2$ arises as $\vec{n}(z)$ for some normalized spinor z_A, and the preimage $\pi^{-1}_{\mathrm{Hopf}}(\vec{n})$ is a circle S^1 of spinors differing only by an overall phase. In other words, the Hopf map realizes S^3 as a principal U(1) bundle over S^2,

$$S^1 \hookrightarrow S^3 \xrightarrow{\pi_{\mathrm{Hopf}}} S^2. \tag{13}$$

Geometrically, we may think of z_A as a "square root" of a unit spatial direction $\vec{n}$: the spinor z lives on the total space S^3, while the classical direction $\vec{n}$ lives on the base space S^2, and the fiber S^1 encodes a phase freedom that will later be related to polarization and internal twistor phases of a photon.

2.3 Spinors as square roots of spatial directions

The Hopf map admits a simple geometric interpretation in terms of three– dimensional Euclidean space. Let $\mathbb{R}^3$ be the space of spatial vectors in a local orthonormal frame, with unit sphere $S^2 = \{\vec{n} \in \mathbb{R}^3 \mid \|\vec{n}\| = 1\}$. The Pauli matrices provide an isomorphism between real vectors and traceless Hermitian 2×2 matrices via

$$\vec{n} \longleftrightarrow n_i \sigma^i, \tag{14}$$

and for any normalized spinor z_A we can rewrite (8) more explicitly as follows.

With the standard normalization

$$\mathrm{tr}(\sigma^i) = 0, \qquad \mathrm{tr}(\sigma^i \sigma^j) = 2\,\delta^{ij}, \tag{15}$$

a short computation using $(z, z) = 1$ shows that the rank–one projector $zz^\dagger$ satisfies

$$zz^\dagger = \frac{1}{2}\left(\mathbf{1} + n_i \sigma^i\right), \tag{16}$$

and hence

$$n_i \sigma^i = 2\, zz^\dagger - \mathbf{1}. \tag{17}$$

In this sense, a unit vector $\vec{n}$ is encoded in the rank–one projector $zz^\dagger$, and z may be regarded as its spinorial "square root".

From the group–theoretic viewpoint, the Hopf map is nothing but the quotient map

$$\mathrm{SU}(2) \to \mathrm{SU}(2)/\mathrm{U}(1) \cong S^2, \tag{18}$$

where U(1) is embedded as the diagonal subgroup. The base S^2 carries the standard action of SO(3), and the double cover SU(2) → SO(3) is implemented at the level of spinors by the usual spin–$\frac{1}{2}$ representation.

In the context of null directions, these facts will allow us to associate to a spinor representative of a null vector a canonically defined spatial direction $\vec{n}$ together with an internal S^1 phase, viewed as the fiber coordinate of the Hopf fibration. We now turn to the null geometry needed to make this statement precise.

3 Null vectors and spatial directions

In four–dimensional Lorentzian spacetime any real vector v^μ can be represented as a Hermitian spinor

$$v_{A\dot{A}} = v^\mu \sigma_{\mu\, A\dot{A}}, \tag{19}$$

where $\sigma_{\mu A\dot{A}}$ are the Infeld–van der Waerden symbols.[1] The vector is null if and only if $v_{A\dot{A}}$ has vanishing determinant, in which case it factorizes as a rank–one outer product of a left– and a right–handed spinor. This is the basic link between null directions and spinors.

3.1 Null vectors as spinor dyads

Let k^μ be a future–directed null vector. Its spinor form $k_{A\dot{A}}$ obeys

$$k^\mu k_\mu = 0 \qquad \Longleftrightarrow \qquad \det(k_{A\dot{A}}) = 0. \tag{20}$$

Since $k_{A\dot{A}}$ is a Hermitian 2×2 matrix of rank one, there exist nonzero spinors λ_A and $\tilde{\lambda}_{\dot{A}}$ such that

$$k_{A\dot{A}} = \lambda_A \, \tilde{\lambda}_{\dot{A}}. \tag{21}$$

The pair $(\lambda_A, \tilde{\lambda}_{\dot{A}})$ is unique up to the complex rescaling

$$\lambda_A \mapsto \alpha \, \lambda_A, \qquad \tilde{\lambda}_{\dot{A}} \mapsto \alpha^{-1} \, \tilde{\lambda}_{\dot{A}}, \qquad \alpha \in \mathbb{C}^\times, \tag{22}$$

which leaves $k_{A\dot{A}}$ invariant. Geometrically, this is the familiar little group freedom for massless momenta. [2]

In the context of electromagnetic waves, k^μ is the wave covector of a photon, and the dyad $(\lambda_A, \tilde{\lambda}_{\dot{A}})$ encodes its helicity and polarization structure in a way that is naturally adapted to twistor methods.

3.2 Observer split of a null direction

To identify a spatial propagation direction associated with a null vector we introduce a timelike observer field u^μ with $u^\mu u_\mu = -1$. In spinor form we write

$$u_{A\dot{A}} = u^\mu \sigma_{\mu A\dot{A}}, \tag{23}$$

and assume that k^μ is future–directed with respect to u^μ, so that $-u_\mu k^\mu > 0$.

The frequency of the photon as measured by this observer is

$$\omega = -u_\mu k^\mu = -u^{A\dot{A}} k_{A\dot{A}} = -u^{A\dot{A}} \lambda_A \tilde{\lambda}_{\dot{A}}, \tag{24}$$

and the corresponding energy is $E = \hbar\omega$. We can use ω to normalize k^μ and define the observer–dependent spatial direction $n^\mu(u, k)$ by

$$n^\mu = \frac{1}{\omega} k^\mu - u^\mu. \tag{25}$$

A short computation using $k^\mu k_\mu = 0$ and $u^\mu u_\mu = -1$ shows that n^μ is unit spacelike and orthogonal to u^μ:

$$n^\mu n_\mu = +1, \qquad u_\mu n^\mu = 0. \tag{26}$$

Thus the pair (u^μ, k^μ) determines a unique unit spatial vector n^μ representing the propagation direction of the photon in the rest frame of u, and we may write

$$k^\mu = \omega \, (u^\mu + n^\mu). \tag{27}$$

In spinor notation we may write

$$n_{A\dot{A}} = n^\mu \sigma_{\mu A\dot{A}} = \frac{1}{\omega} k_{A\dot{A}} - u_{A\dot{A}} = \frac{1}{\omega} \lambda_A \tilde{\lambda}_{\dot{A}} - u_{A\dot{A}}. \tag{28}$$

At each spacetime point, u^μ singles out a local rest frame and n^μ parametrizes a point on the unit sphere S^2 of spatial directions in that frame.

[1] We use signature $(-, +, +, +)$ and conventions compatible with the two–spinor calculus of Penrose and Rindler. [1, 2]

3.3 From spatial direction to Hopf spinor

The unit spatial vector n^μ constructed above has only three independent components; relative to an orthonormal triad $\{e_i^\mu\}$ orthogonal to u^μ we may write

$$n^\mu = n^i e_i^\mu, \qquad \sum_{i=1}^{3}(n^i)^2 = 1. \tag{29}$$

As in Sec. 2, the Pauli matrices σ^i provide an isomorphism between such unit vectors and traceless Hermitian matrices,

$$\vec{n} \longleftrightarrow n_i\sigma^i. \tag{30}$$

Conversely, any unit vector $\vec{n}$ can be written as the image of a normalized spinor z_A under the Hopf map,

$$n^i = z^\dagger\sigma^i z, \qquad (z, z) = 1, \tag{31}$$

with z defined up to an overall complex phase $e^{i\chi}$.

Given (u^μ, k^μ) at a point, the construction therefore proceeds in two steps:

(1) Use (24) and (25) to obtain the unit spatial vector $n^\mu(u, k)$ and its components n^i in the local rest frame of u^μ.

(2) Choose a normalized spinor z_A such that (31) holds. The choice is unique up to a phase $z_A \mapsto e^{i\chi}z_A$, which parametrizes the S^1 fiber over the direction $\vec{n}$.

In this way, each future–directed null vector k^μ and timelike observer u^μ give rise to a Hopf spinor z_A living on the unit spinor sphere S^3, together with a projection $\pi_{\rm Hopf}(z) = \vec{n}$ that coincides with the spatial propagation direction of the photon in the observer frame.

3.4 Internal phase and twistor interpretation

The S^1 freedom $z_A \mapsto e^{i\chi}z_A$ leaves the spatial direction $\vec{n}$ invariant but changes the spinor state along the Hopf fiber. From the spacetime viewpoint, this phase has no effect on the classical null direction k^μ or on the observer–dependent n^μ. From the spinor–twistor perspective, however, it is a natural candidate to encode internal phase information associated with the photon state.

In the geometric–optics limit of Maxwell theory the electromagnetic field can be written locally as [1]

$$F^{(+)}_{AB}(x) \simeq \phi(x)\,\lambda_A(x)\lambda_B(x), \tag{32}$$

with λ_A a principal spinor and ϕ a slowly varying complex amplitude. The overall phase of ϕ and the little–group transformations (22) together generate a U(1) freedom that can be repackaged into the Hopf fiber coordinate of a normalized spinor z_A. In this sense, a single photon ray in a given observer frame carries:

- a spatial propagation direction $\vec{n} \in S^2$,
- and an internal S^1 phase, realized as motion along the Hopf fiber.

In the following sections we use this structure to formulate the notion of "internal directions of light" more precisely. The key point is that the Hopf fibration provides a canonical way to lift classical null directions and observer–dependent spatial directions into a richer spinorial bundle, in which phase and internal orientation are encoded geometrically rather than added by hand.

Remark on conventions and relation to earlier work

For later reference it is useful to summarize the conventions adopted in this section and to note their compatibility with standard two–spinor treatments of null kinematics. [1–3] We work throughout with Lorentzian signature $(-,+,+,+)$ and use Infeld–van der Waerden symbols $\sigma_{\mu\,A\dot{A}}$ to identify real vectors v^μ with Hermitian spinors $v_{A\dot{A}} = v^\mu \sigma_{\mu\,A\dot{A}}$. A real null vector is represented as a rank–one bispinor

$$k_{A\dot{A}} = \lambda_A \tilde{\lambda}_{\dot{A}}, \tag{33}$$

unique up to the little–group rescaling $\lambda_A \mapsto \alpha\lambda_A$, $\tilde{\lambda}_{\dot{A}} \mapsto \alpha^{-1}\tilde{\lambda}_{\dot{A}}$, $\alpha \in \mathbb{C}^\times$.

Given a timelike observer field u^μ with $u^\mu u_\mu = -1$, the frequency measured by this observer is

$$\omega = -u_\mu k^\mu = -u^{A\dot{A}} \lambda_A \tilde{\lambda}_{\dot{A}} > 0, \tag{34}$$

and the associated spatial propagation direction in the observer frame is the unit vector

$$n^\mu = \frac{1}{\omega} k^\mu - u^\mu, \qquad n^\mu n_\mu = +1, \qquad u_\mu n^\mu = 0, \tag{35}$$

so that

$$k^\mu = \omega\,(u^\mu + n^\mu). \tag{36}$$

These conventions agree with those used in previous work on photon kinematics and curvature in a spinor–twistor setting and will be assumed throughout the remainder of the paper.

4 Internal directions of light

The construction of Secs. 2 and 3 associates to each photon ray, represented by a future–directed null covector k_μ and an observer field u^μ, a unit spatial direction $\vec{n} \in S^2$ and a normalized spinor $z_A \in S^3$ defined up to an overall phase. In this section we use this spinor to make precise what we mean by *internal directions of light*: a canonically defined orthonormal triad of directions attached to a null ray via the Hopf fibration and the SU(2) action on spinors.

4.1 Pauli matrices and an internal triad

The three Pauli matrices σ^i form a basis of traceless Hermitian endomorphisms of the spinor space $\mathbb{S}$. For a normalized spinor z_A we defined in (8) the spatial direction

$$n^i(z) = z^\dagger \sigma^i z, \qquad \sum_{i=1}^{3} (n^i)^2 = 1, \tag{37}$$

so that $\vec{n}(z) \in S^2$ may be regarded as the propagation direction of the photon in the local rest frame of an observer u^μ.

To extract more structure, it is useful to note that z_A determines, up to a phase, an element of SU(2). Let $z_A^{\rm ref}$ be a fixed reference spinor, for instance

$$z^{\rm ref} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \vec{n}(z^{\rm ref}) = (0,0,1). \tag{38}$$

For any normalized spinor z_A there exists a matrix $U(z) \in \mathrm{SU}(2)$ such that

$$z_A = U_A^{\ B}(z)\, z_B^{\rm ref}. \tag{39}$$

The matrix $U(z)$ is determined up to right multiplication by a diagonal U(1) phase factor, reflecting the freedom $z \mapsto e^{i\chi} z$.

We now define a rotated set of Pauli operators

$$\Sigma^i(z) = U(z)\, \sigma^i\, U(z)^\dagger, \qquad i = 1, 2, 3, \tag{40}$$

which again form a basis of traceless Hermitian endomorphisms of $\mathbb{S}$. Their mutual commutation relations are the standard ones,

$$\left[\Sigma^i(z), \Sigma^j(z)\right] = 2i\, \epsilon^{ijk} \Sigma^k(z), \tag{41}$$

and they may be viewed as the generators of an orthonormal triad of directions adapted to the spinor z.

The associated internal triad of unit vectors $\vec{n}^{(a)}(z)$, $a = 1, 2, 3$, is defined by

$$n_j^{(a)}(z) = \frac{1}{2} \operatorname{tr}\left(\Sigma^a(z)\, \sigma^j\right), \qquad a, j = 1, 2, 3. \tag{42}$$

A straightforward calculation shows that the 3×3 matrix $R(z) = \left(n_j^{(a)}(z)\right)$ is an element of SO(3), implementing the rotation associated with $U(z)$ at the vector level:

$$\Sigma^a(z) = R^a{}_j(z)\, \sigma^j, \qquad R^a{}_j(z) = n_j^{(a)}(z). \tag{43}$$

Consequently the three vectors $\vec{n}^{(a)}(z)$ form an orthonormal basis of $\mathbb{R}^3$,

$$\vec{n}^{(a)}(z) \cdot \vec{n}^{(b)}(z) = \delta^{ab}, \qquad \sum_{a=1}^{3} n_i^{(a)}(z)\, n_j^{(a)}(z) = \delta_{ij}. \tag{44}$$

By construction the third vector of the triad coincides with the Hopf direction $\vec{n}(z)$ defined earlier,

$$\vec{n}^{(3)}(z) = \vec{n}(z), \tag{45}$$

while $\vec{n}^{(1)}(z)$ and $\vec{n}^{(2)}(z)$ span the plane orthogonal to $\vec{n}(z)$. We will refer to the three vectors $\vec{n}^{(a)}(z)$ as the *internal directions of light* associated with the photon ray represented by (u^μ, k^μ) and its Hopf spinor z.

4.2 Phase freedom and the Hopf fiber

The definition of $\Sigma^i(z)$ and the internal triad $\vec{n}^{(a)}(z)$ is insensitive to the overall phase of z_A. Indeed, if $z_A \mapsto e^{i\chi} z_A$, the corresponding SU(2) matrix $U(z)$ changes by right multiplication with a diagonal element of U(1), which leaves the conjugation (40) and hence the vectors $\vec{n}^{(a)}(z)$ unchanged. Thus the internal triad depends only on the point of S^2 reached by the Hopf map and on the choice of an orthonormal frame adapted to that point; it is constant along the S^1 fibers of the Hopf fibration.

Geometrically, the situation can be summarized as follows. At each spacetime point and for each null direction k^μ and observer u^μ we have:

- a unit spatial vector $\vec{n} \in S^2$ specifying the direction of propagation in the observer frame,
- and an SO(3) matrix $R(z)$ encoding a triad $(\vec{n}^{(1)}, \vec{n}^{(2)}, \vec{n}^{(3)} = \vec{n})$ of internal directions, defined up to an overall rotation around $\vec{n}$ itself.

This residual freedom corresponds to the familiar ambiguity in choosing polarization vectors transverse to $\vec{n}$: any rotation in the plane orthogonal to $\vec{n}$ yields an equally valid basis.

4.3 Relation to polarization and Poincaré sphere

The internal directions introduced above connect naturally to the standard description of photon polarization. In the geometric–optics regime, the electromagnetic field of a monochromatic wave with wave covector k_μ can be characterized, at a fixed point, by two pieces of data:

(a) the null direction k^μ (or equivalently, $\vec{n}$ in the rest frame of u^μ),

(b) the polarization state, usually represented as a point on the Poincaré sphere. [8]

The Poincaré sphere is mathematically identical to the Bloch sphere of normalized spinors: any pure polarization state can be described by a normalized two–component complex vector up to phase. [7] In this sense, the Hopf spinor z_A and the associated direction $\vec{n}(z)$ simultaneously encode both a spatial direction and a point on the polarization sphere.

From the present viewpoint, the internal triad $\{\vec{n}^{(1)}(z), \vec{n}^{(2)}(z), \vec{n}^{(3)}(z)\}$ provides a convenient geometric package for these data:

- $\vec{n}^{(3)}(z)$ is aligned with the direction of propagation of the photon in the observer frame,
- $\vec{n}^{(1)}(z)$ und $\vec{n}^{(2)}(z)$ form a pair of orthonormal transverse directions, und different choices of polarization correspond to different complex combinations of these transverse vectors,
- the point z on the spinor sphere S^3 (modulo phase) can be viewed either as specifying the polarization state for a given $\vec{n}$ or as specifying a triad of internal directions rotated relative to a fixed reference triad.

No modification of Maxwell theory is involved in this identification; it merely reorganizes familiar polarization kinematics in terms of the Hopf fibration associated with spinor representations.

4.4 Summary

To summarize, a single photon ray, specified by a null covector k_μ and an observer field u^μ, naturally carries the following geometric structure:

(i) a unit spatial direction $\vec{n} \in S^2$ in the rest frame of the observer,

(ii) a normalized spinor $z_A \in S^3$ defined up to phase,

(iii) an orthonormal internal triad $\{\vec{n}^{(1)}(z), \vec{n}^{(2)}(z), \vec{n}^{(3)}(z) = \vec{n}\}$ obtained by rotating the Pauli matrices with $U(z) \in \mathrm{SU}(2)$.

The Hopf fibration $S^3 \to S^2$ provides the natural mathematical framework for this structure: it identifies spatial directions with points on S^2 and realizes the spinor sphere S^3 as a U(1) bundle over S^2, with the internal triad constant along each fiber. In subsequent work this internal triad can serve as the geometric backbone for more elaborate constructions in which internal modes of null congruences play a dynamical role; in the present paper we restrict attention to its purely kinematical aspects.

5 Physical examples

To illustrate the structures introduced above we briefly discuss a few simple but representative situations in which the internal directions of light can be made fully explicit. Throughout we work within standard Maxwell theory on fixed background spacetimes; the Hopf fibration and the associated internal triad simply reorganize familiar kinematics in spinor language.

5.1 Plane wave in Minkowski spacetime

We begin with a monochromatic plane electromagnetic wave propagating in Minkowski spacetime. Let (t, x, y, z) be global inertial coordinates with metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{46}$$

and consider a wave whose phase is given by

$$S(t, x, y, z) = -\omega\, t + \omega\, z, \tag{47}$$

with constant frequency $\omega > 0$. The corresponding wave covector is

$$k_\mu = \partial_\mu S = \omega(-1, 0, 0, 1), \tag{48}$$

which is null with respect to the Minkowski metric. The associated null vector obtained by raising the index is

$$k^\mu = g^{\mu\nu} k_\nu = \omega(1, 0, 0, 1). \tag{49}$$

In what follows we use k^μ for the observer split of Sec. 3, while k_μ plays the role of the wave covector in the eikonal picture. (In much of the literature both objects are denoted by k, but keeping track of their index positions is useful here.)

For the standard inertial observer

$$u^\mu = (1, 0, 0, 0), \qquad u_\mu u^\mu = -1, \tag{50}$$

the measured frequency is

$$\omega = -u_\mu k^\mu, \tag{51}$$

in agreement with the definition (24). The spatial propagation direction n^μ defined in (25) becomes

$$n^\mu = \frac{1}{\omega} k^\mu - u^\mu = (1, 0, 0, 1) - (1, 0, 0, 0) = (0, 0, 0, 1). \tag{52}$$

In the observer's rest frame the photon thus propagates along the positive z–axis, so that the unit spatial direction is

$$\vec{n} = (0, 0, 1). \tag{53}$$

At the level of spinors, we choose a standard Infeld–van der Waerden representation in which

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{54}$$

so that the spinor form of k^μ reads

$$k_{A\dot{A}} = k^\mu \sigma_{\mu\, A\dot{A}} = \omega\big(\sigma^0_{A\dot{A}} + \sigma^3_{A\dot{A}}\big) = 2\omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{55}$$

A compatible spinor dyad is

$$\lambda_A = \sqrt{2\omega} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \tilde{\lambda}_{\dot{A}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{56}$$

up to the little–group freedom (22). After removing the overall magnitude we may take as normalized Hopf spinor

$$z_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (z, z) = 1. \tag{57}$$

The Hopf map (8) then yields

$$\vec{n}(z) = \big(0, 0, 1\big), \tag{58}$$

in agreement with the spacetime calculation.

The associated internal triad (42) is particularly simple in this case. With $U(z) = \mathbf{1}$ we have

$$\Sigma^i(z) = \sigma^i, \qquad \vec{n}^{(a)}(z) = \hat{e}_a, \tag{59}$$

where $\hat{e}_a$ denotes the unit vector along the a–th coordinate axis, $a = 1, 2, 3$. Thus

$$\vec{n}^{(1)} = (1, 0, 0), \quad \vec{n}^{(2)} = (0, 1, 0), \quad \vec{n}^{(3)} = (0, 0, 1) = \vec{n}, \tag{60}$$

so that the internal directions of light coincide with the standard Cartesian triad adapted to the propagation direction.

5.2 Linear and circular polarization

Even in flat spacetime the Hopf description offers a convenient way to visualize polarization states. We keep the setup of the previous subsection: a plane wave propagating in the $+z$ direction, so that in the rest frame of the inertial observer the spatial propagation direction is fixed as $\vec{n} = (0, 0, 1)$ and the physical polarization vectors lie in the x–y plane.

A convenient orthonormal basis of transverse polarization vectors is provided by the unit vectors $\hat{e}_x$ and $\hat{e}_y$, corresponding to linear polarization along x and y, respectively. At the level of field amplitudes, any pure polarization state can then be represented by a normalized two–component complex vector (Jones vector)

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad |z_1|^2 + |z_2|^2 = 1, \tag{61}$$

defined up to an overall phase. The space of such spinors is again S^3, and the associated Stokes vector

$$s^i = z^\dagger \sigma^i z, \qquad i = 1, 2, 3, \tag{62}$$

with $\sum_i (s^i)^2 = 1$, defines a point on the Poincaré sphere of pure polarization states. [7, 8] Mathematically, this is identical to the Hopf map discussed in Sec. 2; the interpretation differs only in that the resulting unit vector is now regarded as a Stokes vector rather than a spatial propagation direction.

A useful explicit parametrization is

$$z(\theta, \varphi) = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi} \sin\frac{\theta}{2} \end{pmatrix}, \tag{63}$$

for which

$$\vec{s}(z) = \big(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta\big). \tag{64}$$

Different choices of (θ, φ) label different pure polarization states.

With a suitable convention for the axes on the Poincaré sphere, horizontal and vertical linear polarization (along $\hat{e}_x$ and $\hat{e}_y$) can be represented, for example, by

$$z_{\mathrm{lin},x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad z_{\mathrm{lin},y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{65}$$

while right– and left–circular polarization correspond to

$$z_{\mathrm{circ},\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp i \end{pmatrix}, \tag{66}$$

up to an overall phase. All these spinors live on the same Hopf bundle $S^3 \to S^2$; their Stokes vectors $\vec{s}$ trace out the Poincaré sphere.

In particular, for a fixed propagation direction $\vec{n} = (0, 0, 1)$, different polarization states correspond to different rotations of the pair $(\vec{n}^{(1)}, \vec{n}^{(2)})$ around $\vec{n}^{(3)} = \vec{n}$ in the internal triad introduced in Sec. 4. The Hopf fibration thus makes explicit the close relationship between polarization, internal directions of light and the usual Poincaré–sphere description: the same normalized spinor z that encodes the polarization state also determines the internal triad via (42).

5.3 Radial null rays in Schwarzschild spacetime

As a simple curved–spacetime example we consider radial null propagation in Schwarzschild spacetime with line element [3]

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2), \qquad f(r) = 1 - \frac{2M}{r}. \tag{67}$$

We restrict attention to radial null rays in the equatorial plane $\theta = \pi/2$ and to static observers at fixed radius r.

A static observer has four–velocity

$$u^\mu(r) = \big(f(r)^{-1/2}, 0, 0, 0\big), \tag{68}$$

normalized so that $u_\mu u^\mu = -1$. A radial null geodesic has tangent k^μ with nonvanishing components (k^t, k^r) satisfying $k^\mu k_\mu = 0$, which implies

$$-f(r)(k^t)^2 + f(r)^{-1}(k^r)^2 = 0, \qquad \Rightarrow \quad k^r = \pm f(r)\,k^t. \tag{69}$$

The sign distinguishes outgoing und ingoing rays.

The frequency measured by the static observer is

$$\omega(r) = -u_\mu k^\mu = f(r)^{1/2} k^t, \tag{70}$$

und the spatial direction $n^\mu(r)$ in the observer's local rest frame is obtained from (25) as

$$n^\mu(r) = \frac{1}{\omega(r)} k^\mu + u^\mu(r). \tag{71}$$

Projecting onto an orthonormal triad adapted to u^μ und the radial direction, one finds that $\vec{n}$ points purely radially, $\vec{n} = \pm\hat{e}_r$, with the sign corresponding to outgoing (+) or ingoing (−) rays.

As in the Minkowski case, we can now choose a normalized spinor $z_A(r)$ such that the Hopf map yields

$$\vec{n}\big(z(r)\big) = \pm\hat{e}_r. \tag{72}$$

The internal triad $\{\vec{n}^{(1)}(z(r)), \vec{n}^{(2)}(z(r)), \vec{n}^{(3)}(z(r)) = \pm\hat{e}_r\}$ then provides, at each radius, a set of internal directions of light attached to the radial photon as seen by the static observers.

When comparing two static observers at radii r_e (emitter) und r_o (observer), the gravitational redshift affects the measured frequency $\omega(r)$ but not the spatial direction: both observers see the photon as purely radial. In terms of the Hopf picture, the base point $\vec{n}$ on S^2 remains fixed, while the internal triads at r_e und r_o can be compared by transporting the associated spinors along the null geodesic using the spin connection. Any rotation of the transverse pair $(\vec{n}^{(1)}, \vec{n}^{(2)})$ induced by curvature can be interpreted as a geometric effect on the internal directions of light, closely related to the familiar parallel transport of polarization vectors in curved spacetime.

5.4 Role of the examples

These simple examples illustrate how the Hopf fibration und the associated internal triad appear in familiar situations. In flat spacetime, the construction reproduces the standard description of propagation direction und polarization und packages them into a single spinor z_A on the unit sphere S^3. In curved spacetime, the same spinor bundle can be transported along null geodesics, with curvature entering through the spin connection und inducing holonomies in the internal directions of light.

In all cases, the Hopf structure is already implicit in the spinor description of null vectors; the present framework merely makes it explicit und emphasizes its geometric content. This will be the starting point for subsequent work in which internal modes associated with the spinor sphere S^3 are allowed to participate more directly in the dynamical description of spacetime und matter.

6 Conclusion and outlook

In this paper we have made explicit the role of the Hopf fibration in the two–spinor description of null directions and photon kinematics. Starting from the representation of a future–directed null vector as a bispinor $k_{A\dot{A}} = \lambda_A\tilde{\lambda}_{\dot{A}}$ and from the observer–dependent split into time and space, we showed that a single photon ray, as seen by a given observer field u^μ, naturally carries the following geometric data:

- a unit spatial propagation direction $\vec{n} \in S^2$ in the local rest frame of u^μ, obtained from the null vector via $k^\mu = \omega(u^\mu + n^\mu)$ with $\omega = -u_\mu k^\mu > 0$,
- a normalized spinor $z_A \in S^3$ defined up to phase, serving as a "square root" of $\vec{n}$ via the Hopf map $\pi_{\text{Hopf}} : S^3 \to S^2$, $\vec{n} = z^\dagger\sigma^i z$,
- an associated orthonormal internal triad $\{\vec{n}^{(1)}(z), \vec{n}^{(2)}(z), \vec{n}^{(3)}(z) = \vec{n}\}$ obtained by rotating the Pauli matrices with an SU(2) element $U(z)$ and reading off the corresponding SO(3) rotation matrix.

The Hopf fibration $S^3 \to S^2$ thus appears as the natural fiber structure over the space of spatial directions attached to null congruences. It packages direction, phase and polarization information into a single geometric object on the spinor sphere, while remaining entirely within the standard framework of Maxwell theory and general relativity. No additional fields or modified equations of motion have been assumed; all constructions are purely kinematical reorganizations of familiar two–spinor geometry.

In this sense, the present work does not propose new dynamics, but gives an explicit Hopf–bundle formulation of null congruence kinematics which, to the best of the author's knowledge, has not been spelled out in this combination in the existing literature.

On the physical side, the examples in Minkowski and Schwarzschild spacetimes show that this Hopf picture reproduces standard notions: propagation direction, transverse polarization vectors and the Poincaré sphere can all be recovered as projections or rotations of the internal triad. Curvature enters through the spin connection and the holonomy of the spinor bundle along null geodesics, leading to rotations of the internal directions of light that mirror the parallel transport of polarization in curved spacetime and in optical media. [3, 8] Together with related work on photon kinematics and curvature in a spinor–twistor setting, the present analysis provides a third, complementary kinematical ingredient in which the internal geometry of null directions is made explicit.

Outlook. The constructions presented here are deliberately restricted to kinematics: the background spacetime is fixed and the internal directions of light are not assigned independent dynamics. Nevertheless, the Hopf framework suggests a convenient starting point for several extensions in which the spinor sphere S^3 and the associated internal triad are used more systematically.

(i) *Statistical ensembles and internal mode structure.* Once the spinor sphere S^3 and the internal triad have been identified as part of the kinematics of null congruences, one may consider ensembles of Hopf spinors with prescribed statistical properties. In particular, three–mode decompositions adapted to $\{\vec{n}^{(1)}, \vec{n}^{(2)}, \vec{n}^{(3)}\}$ suggest a natural way to discuss internal mode structure of light rays in a Gaussian or maximum–entropy setting, without leaving the classical Einstein–Maxwell framework.

(ii) *Correlation tensors and effective channels.* Correlations between spinors associated with different null rays and different observers can be organized in terms of tensors built from z_A and its conjugate. In such a description, scalar, vector and tensor channels correspond to different index contractions and different averaging procedures over the spinor sphere. These channels can then be used as a systematic language to encode effective degrees of freedom in correlation–based descriptions, without committing to a specific emergent field model.

(iii) *Two–twistor structures and massive sectors.* While a single null direction is described by one spinor dyad and its Hopf spinor, massive momenta require superpositions of at least two independent null directions. In a spinor/twistor language this leads naturally to two–twistor configurations in which pairs of Hopf spinors are combined or correlated. The present work provides the geometric vocabulary needed to analyze such constructions, for example in the study of how massive sectors can be built from null data.

(iv) *Geometric organization of polarization optics.* Even within classical electrodynamics there are situations where a Hopf–based organization of polarization and internal directions may be helpful, such as the analysis of optical vortices, geometric phases and the transport of polarization in complex media or curved spacetimes. [8] In such applications, the internal triad offers a compact way to track how transverse modes are rotated relative to a given propagation direction.

In summary, the Hopf fibration and the associated internal directions of light are already implicit in the two–spinor description of null vectors. Making them explicit clarifies the geometric structure underlying photon kinematics and provides a clean kinematical backbone on which more elaborate constructions can be mounted. Subsequent work will build on this backbone by introducing statistical ensembles of Hopf spinors and by exploring how multi–twistor configurations and correlation–driven structures can be organized within the same geometric language, while keeping the distinction between kinematics and dynamics clearly separated.

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