

Photon Kinematics in Curved Spacetime: A Spinor–Twistor Perspective

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Abstract

We formulate the propagation of massless spin-1 fields on a fixed curved spacetime entirely in spinor and twistor language and isolate the minimal kinematical structure needed to describe photon energy, redshift, and lensing. Any null direction is represented as a bispinor $k_{A\tilde{A}} = \lambda_A \tilde{\lambda}_{\tilde{A}}$, so that a photon of definite helicity appears as a correlated pair of spin- $\frac{1}{2}$ spinors with field strength $\phi_{AB} \propto \lambda_A \lambda_B$. We show how standard geometric-optics results—null geodesic motion, the eikonal equation, and the observer-dependent energy $E = -\hbar u^\mu k_\mu$ —arise directly from the parallel transport of the underlying spinors via the spin connection, with curvature encoded in their evolution along null congruences.

Within this strictly conventional framework we propose an interpretation of photon energy as a persistent phase gradient between twistor sectors. Relative phases between neighboring null rays and between the photon’s spinor dyad $(\lambda_A, \tilde{\lambda}_{\tilde{A}})$ and an observer’s four-velocity $u^{A\tilde{A}}$ encode measurable frequencies and interference effects, while curvature reshapes these correlations through spinor transport. No modification of general relativity or electrodynamics is introduced; instead, the construction reorganizes familiar kinematics in a form that highlights the role of spinor/twistor correlations and phase in the geometry of light.

1 Introduction

In this paper we restrict attention to massless spin-1 fields (photons) propagating on a fixed, globally hyperbolic Lorentzian spacetime $(\mathcal{M}, g_{\mu\nu})$. Our goal is to reformulate the kinematics of photon propagation using a language entirely based on spinors and twistors, in line with Penrose and Rindler’s two-spinor formalism. [1] and the Newman–Penrose approach to null directions [2], within the standard framework of general relativity [3, 4].

The key observation is that any null vector can be written as a bilinear in two spin- $\frac{1}{2}$ objects. In this sense a spin-1 excitation is naturally represented as a correlated pair of spinors. Curvature then acts on this pair via the spin connection, inducing both the familiar null geodesic motion of light rays and the standard gravitational effects on frequency and polarization. Twistor theory [5]

provides a natural setting in which such spinor pairs are the fundamental objects and spacetime geometry is encoded in their incidence relations.

Within this well-established framework we propose a minimalist interpretational step: we regard the energy of a photon, as measured by a given observer, as arising from a persistent phase gradient between twistor sectors along the null congruence. This does not change any physical prediction of classical or semiclassical gravity; rather, it organizes the known kinematics in a way that emphasizes the spinor/twistor structure and the role of phase.

The structure of the paper is as follows. In Sec. 2 we review the two-spinor description of null vectors and massless spin-1 fields. In Sec. 3 we discuss parallel transport of spinors along null geodesics and its relation to curvature. In Sec. 4 we introduce a phase function associated with a null congruence and connect it to the photon energy as seen by an arbitrary observer. Section 5 illustrates these constructions in Schwarzschild spacetime and recovers the standard formula for gravitational redshift. Section 6 collects interpretational remarks and outlines how these elements can be viewed as a “persistent phase separation” between twistor sectors.

2 Spinor and twistor description of null rays

We work with a four-dimensional Lorentzian spacetime with metric signature $(-, +, +, +)$ and assume the usual two-component spinor formalism as in [1, 3]. Capital Latin indices A, B, \dots and their dotted counterparts \dot{A}, \dot{B}, \dots label components in the left- and right-handed Weyl spinor spaces, respectively. The spacetime index μ is related to a pair of spinor indices via the Infeld-van der Waerden symbols $\sigma_{A\dot{A}}^\mu$,

$$v_{A\dot{A}} \equiv v_\mu \sigma_{A\dot{A}}^\mu, \quad (1)$$

with the inverse relation $v_\mu = \frac{1}{2} \sigma_\mu^{A\dot{A}} v_{A\dot{A}}$.

A crucial fact is that a real vector k^μ is null if and only if the corresponding bispinor $k_{A\dot{A}}$ is a simple outer product of a left-handed and a right-handed spinor [1],

$$k_{A\dot{A}} = \lambda_A \tilde{\lambda}_{\dot{A}}, \quad (2)$$

for some nonzero spinors λ_A and $\tilde{\lambda}_{\dot{A}}$, unique up to simultaneous rescalings

$$\lambda_A \rightarrow \alpha \lambda_A, \quad \tilde{\lambda}_{\dot{A}} \rightarrow \alpha^{-1} \tilde{\lambda}_{\dot{A}}, \quad \alpha \in \mathbb{C}^\times. \quad (3)$$

This decomposition expresses a null direction as a correlated pair of spin- $\frac{1}{2}$ objects. The rescaling freedom encodes the fact that k^μ is invariant under λ - $\tilde{\lambda}$ boosts. In particular, the spinors λ_A and $\tilde{\lambda}_{\dot{A}}$ carry the directional information of the null ray in a way that is especially well suited to twistor methods.

For a massless spin-1 field $F_{\mu\nu}$, the selfdual part can be encoded in a symmetric spinor ϕ_{AB} defined by [1, 6]

$$F_{\mu\nu}^{(+)} = \phi_{AB} \epsilon_{\dot{A}\dot{B}} \sigma_{[\mu}^{A\dot{A}} \sigma_{\nu]}^{B\dot{B}}, \quad (4)$$

where ϵ_{AB} is the antisymmetric spinor used to raise and lower indices. In the geometric optics limit [7, 4], a monochromatic wave of definite helicity can be described locally by

$$\phi_{AB}(x) = \varphi(x) \lambda_A(x) \lambda_B(x), \quad (5)$$

with λ_A a principal spinor field and φ a slowly varying complex amplitude. The associated wave vector is then

$$k_{A\dot{A}}(x) = \lambda_A(x) \tilde{\lambda}_{\dot{A}}(x), \quad (6)$$

where $\tilde{\lambda}_{\dot{A}}$ is determined up to the rescaling above. Equations (5) and (6) make explicit how a helicity eigenstate of the electromagnetic field is supported on a null spinor dyad $(\lambda_A, \tilde{\lambda}_{\dot{A}})$.

From the twistor point of view, a (projective) twistor is a pair $Z^\alpha = (\omega^A, \pi_{\dot{A}})$ satisfying the incidence relation [5]

$$\omega^A = i x^{A\dot{A}} \pi_{\dot{A}}. \quad (7)$$

Null geodesics correspond to certain integral curves in twistor space. For the present discussion we do not need the full twistor machinery; it suffices to note that the pair $(\lambda_A, \tilde{\lambda}_{\dot{A}})$ associated with a null vector plays the role of a twistor dyad. A photon can thus be viewed as an excitation supported on such null dyads, with polarization and phase encoded in their detailed structure.

3 Parallel transport and curvature

We now consider the propagation of a photon along a null geodesic $\gamma : \lambda \mapsto x^\mu(\lambda)$ with tangent $k^\mu = \frac{dx^\mu}{d\lambda}$. In the geometric optics approximation the electromagnetic field is sharply peaked around such curves, and the wave vector k^μ is null and geodesic,

$$g_{\mu\nu} k^\mu k^\nu = 0, \quad k^\nu \nabla_\nu k^\mu = 0, \quad (8)$$

as in the standard treatment of light rays in curved spacetime [3, 4]. In spinor notation the tangent can be written as in (2),

$$k_{A\dot{A}} = \lambda_A \tilde{\lambda}_{\dot{A}}. \quad (9)$$

The covariant derivative acting on spinor fields involves the spin connection; we write

$$\nabla_{A\dot{A}} \lambda_B = \partial_{A\dot{A}} \lambda_B + \Gamma_{A\dot{A}B}{}^C \lambda_C, \quad (10)$$

with analogous expressions for dotted indices. Along the null curve γ we define the directional derivative

$$\frac{D}{D\lambda} \equiv k^{A\dot{A}} \nabla_{A\dot{A}}, \quad (11)$$

so that parallel transport of λ_A and $\tilde{\lambda}_{\dot{A}}$ is expressed by

$$\frac{D\lambda_A}{D\lambda} = 0, \quad \frac{D\tilde{\lambda}_{\dot{A}}}{D\lambda} = 0. \quad (12)$$

These equations encode the effect of curvature on the spinor dyad underlying the null direction: the spin connection, via $\Gamma_{AAB}{}^C$, couples the spinors to the spacetime geometry. In particular, any focusing, shearing, or twisting of a null congruence can be read off from how neighboring solutions of (12) deviate from one another.

The curvature itself can be represented by the Weyl and Ricci spinors ($\Psi_{ABCD}, \Phi_{A\dot{A}B\dot{B}}$) [1, 6]. Their action on null congruences manifests, for example, in the focusing and shearing of bundles of null geodesics, as summarized by the Raychaudhuri and Sachs equations [3]. In the present setting, this information is equivalently contained in the evolution of the spinor fields $(\lambda_A, \bar{\lambda}_{\dot{A}})$ along γ and in their variation between neighboring rays. Thus the spinor transport equations provide a compact “square root” of the usual tensorial optics of light in curved spacetime.

4 Phase and energy along null geodesics

To connect the spinor description to energy measurements, we introduce a scalar phase function $S(x)$ defined in a neighborhood of the null congruence. In the eikonal approximation one writes the electromagnetic potential locally as [7, 4]

$$A_\mu(x) \simeq \Re \left\{ \epsilon_\mu(x) \exp \left(\frac{i}{\hbar} S(x) \right) \right\}, \quad (13)$$

where the polarization vector ϵ_μ varies slowly compared to the phase. The wave covector is then given by

$$k_\mu = \nabla_\mu S, \quad (14)$$

and the eikonal equation $g^{\mu\nu} k_\mu k_\nu = 0$ is equivalent to the null condition for the rays. In this picture the phase S plays the role of a Hamilton–Jacobi function, and k_μ is the corresponding canonical momentum.

Along the null geodesic γ we can choose the affine parameter λ such that

$$\frac{dS}{d\lambda} = k^\mu k_\mu = 0, \quad (15)$$

so that S is constant along each ray. This reflects the freedom to reparametrize the rays without changing the physical content; what is relevant for energy measurements is not the absolute value of S along a single ray, but its variation with respect to an observer’s worldline.

Let u^μ be the four–velocity of an observer. The frequency of the photon as measured by this observer is

$$\omega = -u^\mu k_\mu, \quad (16)$$

and the corresponding energy is

$$E = \hbar\omega = -\hbar u^\mu \nabla_\mu S. \quad (17)$$

Equation (17) is simply the familiar relation $E = \hbar\omega$ together with the identification of the local frequency as the rate of phase change along the observer’s

trajectory, $\omega = dS/d\tau$, with $d\tau$ the observer's proper time [3, 4]. It makes explicit that what we call “photon energy” is entirely determined by the relative orientation of the observer's four-velocity and the null wave covector.

In spinor form, using $k_{A\dot{A}} = \lambda_A \tilde{\lambda}_{\dot{A}}$, this can be written as

$$\omega = -u^{A\dot{A}} \lambda_A \tilde{\lambda}_{\dot{A}}, \quad (18)$$

where $u^{A\dot{A}} = u^\mu \sigma_\mu^{A\dot{A}}$ is the spinor representation of the observer's four-velocity. The scalar product $u^{A\dot{A}} \lambda_A \tilde{\lambda}_{\dot{A}}$ thus measures the overlap between the observer's spinor sector and the null twistor dyad associated with the photon.

Gravitational redshift und blueshift arise when either u^μ or k^μ (or both) vary along the worldlines involved. For two events p and q on the observer's worldline, connected by a null geodesic segment, the ratio of measured frequencies is determined entirely by the change in the scalar product $u^\mu k_\mu$, which in turn depends on the curvature along the connecting null geodesic [3, 4]. In this picture the effect of curvature is mediated by the evolution of the spinor pair $(\lambda_A, \tilde{\lambda}_{\dot{A}})$ under the transport equation (12) and by the transport of $u^{A\dot{A}}$ along the observer's path.

For later use it is convenient to consider also relative phases between neighboring rays within a congruence. Let $S_1(x)$ and $S_2(x)$ be zwei Lösungen der Eikonalgleichung, die zwei benachbarten Nullstrahlen zugeordnet sind, und definieren wir ihre Phasendifferenz

$$\Delta S(x) = S_1(x) - S_2(x). \quad (19)$$

The gradient $\nabla_\mu \Delta S$ measures the relative shift of the corresponding wave covectors and encodes interference properties. In a pure spinor/twistor language, this relative phase can be viewed as arising from the mismatch between the spinor pairs $(\lambda_A^{(1)}, \tilde{\lambda}_{\dot{A}}^{(1)})$ and $(\lambda_A^{(2)}, \tilde{\lambda}_{\dot{A}}^{(2)})$ associated with the two rays. Curvature then controls the evolution of ΔS by dictating how these spinors are transported and compared.

5 Gravitational redshift in Schwarzschild space-time

To illustrate the spinor formulation in a concrete setting, we consider photon propagation in the Schwarzschild spacetime

$$ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad f(r) = 1 - \frac{2M}{r}, \quad (20)$$

and recover the standard gravitational redshift formula in two complementary ways. We restrict attention to radial motion in the equatorial plane, $\theta = \pi/2$, and work in the geometric-optics limit, following the textbook treatments in [3, 4].

5.1 Radial null geodesics and conserved energy

A radial null geodesic has tangent $k^\mu = \frac{dx^\mu}{d\lambda}$ with components

$$k^\mu = \left(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, 0, 0 \right), \quad (21)$$

subject to the null condition $g_{\mu\nu}k^\mu k^\nu = 0$. In Schwarzschild coordinates this reads

$$-f(r) \left(\frac{dt}{d\lambda} \right)^2 + f(r)^{-1} \left(\frac{dr}{d\lambda} \right)^2 = 0, \quad (22)$$

or equivalently

$$\frac{dr}{d\lambda} = \pm f(r) \frac{dt}{d\lambda}, \quad (23)$$

where the plus (minus) sign corresponds to outgoing (ingoing) radial null rays.

Because ∂_t is a timelike Killing vector, there is a conserved quantity along the geodesic,

$$E_\infty \equiv -k_\mu(\partial_t)^\mu = -k_t = f(r) \frac{dt}{d\lambda}, \quad (24)$$

which can be interpreted as the photon energy as seen by an observer at infinity [3]. Solving for $dt/d\lambda$ yields

$$\frac{dt}{d\lambda} = \frac{E_\infty}{f(r)}, \quad \frac{dr}{d\lambda} = \pm E_\infty. \quad (25)$$

The radial velocity with respect to coordinate time is

$$\frac{dr}{dt} = \pm f(r), \quad (26)$$

showing the familiar coordinate slowing of outgoing light near the horizon.

5.2 Static observers and gravitational redshift

A static observer at radius r has four-velocity

$$u^\mu(r) = \left(f(r)^{-1/2}, 0, 0, 0 \right), \quad (27)$$

normalized such that $g_{\mu\nu}u^\mu u^\nu = -1$. The photon frequency measured by this observer is

$$\omega(r) = -u^\mu k_\mu = -g_{\mu\nu}u^\mu k^\nu. \quad (28)$$

Using k^μ and u^μ as above,

$$\omega(r) = -[-f(r)u^t k^t] = f(r)u^t k^t = f(r)f(r)^{-1/2} \frac{dt}{d\lambda} = f(r)^{1/2} \frac{dt}{d\lambda}. \quad (29)$$

With (24) this becomes

$$\omega(r) = \frac{E_\infty}{\sqrt{f(r)}}. \quad (30)$$

For two static observers at radii r_e (emitter) and r_o (observer), the ratio of measured frequencies for the same photon is

$$\frac{\omega(r_o)}{\omega(r_e)} = \sqrt{\frac{f(r_e)}{f(r_o)}} = \sqrt{\frac{1 - \frac{2M}{r_e}}{1 - \frac{2M}{r_o}}}. \quad (31)$$

In particular, for an observer at infinity ($r_o \rightarrow \infty$, $f(r_o) \rightarrow 1$) one finds

$$\omega_\infty = \omega(r_e) \sqrt{1 - \frac{2M}{r_e}}. \quad (32)$$

This is the standard gravitational redshift in Schwarzschild spacetime: photons climbing out of the gravitational potential well are observed with lower frequency at infinity [3, 4].

5.3 Spinor and twistor representation

We now recast the above in the spinor language of Secs. 2 and 4. It is convenient to introduce an orthonormal tetrad ($e_{(a)}^\mu$) adapted to the static observers,

$$e_{(0)}^\mu = u^\mu = (f^{-1/2}, 0, 0, 0), \quad e_{(1)}^\mu = (0, f^{1/2}, 0, 0), \quad (33)$$

with $e_{(2)}^\mu$ and $e_{(3)}^\mu$ spanning the angular directions. From this tetrad one can construct Infeld–van der Waerden symbols σ_{AA}^μ and the associated spinor representation of vectors [1, 6].

For a radial null ray we choose a Newman–Penrose null tetrad ($\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu$) aligned with the radial direction, for example [2, 6]

$$\ell^\mu = (f^{-1}, 1, 0, 0), \quad n^\mu = \frac{1}{2}(1, -f, 0, 0), \quad (34)$$

with m^μ tangent to the two–spheres of symmetry. The photon wave vector for an outgoing radial ray can then be taken proportional to ℓ^μ ,

$$k^\mu = \kappa \ell^\mu, \quad (35)$$

where κ is a positive scalar factor related to E_∞ .

In the two–spinor formalism there exists a spin frame (o^A, ι^A) such that [1, 6]

$$\ell^{AA} = o^A \bar{o}^{\dot{A}}, \quad n^{AA} = \iota^A \bar{\iota}^{\dot{A}}, \quad (36)$$

and the complex vectors m^μ and \bar{m}^μ are represented by $o^A \bar{\iota}^{\dot{A}}$ and $\iota^A \bar{o}^{\dot{A}}$. In this frame the null wave vector becomes

$$k_{A\dot{A}} = \kappa o_A \bar{o}_{\dot{A}}, \quad (37)$$

so that in the notation of Sec. 2 we may identify

$$\lambda_A = \sqrt{\kappa} o_A, \quad \tilde{\lambda}_{\dot{A}} = \sqrt{\kappa} \bar{o}_{\dot{A}}. \quad (38)$$

The photon is thus described by a spinor dyad $(\lambda_A, \tilde{\lambda}_{\dot{A}})$ aligned with the outgoing radial null direction.

The four-velocity of a static observer can similarly be written in spinor form as

$$u^{A\dot{A}} = \alpha(r) o^A \bar{o}^{\dot{A}} + \beta(r) \iota^A \bar{\iota}^{\dot{A}}, \quad (39)$$

with real coefficients $\alpha(r)$ and $\beta(r)$ chosen so that $u^\mu u_\mu = -1$ and u^μ points along $e_{(0)}^\mu$. The measured frequency is then

$$\omega(r) = -u^{A\dot{A}} k_{A\dot{A}} = -\kappa \left[\alpha(r) o^A o_A \bar{o}^{\dot{A}} \bar{o}_{\dot{A}} + \beta(r) \iota^A o_A \bar{\iota}^{\dot{A}} \bar{o}_{\dot{A}} \right]. \quad (40)$$

Using the standard normalization $o^A \iota_A = 1$ and the fact that $o^A o_A = 0$ while $\iota^A o_A = 1$, this reduces to

$$\omega(r) = -\kappa \beta(r), \quad (41)$$

so that the radial dependence of the frequency is entirely encoded in the coefficient $\beta(r)$ that relates the observer's timelike direction to the null dyad. Matching to the vector calculation above fixes

$$\beta(r) = -\frac{E_\infty}{\kappa \sqrt{f(r)}}, \quad (42)$$

and we recover

$$\omega(r) = \frac{E_\infty}{\sqrt{f(r)}}. \quad (43)$$

In this way the familiar gravitational redshift arises from the changing overlap between the observer's four-velocity spinor $u^{A\dot{A}}$ and the null spinor dyad $(\lambda_A, \tilde{\lambda}_{\dot{A}})$ as a function of radius. The curvature of the Schwarzschild space-time controls this overlap via the spin connection, which governs the parallel transport of o^A and ι^A along both the photon trajectory and the observer's worldline.

6 Discussion: twistor correlations and persistent phase separation

The preceding sections contained only standard kinematics in curved space-time, recast in spinor and twistor language. A photon is represented by a null bispinor $k_{A\dot{A}} = \lambda_A \tilde{\lambda}_{\dot{A}}$ and a helicity-carrying field $\phi_{AB} \propto \lambda_A \lambda_B$; curvature acts through the spin connection to transport the underlying spinors, and the energy measured by an observer is $E = -\hbar u^\mu k_\mu$.

This setting admits a compact interpretation in terms of twistor correlations. The pair $(\lambda_A, \tilde{\lambda}_{\dot{A}})$ associated with a given null direction can be regarded as

defining a *twistor sector* of the field: a correlated configuration in the space of spinor degrees of freedom. Neighboring rays correspond to neighboring sectors, and their relative phase ΔS is determined by how the associated spinor pairs fail to match under parallel transport.

From this viewpoint, nonzero photon energy as seen by a given observer can be attributed to a *persistent phase gradient* between the observer’s four-velocity sector and the null twistor sector supporting the photon, as encoded by the scalar product $u^\mu k_\mu$ or, equivalently, by the spinorial contraction $u^{A\dot{A}}\lambda_A\tilde{\lambda}_{\dot{A}}$. Curvature does not create or destroy this correlation, but it reshapes it by altering the transport of the spinors along their respective curves. Gravitational redshift and lensing then appear as manifestations of how this persistent phase relation is deformed by curvature.

We stress that this interpretation does not add new dynamical content to general relativity or electrodynamics; it reorganizes the existing kinematics in a way that emphasizes the role of spinor/twistor structure and phase. In particular, the “splitting” of a spin-1 photon into two spin- $\frac{1}{2}$ objects is not a dynamical process induced by curvature, but a structural feature of the two-spinor formalism: curvature acts by transporting and correlating these spinors, thereby controlling the geometry of null rays and the energy they carry for different observers.

In future work one could extend this construction beyond the geometric optics limit, incorporate quantum aspects more explicitly (for example by promoting the spinor fields to annihilation and creation operators on twistor space), or explore whether similar “phase-separation” pictures can be formulated for other massless fields or for effective degrees of freedom in emergent gravity scenarios.

7 Conclusion

We have revisited the propagation of massless spin-1 fields in curved spacetime entirely within the two-spinor and twistor framework. The central structural point is that every null direction admits a bispinor factorization $k_{A\dot{A}} = \lambda_A\tilde{\lambda}_{\dot{A}}$, so that a photon of definite helicity can be represented as a correlated pair of spin- $\frac{1}{2}$ spinors with field strength $\phi_{AB} \propto \lambda_A\lambda_B$. On this basis, the familiar kinematics of geometric optics—null geodesic motion, the eikonal equation, and the observer-dependent energy $E = -\hbar u^\mu k_\mu$ —emerge directly from the parallel transport of the underlying spinors via the spin connection.

Within this strictly conventional setting we have emphasized an interpretational viewpoint in which photon energy is associated with a persistent phase gradient between twistor sectors. The contraction $u^{A\dot{A}}\lambda_A\tilde{\lambda}_{\dot{A}}$ encodes the overlap between the observer’s four-velocity sector and the null twistor dyad supporting the photon; gravitational redshift and related effects can then be seen as consequences of how curvature reshapes this overlap along null congruences.

The explicit example of radial photon propagation in Schwarzschild spacetime shows how this picture reproduces the standard formula for gravitational redshift while keeping the spinor/twistor structure manifest. In particular, the

redshift factor arises from the radial dependence of the coefficients that relate a static observer's spinor frame to the null dyad aligned with the photon, making the role of curvature in the measurement process transparent.

Several extensions suggest themselves. One could move beyond the geometric-optics limit and analyze corrections to the eikonal approximation in spinor language, incorporate quantized fields on twistor space more explicitly, or apply similar constructions to other massless fields. More speculatively, the “persistent phase separation” viewpoint might provide a useful organizing principle in approaches where spacetime geometry itself is emergent and twistor or spinor degrees of freedom are taken as more fundamental than the metric.

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