Metric Generation by Gravitational Rotors: Systematic Derivations and MOND Implications in a Biquaternionic Framework

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Abstract

We develop the explicit first–order calculus of the gravitational rotor $Q_g \in \mathrm{Spin}(1,3)$ and show how general–relativistic metrics, gravitational flows, and MOND–type galactic dynamics arise directly from its adjoint action in the Dirac–biquaternion (BQ) algebra. Starting from rapidity fields in the rotor exponent, we derive the Schwarzschild, de Sitter, and Kerr forms in Painlevé–Gullstrand coordinates, demonstrating that tetrads, the spin connection, and the metric follow from Q_g without invoking Christoffel symbols or second–order curvature equations. The Q_g current conservation law $D_\mu J^\mu = 0$, together with the Bernoulli–Noether constraint, selects the Constant–Lagrangian with Hubble boundary (CL–H) flow as the unique stationary solution for axisymmetric rotating disks. This flow yields an azimuthal velocity profile $v_\phi^2(r) = \frac{3}{2}\,w^2(R) - w^2(r)$, where $w(r)$ is the mixed Schwarzschild–Hubble river velocity, and produces a covariant acceleration law containing the Newtonian, MOND, and cosmological de Sitter regimes as limiting cases. The formalism therefore links the algebraic structure of the Dirac equation to relativistic metric construction and to observed galactic rotation curves within a single first–order gravitational framework.

Contents

1 Introduction

The dynamics of self–gravitating systems across a wide range of scales suggests that the conventional formulation of general relativity (GR) and Newtonian gravity does not always provide a complete phenomenological description. Galaxy rotation curves in particular exhibit a persistent discrepancy between baryonic mass distributions and observed orbital velocities. Since the seminal proposals of Milgrom [10, 11, 12], a substantial body of work has developed the Modified Newtonian Dynamics (MOND) paradigm and its relativistic extensions [15, 16, 17, 24, 18, 19, 20, 21, 22, 23, 14, 25, 26]. The empirical regularities uncovered in these studies, including the baryonic Tully–Fisher relation and the characteristic acceleration scale a_0, strongly suggest a link between galactic dynamics and cosmological boundary conditions.

Parallel to these developments, algebraic formulations of spacetime and spinor physics have provided alternative avenues for describing gravitational structure. Following the original algebraic insights of Dirac [6] and their geometric generalisation by Hestenes [7], the biquaternion (BQ) algebra has been shown to supply a unified and representation–independent construction of the Weyl and Dirac matrices and their Lorentz transformation operators [2]. When equipped with a local gravitational rotor $Q_g \in \mathrm{Spin}(1,3)$, the Dirac–BQ algebra admits a first–order formulation of gravitational geometry, in which tetrads, the spin connection, and the metric arise from the adjoint action of Q_g. Recent work has demonstrated that this structure leads to an exact linearisation of the Einstein equations and a closed first–order dynamical system for the gravitational frame [4]. The covariant treatment of the Dirac adjoint can be carried out within the same framework, resolving the mismatch between global and local time in curved spacetime [5].

The astrophysical consequences of this first–order rotor formalism have been explored in applications to galactic rotation curves. The "constant Lagrangian" (CL) condition first identified in [1] was recently reinterpreted within the full Q_g framework and tested empirically on late–type galaxies in the SPARC database [3]. In this formulation the gravitational river interpretation of Painlevé–Gullstrand geometries, together with the Bernoulli–Noether constraint on the Q_g current, leads to the CL–H (constant–Lagrangian with Hubble boundary) flow as a unique stationary solution for rotating disks. The resulting azimuthal velocity profile reproduces the observed transition from Newtonian to deep–MOND behaviour without introducing additional fields or interpolation functions.

The present paper develops the explicit calculus of the gravitational rotor Q_g underlying these results. We show how metrics of Schwarzschild, de Sitter, and Kerr type arise directly from rapidity fields; how the CL–H flow follows from the first–order conservation law $D_\mu J^\mu = 0$; and how the associated acceleration law naturally reproduces the MOND phenomenology within a fully covariant Dirac–BQ framework. This provides a unified algebraic pathway connecting the structure of the Dirac equation, the construction of general relativistic metrics, and the large–scale dynamics of galactic disks.

2 The metrics

2.1 Explicit construction of the Schwarzschild tetrad from the rotor rapidity

We work in the Dirac/BQ basis $\{\beta_\mu\}$ with Minkowski $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. The gravitational field is encoded by a *radial* rapidity $\psi_r(r)$ through the rotor

$$Q_g(r) = \exp\Big[\tfrac{1}{2}\,\psi_r(r)\,\beta_r\beta_0\Big], \qquad v_r(r) := \tanh\psi_r(r), \qquad \gamma(r) := \cosh\psi_r = \frac{1}{\sqrt{1 - v_r^2}}.$$

Its adjoint action defines the rotated basis $\hat\beta_a$ that enters $/G_\mu := Q_g\beta_\mu Q_g^{-1} = e_\mu{}^a\,\hat\beta_a$. For a pure boost in the $0-r$ plane we have the standard relations

$$Q_g\,\beta_0\,Q_g^{-1} = \gamma\,\beta_0 + \gamma v_r\,\beta_r, \tag{1}$$

$$Q_g\,\beta_r\,Q_g^{-1} = \gamma v_r\,\beta_0 + \gamma\,\beta_r, \tag{2}$$

$$Q_g\,\beta_\theta\,Q_g^{-1} = \beta_\theta, \qquad Q_g\,\beta_\phi\,Q_g^{-1} = \beta_\phi, \tag{3}$$

which can be written as a Lorentz matrix acting on the tangent index $(a, b = 0, r, \theta, \phi)$:

$$\hat\beta_a = \Lambda_a{}^b(\psi_r)\,\beta_b, \qquad \Lambda(\psi_r) = \begin{pmatrix} \gamma & \gamma v_r & 0 & 0 \\ \gamma v_r & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Coordinate coframe. Adopt spherical coordinates $x^\mu = (t, r, \theta, \phi)$ with the diagonal (co)frame

$$E_{\mathrm{diag}} = \mathrm{diag}(1,\ 1,\ r,\ r\sin\theta),$$

meaning the one–forms $\theta^0 = dt,\ \theta^r = dr,\ \theta^\theta = r\,d\theta,\ \theta^\phi = r\sin\theta\,d\phi$.

Lorentz–display tetrad. Combine the diagonal frame with the boost on the right to obtain the "Lorentz display"

$$E_{\mathrm{Lor}} = E_{\mathrm{diag}}\Lambda(\psi_r) = \begin{pmatrix} \gamma & \gamma v_r & 0 & 0 \\ \gamma v_r & \gamma & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}.$$

This is the tetrad $e_\mu{}^a$ such that $/G_\mu = e_\mu{}^a\hat\beta_a$. Contracting with η_{ab} yields the metric $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b\eta_{ab}$, which already displays a mixed $t-r$ block induced by the boost.

Gauge to Painlevé–Gullstrand (PG). To expose the river/shift form, act with a lower–triangular *coframe* gauge on the μ-index (left multiplication)

$$S = \begin{pmatrix} 1 & -v_r & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad E_{\rm PG} := S\, E_{\rm Lor}.$$

A short multiplication, using $\gamma(1 - v_r^2) = 1/\gamma$, shows that the first two rows can be written in the clean ADM/PG form with unit lapse and shift $N^r = v_r$:

$$\boxed{E_{\rm PG}(r,\theta) = \begin{pmatrix} 1 & v_r(r) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}}.$$

Interpreting rows as one–forms, this corresponds to $\theta^0 = dt$, $\theta^r = dr - v_r\, dt$, $\theta^\theta = r\, d\theta$, $\theta^\phi = r\sin\theta\, d\phi$.

Metric reconstruction. From $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ with the PG tetrad we obtain immediately

$$\boxed{ds^2 = -dt^2 + \big(dr - v_r(r)\, dt\big)^2 + r^2\, d\Omega^2,} \qquad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

Schwarzschild identification. Choosing the *free–fall (river) speed*

$$v_r(r) = \sqrt{\frac{2GM}{r}} = \tanh\psi_r(r)$$

yields the standard Schwarzschild metric in PG coordinates,

$$\boxed{ds^2 = -dt^2 + \Big(dr - \sqrt{\tfrac{2GM}{r}}\, dt\Big)^2 + r^2 d\Omega^2,}$$

constructed here *entirely* from the rotor rapidity. The dimensionful ($c \neq 1$) Painlevé–Gullstrand metric derived from the rotor can be written as:

$$\boxed{ds^2 = -c^2 dt^2 + \Big(dr - \sqrt{\tfrac{2GM}{r}}\, dt\Big)^2 + r^2 d\Omega^2,}$$

In summary:

$$Q_g(\psi_r) \xrightarrow{\text{adjoint}} \hat{\beta}_a, \qquad E_{\rm diag}\cdot\Lambda(\psi_r) \xrightarrow{S} E_{\rm PG} \xrightarrow{\eta} g_{\mu\nu}.$$

This shows explicitly, step by step, how a single radial rapidity $\psi_r(r)$ determines the tetrad E and hence the Schwarzschild geometry.

2.2 Pure Hubble–Flow Rotor (ψ_H) in Spherical Symmetry

We now derive the line element corresponding to a purely Hubble–type flow, generated by a single rapidity field $\psi_H(t,r)$ that defines a local velocity

$$v_r(t,r) = H(t)\, r, \qquad \tanh\psi_H(t,r) = \frac{v_r}{c} = \frac{H(t)\, r}{c}.$$

This corresponds to the case where the gravitational rotor represents a purely cosmological expansion, without additional local curvature.

1. Rotor and adjoint action. The gravitational rotor acts as a local boost in the (t, r) plane:

$$Q_g(t,r) = \exp\left[-\frac{1}{2}\,\psi_H(t,r)\,\beta_0\beta_r\right].$$

Its adjoint action transforms the Dirac basis elements as

$$\tilde{\beta}_0 = Q_g\,\beta_0\,Q_g^{-1} = \cosh\psi_H\,\beta_0 + \sinh\psi_H\,\beta_r, \qquad \tilde{\beta}_r = Q_g\,\beta_r\,Q_g^{-1} = \cosh\psi_H\,\beta_r + \sinh\psi_H\,\beta_0,$$

with $\tanh\psi_H = v_r/c$ and $\gamma_H = \cosh\psi_H = (1 - v_r^2/c^2)^{-1/2}$. This operation defines the local coframe of observers comoving with the cosmic flow.

2. Tetrad (orthonormal coframe). In spherical coordinates, the orthonormal one–forms can be written directly as

$$\boxed{\theta^0 = c\,dt, \qquad \theta^1 = dr - v_r\,dt = dr - H(t)\,r\,dt, \qquad \theta^2 = r\,d\theta, \qquad \theta^3 = r\sin\theta\,d\phi.}$$

These one–forms are the rows of the tetrad $e^a{}_\mu$ defined by $\theta^a = e^a{}_\mu\,dx^\mu$. Explicitly,

$$e^a{}_\mu = \begin{pmatrix} c & 0 & 0 & 0 \\ -\,v_r & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}, \qquad dx^\mu = (dt,\,dr,\,d\theta,\,d\phi).$$

3. Metric from the coframe. Using the orthonormal form of the line element,

$$ds^2 = \eta_{ab}\,\theta^a\theta^b = -(\theta^0)^2 + (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2,$$

we obtain the metric for the Hubble–flow case:

$$\boxed{ds^2 = -c^2\,dt^2 + \big(dr - H(t)\,r\,dt\big)^2 + r^2\,d\Omega^2, \qquad d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2.}$$

Expanding the square gives the mixed form,

$$\boxed{ds^2 = -\big(c^2 - H^2 r^2\big)\,dt^2 - 2H\,r\,dr\,dt + dr^2 + r^2\,d\Omega^2.}$$

This is the Painlevé–Gullstrand form of a spatially–flat cosmological metric.

4. Transformation to comoving coordinates. Introduce the comoving coordinate χ via $r = a(t)\,\chi$, where $a(t)$ is the cosmological scale factor and $H(t) = \dot{a}/a$. Then

$$dr = a\,d\chi + \dot{a}\,\chi\,dt = a\,d\chi + H\,r\,dt.$$

Substituting this into the expression for the metric, we find

$$(dr - H\,r\,dt)^2 = (a\,d\chi)^2,$$

and thus

$$\boxed{ds^2 = -c^2\,dt^2 + a^2(t)\big(d\chi^2 + \chi^2\,d\Omega^2\big).}$$

This is exactly the spatially–flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric, showing that the Hubble rapidity field ψ_H reproduces cosmological expansion.

5. Interpretation and validity.

- The rotor rapidity ψ_H is locally well–defined wherever $|H(t)\,r| < c$. The surface where $|H(t)\,r| = c$ corresponds to the cosmological horizon.
- The Q_g rotor here generates the orthonormal coframe algebraically, without assuming a pre–defined curvature tensor. The geometry emerges directly from the rotor action.
- Setting $H = \text{const}$ reproduces the Painlevé–Gullstrand form of de Sitter space. Taking $H(t) = \dot{a}/a$ gives the general flat FRW cosmology.

Summary. With only the Hubble rapidity active, the Q_g rotor produces the orthonormal coframe

$$\theta^0 = c\,dt, \qquad \theta^1 = dr - H(t)\,r\,dt,$$

whose metric is

$$\boxed{ds^2 = -c^2\,dt^2 + (dr - H(t)\,r\,dt)^2 + r^2\,d\Omega^2.}$$

This is the Painlevé–Gullstrand form of the spatially–flat FRW line element, and transforms exactly to the standard form

$$ds^2 = -c^2\,dt^2 + a^2(t)\,d\vec{x}^{\,2}$$

under the substitution $r = a(t)\,\chi$. Thus, the cosmological geometry arises purely from the rotor–defined Hubble rapidity field, within the linear Q_g algebra.

2.3 Combined radial rapidities ($\psi_M + \psi_H$): Schwarzschild inflow plus Hubble outflow

We now superpose the two radial rapidities derived in the previous subsections: a local Schwarzschild / free–fall inflow $\psi_M(r)$ and a Hubble outflow $\psi_H(t)$. Because both boosts act in the same (t, r) plane, their rapidities *add*:

$$\psi_{\text{tot}}(t,r) = \psi_M(r) + \psi_H(t), \qquad \tanh\psi_{\text{tot}} = \frac{w(t,r)}{c},$$

with the total physical radial *flow speed*

$$w(t,r) = v_M(r) - v_H(t,r) = \sqrt{\frac{2GM}{r}} \;-\; H(t)\,r,$$

where $v_M(r) = \sqrt{2GM/r}$ (inflow magnitude) and $v_H(t,r) = H(t)r$.

1. Rotor and adjoint action. The gravitational rotor for collinear boosts is

$$Q_g(t,r) = \exp\!\left[-\frac{1}{2}\,\psi_{\text{tot}}(t,r)\,\beta_0\beta_r\right],$$

so that

$$\tilde{\beta}_0 = Q_g\beta_0Q_g^{-1} = \cosh\psi_{\text{tot}}\,\beta_0 + \sinh\psi_{\text{tot}}\,\beta_r, \quad \tilde{\beta}_r = Q_g\beta_rQ_g^{-1} = \cosh\psi_{\text{tot}}\,\beta_r + \sinh\psi_{\text{tot}}\,\beta_0,$$

with $\tanh\psi_{\text{tot}} = w/c$ and $\gamma_{\text{tot}} = \cosh\psi_{\text{tot}} = (1 - w^2/c^2)^{-1/2}$.

2. Orthonormal coframe (tetrad). Exactly as in the single–rapidity cases, the orthonormal one–forms $\theta^a = e^a{}_\mu dx^\mu$ adapted to the flow are

$$\boxed{\theta^0 = c\,dt, \qquad \theta^1 = dr - w(t,r)\,dt = dr - \big(\sqrt{2GM/r} - H(t)r\big)\,dt, \qquad \theta^2 = r\,d\theta, \qquad \theta^3 = r\sin\theta\,d\phi.}$$

Equivalently,

$$e^a{}_\mu = \begin{pmatrix} c & 0 & 0 & 0 \\ -w & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}, \qquad (x^\mu) = (t, r, \theta, \phi).$$

3. Metric in Painlevé–Gullstrand (PG) form. Using $ds^2 = \eta_{ab}\,\theta^a\theta^b$ with $\eta = \mathrm{diag}(-1,1,1,1)$,

$$\boxed{ds^2 = -\,c^2\,dt^2 + \big(dr - w\,dt\big)^2 + r^2\,d\Omega^2, \qquad d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2.}$$

Expanding,

$$\boxed{ds^2 = -(c^2 - w^2)\,dt^2 - 2w\,dr\,dt + dr^2 + r^2\,d\Omega^2, \quad w(t,r) = \sqrt{\frac{2GM}{r}} - H(t)r.}$$

Thus,

$$g_{tt} = -(c^2 - w^2), \quad g_{tr} = g_{rt} = -w, \quad g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2\sin^2\theta.$$

4. Inverse metric and 3+1 split. The inverse in the (t,r) block is

$$g^{tt} = -\frac{1}{c^2}, \qquad g^{tr} = g^{rt} = -\frac{w}{c^2}, \qquad g^{rr} = 1 - \frac{w^2}{c^2},$$

with $g^{\theta\theta} = 1/r^2$ and $g^{\phi\phi} = 1/(r^2\sin^2\theta)$. In 3+1 language this is a flat spatial metric in PG gauge with shift vector $\vec{w} = w\,\hat{\mathbf{r}}$ and unit lapse.

5. Limits, horizons and remarks.

- **Schwarzschild limit:** $H \to 0 \Rightarrow w = \sqrt{2GM/r}$, recovering the Schwarzschild–PG line element.
- **Cosmological limit:** $M \to 0 \Rightarrow w = -H(t)r$, giving the PG form of spatially–flat FRW; with $r = a(t)\chi$ it reduces to $-c^2dt^2 + a^2(t)\,d\vec{x}^2$.
- **Horizon condition:** the local flow speed bound $|w(t,r)| = c$ marks a PG horizon (black–hole, cosmological, or combined "McVittie–like" surface).
- **Superposition caveat:** the algebra shows rapidities add because the boosts are collinear. The resulting metric is the natural PG–slice realisation of a central mass embedded in an expanding background.

Summary. With both rapidities active, the Q_g rotor yields the single PG–type metric

$$\boxed{ds^2 = -c^2dt^2 + \big(dr - [\sqrt{2GM/r} - H(t)r]\,dt\big)^2 + r^2d\Omega^2,}$$

which reduces smoothly to Schwarzschild–PG or flat FRW in the appropriate limits, and keeps c explicit throughout.

2.4 Adding an azimuthal flow: ψ_r (inflow$\oplus$Hubble) plus ψ_ϕ (rotation)

We now extend the PG coframe to include an *unspecified* azimuthal velocity v_ϕ (with rapidity ψ_ϕ, $\tanh\psi_\phi = v_\phi/c$) in addition to the radial flow built from the Schwarzschild inflow and the Hubble outflow. As before, we keep the speed of light c explicit (SI units).

1. Kinematic inputs and sign conventions. We adopt the PG "river" sign convention where the metric is built from orthonormal one–forms θ^a carrying a shift (flow) relative to the static coordinate basis (t, r, θ, ϕ):

$$\theta^0 = c\,dt, \qquad \theta^1 = dr - w(t,r)\,dt, \qquad \theta^2 = r\,d\theta, \qquad \theta^3 = r\sin\theta\big(d\phi - \Omega(t,r,\theta)\,dt\big).$$

Here

$$w(t,r) \;=\; \underbrace{\sqrt{\frac{2GM}{r}}}_{\text{Schwarzschild inflow}} \;-\; \underbrace{H(t)\,r}_{\text{Hubble outflow}}\;, \qquad \Omega(t,r,\theta) \;:=\; \frac{v_\phi(t,r,\theta)}{r\sin\theta},$$

so that v_ϕ is the physical azimuthal speed and Ω the corresponding angular velocity (left unspecified throughout). With this sign choice, inward gravitational flow $(+\sqrt{2GM/r})$ competes against outward Hubble flow $(-Hr)$ inside the same radial shift w.

2. Orthonormal coframe and tetrad. The one–forms above define the orthonormal coframe $\{\theta^a\}$ and the tetrad $e^a{}_\mu$ via $\theta^a = e^a{}_\mu dx^\mu$:

$$\boxed{\theta^0 = c\,dt, \quad \theta^1 = dr - w\,dt, \quad \theta^2 = r\,d\theta, \quad \theta^3 = r\sin\theta\,(d\phi - \Omega\,dt).}$$

Equivalently,

$$e^a{}_\mu = \begin{pmatrix} c & 0 & 0 & 0 \\ -\,w & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ -\,r\sin\theta\,\Omega & 0 & 0 & r\sin\theta \end{pmatrix}, \qquad dx^\mu = (dt, dr, d\theta, d\phi).$$

(We refrain from composing non–collinear boosts at the rotor level to avoid Wigner rotations; in PG gauge the full kinematics is cleanly encoded by the *shift* components w and $r\sin\theta\,\Omega$ in the coframe.)

3. Metric with radial *and* azimuthal shift. Using $ds^2 = \eta_{ab}\,\theta^a\theta^b$ with $\eta = \mathrm{diag}(-1,1,1,1)$, we obtain the PG–type line element with two flows:

$$\boxed{ds^2 = -c^2\,dt^2 + \big(dr - w\,dt\big)^2 + r^2\,d\theta^2 + r^2\sin^2\theta\big(d\phi - \Omega\,dt\big)^2.}$$

Expanding,

$$\boxed{\begin{aligned} ds^2 = &-\Big(c^2 - w^2 - r^2\sin^2\theta\,\Omega^2\Big)\,dt^2 - 2\,w\,dr\,dt - 2\,r^2\sin^2\theta\,\Omega\,d\phi\,dt \\ &+\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2, \end{aligned}}$$

with

$$\boxed{w = \sqrt{\frac{2GM}{r}} - H(t)\,r, \quad \Omega = \frac{v_\phi}{r\sin\theta}.}$$

Hence the nonzero metric components are

$$\begin{aligned} &g_{tt} = -(c^2 - w^2 - r^2\sin^2\theta\,\Omega^2), \quad g_{tr} = g_{rt} = -\,w, \quad g_{t\phi} = g_{\phi t} = -\,r^2\sin^2\theta\,\Omega, \\ &g_{rr} = 1, \qquad g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = r^2\sin^2\theta. \end{aligned}$$

4. Inverse metric and 3+1 interpretation. In PG gauge the lapse is unity, and the *shift one–form* is

$$\beta_\mu \, dx^\mu = w \, dr + r^2 \sin^2\theta \, \Omega \, d\phi,$$

so the spatial slices are flat in (r, θ, ϕ) while the flow lives in the shift. The inverse metric in the (t, r, ϕ) block is

$$g^{tt} = -\frac{1}{c^2}, \qquad g^{tr} = g^{rt} = -\frac{w}{c^2}, \qquad g^{t\phi} = g^{\phi t} = -\frac{\Omega}{c^2}, \qquad g^{rr} = 1 - \frac{w^2}{c^2}, \qquad g^{\phi\phi} = \frac{1}{r^2 \sin^2\theta} - \frac{\Omega^2}{c^2},$$

with $g^{\theta\theta} = 1/r^2$. (These follow from the standard 3+1 identities with unit lapse and flat 3–metric.)

5. Limits, horizons, and remarks.

- **Pure Schwarzschild PG:** $H = 0,\ \Omega = 0 \Rightarrow ds^2 = -c^2 dt^2 + \big(dr - \sqrt{2GM/r}\, dt\big)^2 + r^2 d\Omega^2$.
- **Flat FRW in PG gauge:** $M = 0,\ \Omega = 0 \Rightarrow ds^2 = -c^2 dt^2 + \big(dr - Hr\, dt\big)^2 + r^2 d\Omega^2$, which becomes $-c^2 dt^2 + a^2(t)\, d\vec{x}^{\,2}$ under $r = a(t)\chi$.
- **Pure azimuthal flow:** $w = 0 \Rightarrow ds^2 = -(c^2 - r^2 \sin^2\theta\, \Omega^2) dt^2 - 2r^2 \sin^2\theta\, \Omega\, d\phi\, dt + \cdots$, a PG slicing with frame–drag–like cross term $g_{t\phi}$ sourced purely by the chosen v_ϕ.
- **Effective horizon (PG):** where the *net* flow speed $\sqrt{w^2 + v_\phi^2}$ reaches c, the Killing vector ∂_t changes character and $g_{tt} \to 0$:

$$c^2 = w^2 + r^2 \sin^2\theta\, \Omega^2 \quad \Leftrightarrow \quad c^2 = w^2 + v_\phi^2.$$

- **On rapidities:** one may define $\tanh\psi_r = w/c$ and $\tanh\psi_\phi = v_\phi/c$. Non–collinear boosts do not add linearly at the rotor level (Wigner rotation). The PG construction sidesteps this by encoding both flows as shift components in the coframe, which is algebraically exact for the metric.

Summary. With a general azimuthal velocity v_ϕ left unspecified and a radial flow $w = \sqrt{2GM/r} - H(t)r$ (inflow + outflow, with $v_H < 0$), the combined PG–type metric reads

$$\boxed{ds^2 = -c^2\, dt^2 + \big(dr - w\, dt\big)^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, \big(d\phi - \Omega\, dt\big)^2, \quad \Omega = \frac{v_\phi}{r \sin\theta}.}$$

This reduces to each single–rapidity case in the appropriate limit and keeps all dependences on c, M, $H(t)$, and v_ϕ explicit.

2.5 Existing GR Metrics and the Role of Ω in the Q_g Framework

Before proceeding to the derivation of the Constant–Lagrangian–Hubble (CL–H) metric, it is instructive to identify which of the previously constructed flow metrics correspond to known general–relativistic solutions and to clarify why the azimuthal component Ω cannot be left arbitrary. This also allows us to explain why, in the Q_g framework, the Bernoulli–Noether Constant (BNC) or the current conservation law $D_\mu J^\mu = 0$ automatically ensures compatibility with the Einstein equations.

1. Correspondence with known GR metrics. The Painlevé–Gullstrand–type line element

$$ds^2 = -c^2\,dt^2 + (dr - w\,dt)^2 + r^2 d\theta^2 + r^2 \sin^2\theta\,(d\phi - \Omega\,dt)^2,$$

represents a general flow field of radial velocity $w(r,t)$ and azimuthal velocity $v_\phi = r\sin\theta\,\Omega(r,\theta,t)$. Different specifications of these functions reproduce well–known general–relativistic metrics:

- $w = \sqrt{2GM/r},\ \Omega = 0$: the Schwarzschild metric in Painlevé–Gullstrand form.
- $w = -H(t)r,\ \Omega = 0$: the spatially flat FRW (or de Sitter) metric in PG coordinates.
- $w = \sqrt{2GM/r} - Hr,\ \Omega = 0$: for constant H, this is equivalent to the Schwarzschild–de Sitter (Kottler) solution; for variable $H(t)$ it approaches the McVittie family, but is no longer an exact GR solution unless special conditions are met.
- $w = \sqrt{2GM/r},\ \Omega \neq 0$: only when Ω satisfies the specific Kerr frame–dragging law does this metric reproduce the Kerr geometry (or Kerr–de Sitter if $H \neq 0$). Arbitrary Ω otherwise violates the Einstein field equations and represents merely a kinematic ansatz.

2. Why Ω cannot be arbitrary in GR. In standard general relativity, the ten coupled field equations $G_{\mu\nu} = (8\pi G/c^4)\,T_{\mu\nu}$ fix the functional dependence of w and Ω. In vacuum ($T_{\mu\nu} = 0$), only a very restricted set of $\Omega(r,\theta)$ functions lead to consistent solutions:

$$\begin{aligned}\text{Kerr:}\quad \Omega(r,\theta) &= \frac{ar}{r^2 + a^2\cos^2\theta},\\ \text{Kerr–de Sitter:}\quad \Omega(r,\theta) &= \frac{a(1-\Lambda r^2/3)}{r^2 + a^2\cos^2\theta}.\end{aligned}$$

Any other choice of Ω fails to satisfy $G_{\mu\nu} = 0$ and therefore represents only a formal *ansatz* rather than a physical space–time. Hence, in GR, the rotational flow law is tightly constrained by the curvature equations.

3. The $\mathbf{Q}_g$ replacement for Einstein's constraints. In the Q_g framework, the governing equations are first–order in form:

$$D_\mu J^\mu = 0, \qquad M_\mu{}^\nu = E_\mu{}^a\,\Phi_a{}^\nu, \qquad J^\nu = u^\mu M_\mu{}^\nu.$$

Here, the conserved adjoint current J^ν plays the role of a combined energy–momentum and curvature carrier. The radial and azimuthal flow components (w,Ω) are no longer arbitrary metric functions, but dynamical quantities constrained by the conservation law itself. The Bernoulli–Noether constant (BNC) fixes the relation between energy flow and metric generation in each direction,

$$\mathcal{L}_{\text{BNC}} = \text{const.} \;\Rightarrow\; \Psi^\dagger Q_g\,\beta_\mu Q_g^{-1}\,\beta^\mu\Psi = \text{inv.},$$

which, when decomposed along radial and azimuthal coordinates, produces the required $\Omega(r,\theta)$ that maintains covariant current balance.

4. Automatic consistency with Einstein's field equations. Because the Q_g adjoint and its current are constructed to obey $D_\mu J^\mu = 0$, their solutions automatically satisfy the differential identities underlying the Einstein tensor, $D_\mu G^{\mu\nu} = 0$. The conservation law therefore replaces the explicit metric constraint equations of GR by an algebraic condition on the rotor field. When J^ν and $M_\mu{}^\nu$ are self–consistent, the induced metric $g_{\mu\nu} = \langle Q_g\beta_\mu Q_g^{-1}\beta_\nu\rangle_S$ has the same curvature structure as a corresponding Einstein solution. Hence, the Q_g field does not require an external verification step by $G_{\mu\nu} = 0$; its conservation laws implicitly ensure compliance.

5. Summary and implication for the CL–H metric.

- The radial and azimuthal flow forms of the previous section reproduce known GR metrics (Schwarzschild, de Sitter, Kottler, Kerr) when their flow functions satisfy the Einstein constraints.
- Arbitrary Ω is not a solution of GR and only a kinematic ansatz.
- In the Q_g rotor formalism, Ω (and v_ϕ) follow uniquely from the covariant conservation of the adjoint current $D_\mu J^\mu = 0$, which ensures that all resulting metrics satisfy the Einstein identities automatically.

This establishes that the Constant–Lagrangian–Hubble (CL–H) metric derived from Q_g represents a dynamically constrained flow geometry, not an arbitrary combination of Schwarzschild, Hubble, and rotational terms. The BNC and continuity conditions play the role of Einstein's constraints, guaranteeing that the resulting space–time remains fully covariant and physically consistent.

3 CL–H specification of the azimuthal flow v_ϕ from the rotor, BNC, and continuity

We give a step–by–step derivation—from the rotor level to an explicit $v_\phi(r)$—of the Constant–Lagrangian with Hubble boundary (CL–H) azimuthal law that, together with the radial CL–H flow, solves the Q_g conservation law $D_\mu J^\mu = 0$ and the Bernoulli–Noether constraint (BNC). In the $Q_g \to$GR translation, this construction reproduces the known Schwarzschild/de Sitter limits and satisfies Einstein's constraint identities (Bianchi) in the appropriate regimes. The derivation follows the algorithmic steps laid out in the collected notes.

3.1 Rotor and adjoint: from rapidities to the PG coframe

Consider collinear boosts in the (t, r)–plane generated by the total rapidity $\psi_{\text{tot}} = \psi_M(r) + \psi_H(t)$, with physical radial flow

$$w(t, r) = \sqrt{\frac{2GM}{r}} \; - \; H(t)\, r.$$

The gravitational rotor and its adjoint action are

$$Q_g = \exp\Big[-\tfrac{1}{2}\, \psi_{\text{tot}}\, \beta_0 \beta_r \Big], \qquad /G_\mu = Q_g\, \beta_\mu\, Q_g^{-1}.$$

In Painlevé–Gullstrand (PG) gauge the orthonormal one–forms (unit lapse) are

$$\theta^0 = c\, dt, \qquad \theta^1 = dr - w\, dt, \qquad \theta^2 = r\, d\theta, \qquad \theta^3 = r \sin\theta\, (d\phi - \Omega\, dt),$$

with $\Omega = v_\phi/(r \sin\theta)$. The metric then reads

$$\boxed{ds^2 = -c^2 dt^2 + (dr - w\, dt)^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, (d\phi - \Omega\, dt)^2.} \tag{4}$$

3.2 Dirac bilinears and the BNC

Inside the BQ/Dirac algebra define the density and spatial current bilinears

$$\rho := \Psi^\dagger \Psi, \qquad j^a := \Psi^\dagger \hat{\beta}^{\,a} \Psi = \rho(\gamma, \gamma v_i),$$

so that $u^a = j^a/\rho = (\gamma, \gamma v_i)$ is the normalized 4–velocity. For any Killing vector ξ, the Bernoulli–Noether scalar invariant is (PG time symmetry $\xi = \partial_t$)

$$B[\xi] := \frac{1}{\rho}\,\Psi^\dagger(\xi^\mu/G_\mu)\Psi = u^a\xi_a = \gamma\,(1 - w_i v^i). \tag{5}$$

In spherical/axisymmetric situations with radial river $w_i = (w, 0, 0)$ this simplifies to

$$\boxed{B = \gamma\,(1 - w\,v_r) = \text{const.}} \tag{6}$$

Equations (4)–(6) summarize the rotor/coframe and BNC ingredients we will use.

3.3 Low–velocity reduction and the Bernoulli invariant

In the weak–field, galactic regime $v_r^2 + v_\phi^2 \ll c^2$ we expand $\gamma \simeq 1 + \frac{1}{2}(v_r^2 + v_\phi^2)/c^2$ and obtain from (6)

$$B \simeq 1 + \frac{v_r^2 + v_\phi^2}{2c^2} - \frac{w\,v_r}{c^2}.$$

Up to an additive background normalization (lapse) that can be absorbed in the constant of motion,[1] we arrive at the *Bernoulli invariant* in PG form:

$$\boxed{v_\phi^2 + (v_r - w)^2 = C, \qquad C = \text{const along streamlines.}} \tag{7}$$

This is the CL–H form of the Bernoulli–Noether constant in the low–velocity limit.

3.4 Boundary condition at the smooth bulge edge $r = R$

For a regular bulge matched to a thin disk, impose at the bulge–disk interface $r = R$ a virial circular condition in the PG frame:

$$v_\phi^2(R) = v_{\text{orb}}^2(R) = \frac{1}{2}\left(\sqrt{\tfrac{2GM}{R}} - H_z\,R\right)^2 = \frac{1}{2}w^2,$$

so that the Bernoulli constant becomes

$$\boxed{C = v_\phi^2(R) + (\sqrt{\tfrac{2GM}{R}} - H_z R)^2 = \frac{3}{2}\left(\sqrt{\tfrac{2GM}{R}} - H_z\,R\right)^2 = \frac{3}{2}w(R)^2.} \tag{8}$$

Here H_z is the (epoch–dependent) Hubble rate at which the disk is embedded; the same subtractions appear inside the bulge so that the matching is smooth.

3.5 CL–H azimuthal law $v_\phi(r)$ and effective radial speed

In the exterior disk the matter is nearly steady in the PG frame, so $v_r \simeq 0$ (the inflow is carried by the geometry). Using (7) and (8) we obtain the explicit CL–H azimuthal profile

$$\boxed{v_\phi^2(r) = \frac{3}{2}\left(\sqrt{\tfrac{2GM}{R}} - H_z R\right)^2 - \left(\sqrt{\tfrac{2GM}{r}} - H_z r\right)^2, \qquad R \le r \le r_c,} \tag{9}$$

together with the *effective* radial term relative to the river,

$$\boxed{v_{r,\text{eff}}^2(r) = (\sqrt{\tfrac{2GM}{r}} - H_z r)^2,} \tag{10}$$

[1]The conserved streamline combination is $(v_r - w)^2 + v_\phi^2 - w^2 = \text{const}$, so the w^2 piece does not disappear even if $v_r = 0$; see the detailed note. :contentReference[oaicite:2]index=2

and the turnaround radius r_c from $v_{r,\text{eff}}(r_c) = 0$:

$$\boxed{r_c = \left(\frac{2GM}{H_z^2}\right)^{1/3}, \qquad v_\phi^2(r_c) = \frac{3}{2}\left(\sqrt{\frac{2GM}{R}} - H_z R\right)^2.} \tag{11}$$

These are precisely the CL–H forms used in the fits and in the MOND–limit discussion. The most concice expression of the CL-H condition is

$$\boxed{v_\phi^2(r) + w(r)^2 = v_L^2 \qquad \frac{L}{2m} = v_L^2 = \frac{3}{2}w(R)^2 \qquad R \le r \le r_c} \tag{12}$$

thus the name 'constant Lagrangian' with the Hubble parameter included CL-H.

3.6 Pitch–angle rule from the BNC: $\tan\alpha = w^2/v_\phi^2$

The Bernoulli–Noether constant (BNC) in the low–velocity CL–H regime gives the streamline invariant

$$v_\phi^2 + (v_r - w)^2 \;=\; C \quad \text{(constant } v_L \text{ along a streamline).} \tag{13}$$

In the thin, nearly steady disk outside the bulge we have $v_r \simeq 0$ and we can write $C = v_L^2$, hence

$$v_\phi^2(r) + w^2(r) \;=\; v_L^2, \qquad w(r) = \sqrt{\frac{2GM}{r}} - H_z r. \tag{14}$$

Introduce a *pitch* (or *partition*) angle $\alpha(r) \in (0, \pi/2)$ that encodes how the BNC budget C is split between the azimuthal motion (v_ϕ) and the radial *relative* motion with respect to the river (w). We *define* α by the *energy–fraction* (square–amplitude) projection:

$$\frac{v_\phi^2(r)}{v_L^2} \;=\; \cos\alpha(r), \qquad \frac{w^2(r)}{v_L^2} \;=\; \sin\alpha(r), \tag{15}$$

which is consistent with (14) and makes α a purely geometric label of the BNC partition. From (15) we obtain immediately the *pitch–angle rule*

$$\boxed{\tan\alpha(r) \;=\; \frac{w^2(r)}{v_\phi^2(r)}} \tag{16}$$

and, using $v_\phi^2(r) = v_L^2 - w^2(r)$,

$$\tan\alpha(r) \;=\; \frac{w^2(r)}{v_L^2 - w^2(r)}. \tag{17}$$

Why the rule follows from the BNC. Equation (14) is the low–velocity form of the BNC, i.e. the covariant Bernoulli constant obtained from the rotor adjoint and the conserved Dirac current. It states that, along a streamline, the sum of the squared *relative radial* speed (with respect to the river) and the squared azimuthal speed is conserved. Therefore the *entire* partition between these two channels is fixed by a single angle α, and the natural "direction cosines" are the *ratios of squared amplitudes* (energy fractions), not the raw velocities. This is why $\tan\alpha$ comes out as a *square ratio* w^2/v_ϕ^2 rather than w/v_ϕ.

CL–H specialization. With the CL–H boundary at the bulge rim $r = R$ (smooth virial matching), one has $v_\phi^2(R) = \frac{1}{2}w^2(R)$ (see Sec. §4–5 in the CL–H derivation). Hence, at $r = R$,

$$\tan\alpha(R) \;=\; \frac{w^2(R)}{v_\phi^2(R)} \;=\; 2 \quad\Longrightarrow\quad \alpha(R) \;=\; \arctan 2 \;\approx\; 63.43^\circ. \tag{18}$$

Away from the rim, $w(r)$ evolves as $w(r) = \sqrt{2GM/r} - H_z r$ while C is fixed by the boundary, so $\alpha(r)$ follows from (17). The pitch therefore *tracks* the local redistribution between azimuthal support and (relative) radial river contribution under the single BNC invariant.

Summary. The BNC fixes the quadratic partition $v_\phi^2 + w^2 = v_L^2$ along streamlines. Defining the pitch angle α by energy–fraction projections yields the CL–H pitch rule $\tan\alpha = w^2/v_\phi^2$. This provides a compact diagnostic of how the Hubble–subtracted free–fall w and the azimuthal support v_ϕ co–determine the disk kinematics in the conserved Q_g flow.

Pitch angle in rapidity variables In the CL–H disk, the Bernoulli–Noether constant fixes

$$v_\phi^2(r) + w^2(r) = C, \qquad \tan\alpha(r) = \frac{w^2(r)}{v_\phi^2(r)}.$$

Expressing the velocities in terms of rapidities,

$$\tanh\psi_r(r) = \frac{w(r)}{c}, \qquad \tanh\psi_\phi(r) = \frac{v_\phi(r)}{c},$$

we obtain

$$\frac{w^2(r)}{v_\phi^2(r)} = \frac{c^2\tanh^2\psi_r(r)}{c^2\tanh^2\psi_\phi(r)} = \frac{\tanh^2\psi_r(r)}{\tanh^2\psi_\phi(r)}.$$

Hence the pitch angle can be written purely in rapidity variables as

$$\boxed{\sqrt{\tan\alpha(r)} = \frac{\tanh\psi_r(r)}{\tanh\psi_\phi(r)},} \tag{19}$$

This shows that the CL–H pitch angle is entirely determined by the ratio of the squared rapidity tanh functions associated with the radial river (ψ_r) and the azimuthal flow (ψ_ϕ), making the BNC partition explicitly relativistic at the rotor level.

3.7 Free Fall, the Lagrangian Velocity v_L, and the Physical Flow in the CL–H Disk

In the standard GR (Painlevé–Gullstrand) picture, a test particle released from rest at infinity follows the radial "river" velocity

$$w(r) = \sqrt{\tfrac{2GM}{r}} - Hr,$$

which represents the geometric free–fall speed of space itself. However, in the Clock Equivalence Principle (CEP) and Q_g rotor framework, the concept of free fall is formulated differently: *a freely falling particle is one that experiences no local Lorentz boost relative to its local flow of space.* This difference in definition leads to an important refinement of what the "riverbed" velocity actually is.

1. CEP definition of free fall. In the CEP, free fall 'from infinity/boundary' is characterised by the condition

$$\psi_{\text{rel}}(r) = 0,$$

meaning that the particle has zero rapidity relative to the local flow field. A free–falling particle therefore follows the *full flow* defined by the adjoint current J^ν and the mixed tensor $M_\mu{}^\nu$,

$$J^\nu = u^\mu M_\mu{}^\nu,$$

whose spatial norm defines the total flow speed at a given radius. This total flow is *not* just the radial component $w(r)$, but the complete streamline velocity arising from the Constant–Lagrangian condition.

2. Constant–Lagrangian flowlines. In the CL–H region of a galaxy disk, the Bernoulli–Noether constant reduces at low velocities to

$$v_\phi^2(r) + \big(v_r(r) - w(r)\big)^2 = C,$$

and in the stationary disk where $v_r \simeq 0$,

$$v_\phi^2(r) + w^2(r) = v_L^2, \tag{20}$$

where we write the constant C as the square of a characteristic velocity v_L. Equation (20) is the relativistic analogue of the Bernoulli condition for a steady inviscid flow: the *sum* of the squared radial and squared azimuthal velocity components along a streamline is constant.

3. The true "riverbed": the total flow magnitude v_L. In a Bernoulli flow, a passive tracer that feels no external force follows the *full flowline*, not one of its projections. Analogously, in the Q_g framework the freely falling (boost-free) particle follows the flow vector

$$\boldsymbol{v}_{\text{flow}}(r) = \big(w(r),\, v_\phi(r)\big)$$

whose magnitude is

$$|\boldsymbol{v}_{\text{flow}}(r)| = v_L.$$

Thus the physically correct "riverbed velocity" for a test mass is *not* the radial component $w(r)$ alone, but the full Lagrangian velocity v_L. In the CL–H regime the magnitude v_L is constant along streamlines, so a boost–free particle rides a flow of constant speed. The radial drift $w(r)$ is merely the projection of this flow onto the radial direction, while $v_\phi(r)$ supplies the complementary azimuthal support.

4. Free fall as motion along a constant–v_L streamline. The CEP implies that free fall means remaining unboosted relative to the local space flow, so a test mass follows the trajectory

$$v_{\text{ff}}(r) = v_L \quad \text{(magnitude of the flow)},$$

not

$$v_{\text{ff}}(r) = w(r) \quad \text{(radial GR projection)}.$$

This distinction is crucial:

- $w(r)$ describes the geometric gravitational inflow/outflow component;
- $v_\phi(r)$ supplies rotational support;
- v_L is the total flow velocity that ensures the particle has $\psi_{\text{rel}} = 0$.

A particle that stays at rest with respect to its surrounding space therefore maintains the same clock frequency along the flowline, explaining why CL–H disks form self-synchronising relativistic platforms.

5. Summary. In the CL–H flow, the radial velocity $w(r)$ is only a projection of the full flow vector. The true free–fall riverbed velocity is the constant Lagrangian speed v_L, determined by the Bernoulli–Noether invariant and representing the full Q_g flow. A boost-free test particle rides exactly this flow, just as a tracer in a classical Bernoulli streamline follows the full velocity field rather than any single component. This interpretation aligns the CL–H disk with a relativistic Bernoulli flow and clarifies why v_L constitutes the physical "riverbed" for clock synchronisation and free fall in the Q_g framework.

4 The Role of the Rim Boundary Condition in the CL–H Solution

The Constant–Lagrangian with Hubble boundary (CL–H) solution arises from the first–order conservation law $D_\mu J^\mu = 0$ in the Q_g rotor framework. Unlike in standard general relativity, where one specifies initial data for $g_{\mu\nu}$ and $\partial g_{\mu\nu}$ on a Cauchy surface, the Q_g formalism determines the entire geometry from a single conserved current,

$$J^\nu = u_\mu \, M^{\mu\nu}, \qquad M^{\mu\nu} = E_\mu{}^a \, \Phi_a{}^\nu, \qquad /G_\mu = Q_g \beta_\mu Q_g^{-1}.$$

In this first–order system the boundary condition at the bulge rim plays a crucial role, as it fixes the flow structure uniquely and thereby selects the corresponding Einstein–consistent metric.

4.1 Free–fall flow and the need for boundary data

In the Q_g framework the physical flow of space is encoded in the adjoint current J^ν, whose spatial part determines the local flow velocity. A freely falling (boost–free) particle follows this flow, meaning that the physical "riverbed" for test masses is the *total* Lagrangian flow speed v_L, not merely its radial projection $w(r)$. The full flow magnitude satisfies

$$|\boldsymbol{v}_{\text{flow}}(r)| = v_L, \qquad v_L^2 = v_\phi^2(r) + w^2(r),$$

and a particle with $\psi_{\text{rel}} = 0$ must follow this streamline exactly. To determine v_L and the functions $v_\phi(r)$ and $w(r)$ uniquely, boundary data are required.

4.2 The Bernoulli–Noether constraint

In the stationary CL–H disk, the Bernoulli–Noether invariant reduces to

$$v_\phi^2(r) + w^2(r) = C \equiv v_L^2,$$

which is the relativistic analogue of a Bernoulli flow: each streamline carries a constant velocity–squared sum, and the flow must redistribute between radial drift and azimuthal support while keeping this constant fixed. However, the constant v_L itself cannot be chosen freely; it is fixed by the matching conditions at the bulge rim.

4.3 Rim matching as the governing boundary condition

At the bulge radius $r = R$ two physical conditions must hold:

(i) the orbit must satisfy the effective Newtonian virial relation

$$v_{\text{orb}}^2(R) = \frac{1}{2}\left(\sqrt{\tfrac{2GM}{R}} - H_z\, R\right)^2 = \frac{1}{2} w(R)^2,$$

(ii) the flow must satisfy the CL–H partition

$$v_\phi^2(R) + w^2(R) = v_L^2, \qquad w(R) = \sqrt{\frac{2GM}{R}} - HR.$$

These two equations fix v_L uniquely:

$$v_L^2 = \frac{3}{2}\left(\sqrt{\frac{2GM}{R}} - HR\right)^2.$$

Once v_L is known, the entire streamline family is uniquely determined. Thus the bulge rim boundary condition is not optional; it is required for the flow to be physically and mathematically self–consistent.

4.4 From the flow to the current and the metric

Given v_L, the partition

$$v_\phi^2(r) = v_L^2 - w^2(r), \qquad w(r) = \sqrt{\frac{2GM}{r}} - Hr,$$

determines the spatial part of the adjoint current J^ν. Because the Q_g equations impose the single conservation law $D_\mu J^\mu = 0$, the current and the induced mixed tensor $M_\mu{}^\nu = E_\mu{}^a \Phi_a{}^\nu$ evolve coherently throughout the disk. The rotated basis

$$/G_\mu = Q_g \beta_\mu Q_g^{-1}$$

then yields the tetrads $e_\mu{}^a$ and metric

$$g_{\mu\nu} = \langle /G_\mu \, /G_\nu \rangle_S.$$

With the rim–determined flow, this metric automatically satisfies the Einstein–Cartan identities, since the adjoint conservation law imposes the same differential constraints as the contracted Bianchi identities.

4.5 Limited freedom and uniqueness of the CL–H solution

Once the physical parameters (M, R, H) of a galaxy are provided, the CL–H flow is fully determined:

$$(M, R, H) \quad \Longrightarrow \quad v_L \quad \Longrightarrow \quad (v_\phi(r), w(r)) \quad \Longrightarrow \quad J^\nu \quad \Longrightarrow \quad g_{\mu\nu}.$$

There is no remaining dynamical freedom beyond the spin sign of the disk. This is analogous to GR, where a mass distribution determines the metric, but even more rigid, because the first–order structure of Q_g leaves no freedom to specify independent potentials or lapse–shift functions.

Conclusion. The rim boundary condition at $r = R$ is essential for the CL–H solution. It fixes the Lagrangian velocity v_L, which in turn determines the entire flow structure and thereby the adjoint current and metric. Without this condition the Q_g system is underdetermined; with it, the resulting flow is a unique Einstein–consistent solution. The boundary condition thus plays the same role in the Q_g formalism as the full set of Einstein constraint equations in GR, but enforced through the single conservation law $D_\mu J^\mu = 0$ and the Bernoulli–Noether invariant.

5 Why the CL–H Flow Satisfies the Einstein Field Constraints

In Sections 1–3 we derived the CL–H flow from the Q_g rotor algebra: the mixed radial–Hubble river

$$w(r) = \sqrt{2GM/r} \; - \; H_z r,$$

the Bernoulli–Noether invariant

$$v_\phi^2 + (v_r - w)^2 = C,$$

and the resulting azimuthal law

$$v_\phi^2(r) = \frac{3}{2}\left(\sqrt{\frac{2GM}{R}} - H_z R\right)^2 - \left(\sqrt{\frac{2GM}{r}} - H_z r\right)^2.$$

The remaining task is to show that this pair (w, Ω) is *not* a free ansatz but an *Einstein-admissible* solution: it satisfies the Q_g conservation laws and, under the $Q_g \to$GR translation, satisfies the Einstein field constraints automatically.

This section provides that demonstration. We proceed in three steps: (1) the Q_g (first–order) side; (2) the GR (second–order) side; (3) the role and necessity of the bulge boundary condition. We close with a summary unifying these ingredients.

5.1 (i) Q_g side: first–order constraints

The Q_g framework obeys the single covariant adjoint conservation law

$$D_\mu J^\mu = 0, \qquad J^\nu = u_\mu \, M^{\mu\nu}, \qquad M = (\Psi^\dagger Q_g)[\beta_\mu \widetilde{\beta}_\nu](Q_g^{-1}\Psi).$$

Because J^μ is generated algebraically from the rotor Q_g, *all gravitational flow degrees of freedom come from the rotor's rapidity fields.*

In a stationary, axisymmetric configuration, the conservation law reduces to the Bernoulli–Noether invariant

$$v_\phi^2 + (v_r - w)^2 = C, \tag{4.1}$$

obtained by integrating $D_\mu J^\mu = 0$ along streamlines. Thus neither w nor $\Omega = v_\phi / r$ is free: they must satisfy (4.1).

The CL–H radial river

$$w(r) = \sqrt{2GM/r} \; - \; H_z r \tag{4.2}$$

is the unique smooth combination of the Newtonian ($\propto r^{-1/2}$) and Hubble ($\propto r$) flows that produces a divergence–free J^μ at large r and matches a regular boundary at $r = R$. Inserting (4.2) into (4.1) and imposing the virial boundary condition

$$v_\phi(R)^2 = \frac{1}{2} w(R)^2 \tag{4.3}$$

generates the azimuthal velocity law (CL–H):

$$v_\phi^2(r) = \frac{3}{2}\left(\sqrt{\frac{2GM}{R}} - H_z R\right)^2 - \left(\sqrt{\frac{2GM}{r}} - H_z r\right)^2. \tag{4.4}$$

Thus, on the Q_g side, the azimuthal law is *not postulated*: it is produced by the algebraic conservation law $D_\mu J^\mu = 0$ combined with the smooth bulge boundary (4.3). The entire flow (w, Ω) is fixed by the conserved current.

5.2 (ii) GR side: satisfying the Einstein constraints

To pass to GR we use the adjoint action

$$/G_\mu = Q_g \, \beta_\mu \, Q_g^{-1} = e_\mu{}^a \, \widehat{\beta}_a,$$

from which the tetrad $e_\mu{}^a$ and metric $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ follow. Because the tetrad comes from a rotor, the Bianchi identities are automatic:

$$D_\mu G^{\mu\nu} \equiv 0,$$

so Einstein's left–hand side is identically divergence–free. Hence, if the matter side produced by J^μ is divergence–free (as enforced by $D_\mu J^\mu = 0$), then the Einstein constraints are satisfied *by construction.*

In the radial sector, the flow (4.2) reproduces the standard Painlevé–Gullstrand form of Schwarzschild–de Sitter (Kottler) in the limits $H_z \to 0$, $M \to 0$, or constant H_z. In the rotational sector, the CL–H azimuthal flow (4.4) fixes the frame–drag function in exactly the same way that Kerr solutions fix the angular velocity $\Omega(r)$ by ensuring stationarity and divergence–free stress–energy.

Thus the pair (w, Ω) derived on the Q_g side *automatically satisfies* the Einstein field constraints on the GR side: no Einstein equation needs to be solved separately.

5.3 (iii) Why the bulge boundary is necessary and unique

The bulge radius R enters only through

$$v_\phi(R)^2 = \frac{1}{2} w(R)^2 \tag{4.3}$$

the physical statement that the last closed azimuthal streamline connects smoothly to the bulge. This condition:

- removes the additive constant in the Bernoulli invariant;
- fixes the integration constant C in a gauge–independent manner;
- selects the unique rotational law consistent with $D_\mu J^\mu = 0$;
- ensures the resulting tetrad is smooth at $r = R$ and has no torsion singularity;
- matches exactly the role of boundary data in standard GR solutions (e.g. fixing $J(r)$ in Kerr, or specifying $H(r)$ in LTB).

Any other boundary condition produces either (i) a non–integrable flow that violates $D_\mu J^\mu = 0$, or (ii) a metric that fails the Einstein constraints (typically via non–vanishing off–diagonal torsion terms). Thus the smooth bulge boundary does not restrict physical freedom: it selects the only solution consistent with both the Q_g and GR constraints.

5.4 Summary

From the rotor algebra and the single conservation law $D_\mu J^\mu = 0$, the CL–H radial river (4.2) and azimuthal law (4.4) follow uniquely. Under the $Q_g \to$GR map, this pair yields a metric that (i) reduces in the appropriate limits to Schwarzschild, FRW, and Kottler; (ii) satisfies the Einstein constraints automatically by virtue of the divergence–free current; and (iii) incorporates both the Newtonian and Hubble flows in a single covariant structure.

The CL–H solution therefore represents the unique axisymmetric, stationary, divergence–free flow compatible with the Q_g algebra, and the metric it generates is an Einstein–admissible solution fixed entirely by the rotor structure itself.

6 MOND–type acceleration laws derived from the CL_H orbital velocity

6.1 Background: origin of MOND and its covariant problem

Modified Newtonian Dynamics (MOND) was introduced by Milgrom [10, 11, 12] to explain the flat rotation curves of spiral galaxies without invoking dark matter. In its original, nonrelativistic form the dynamical law

$$a\,\mu\left(\frac{a}{a_0}\right) = a_N = \frac{GM}{r^2}$$

introduced a new constant acceleration scale $a_0 \sim 10^{-10}\,\mathrm{m\,s^{-2}}$ and an interpolation function $\mu(x)$ connecting the Newtonian regime ($\mu \to 1$) to the "deep–MOND" limit ($\mu \approx x$). Although phenomenologically successful [17, 14], this construction was not relativistic and could not be embedded consistently in a cosmological context.

Relativistic and covariant extensions were later developed, most notably the Tensor–Vector–Scalar (TeVeS) theory [16] and its descendants [21, 22]. These frameworks aimed to reproduce the MOND limit while providing a metric description compatible with cosmology and gravitational lensing. A remaining challenge has been to obtain the MOND regimes *and* their transitions directly from a geometric flow or covariant Lagrangian without introducing auxiliary fields or interpolation functions by hand.

The CL_H formalism derived in this paper offers such a route: it provides a single kinematic law—based on a constant–Lagrangian condition with a cosmological (Hubble) boundary—that reduces to MOND–type behaviour in the appropriate acceleration range while remaining fully covariant and explicitly cosmological through its dependence on the Hubble parameter H.

6.2 Derivation of MOND–type acceleration laws

Starting point (CLH kinematics). In the constant–Lagrangian with Hubble boundary (CLH) configuration, the Painlevé–Gullstrand (PG) river model gives the effective radial flow

$$v_r^{\mathrm{eff}}(r) = \sqrt{\frac{2GM}{r}} - Hr,$$

and the disk azimuthal velocity (for the matter–free baseline) follows from the CL closure as

$$v_\phi^2(r) = \frac{3}{2}\, v_r^{\mathrm{eff}2}(R) - v_r^{\mathrm{eff}2}(r) = v_{\mathrm{flat}}^2 - v_r^{\mathrm{eff}2}(r),$$

where the bulge rim $r = R$ fixes the integration constant through the virial–continuity condition. The quantity v_{flat}^2 is the empirical plateau velocity at the disk edge.

From velocity to acceleration. The centripetal acceleration observed in the disk is

$$a(r) = \frac{v_\phi^2(r)}{r}.$$

Expanding $v_r^{\mathrm{eff}2}(r)$ gives

$$v_r^{\mathrm{eff}2}(r) = \frac{2GM}{r} - 2H\sqrt{2GM\,r} + H^2 r^2,$$

so that

$$a(r) = \underbrace{\frac{GM}{r^2}}_{\text{Newtonian}} - \underbrace{2H\sqrt{\frac{2GM}{r}}\frac{1}{r}}_{\text{transition (MOND-like)}} + \underbrace{H^2 r}_{\text{cosmic tail}} + \underbrace{\frac{v_{\mathrm{flat}}^2}{r}}_{\text{rim constant}}. \qquad (21)$$

Interpretation of the terms. Equation (21) contains all four components required for a complete galactic acceleration law:

- The *Newtonian term* $\frac{GM}{r^2}$ dominates inside the bulge and recovers standard gravity.
- The *geometric–mean transition term* $-2H\sqrt{2GM/r}\,1/r$ reproduces the MOND interpolation behaviour $a \propto \sqrt{a_N\, a_0}$, with H acting as the natural acceleration scale.
- The *cosmic tail* $H^2 r$ describes the large–radius coupling to the global expansion field and approaches a de–Sitter background.
- The *rim constant* v_{flat}^2/r retains the boundary condition from $r = R$ and fixes the asymptotic flat velocity of the disk. It provides an explicit algebraic realization of the MOND plateau.

Relation to MOND. Up to the rim constant, Eq. (21) already exhibits the canonical MOND structure: a Newtonian term, a geometric–mean transition, and a large–radius cosmological tail. Including the rim term makes the correspondence exact: in MOND, the flat–velocity plateau emerges implicitly through the interpolation function, whereas in the CLH law it appears explicitly through the boundary condition $v_{\text{flat}}^2 = \frac{3}{2}(\sqrt{2GM/R} - HR)^2$. The CLH expression thus provides a direct algebraic derivation of the MOND form without introducing a phenomenological constant a_0:

$$a_0 \equiv cH.$$

Unified interpretation. The complete acceleration law

$$a(r) = \frac{GM}{r^2} - 2H\sqrt{\frac{2GM}{r}}\frac{1}{r} + \frac{v_{\text{flat}}^2}{r} + H^2 r$$

naturally spans four dynamical regimes:

Regime	Dominant term	Scaling	Physical meaning
Bulge	$\frac{GM}{r^2}$	r^{-2}	Newtonian core
Disk (transition)	$-2H\sqrt{2GM/r}\,1/r$	$r^{-3/2}$	MOND-like interpolation
Outer rim	v_{flat}^2/r	r^{-1}	Flat velocity plateau
Cosmic	$H^2 r$	r^{+1}	Cosmological tail

Summary. The CLH acceleration law unifies the Newtonian, MOND, and cosmological regimes in a single analytic expression derived from the Q_g rotor framework. The flat–velocity plateau arises from the rim constant, the MOND scaling from the Hubble–coupled term, and the cosmic expansion from $H^2 r$. No empirical interpolation function or new scale parameter is required: the observed MOND phenomenology emerges as the low–acceleration limit of the first–order CLH invariant.

6.3 Interpretations of the baryonic mass term M

The baryonic mass M in the CLH expression no longer appears as a purely Newtonian parameter. It participates in a composite dynamical structure where the local baryonic field, the Hubble coupling, and the rim constant jointly define the effective gravitational potential. From the acceleration law (21),

$$a(r) = \frac{GM}{r^2} - 2H\sqrt{\frac{2GM}{r}}\frac{1}{r} + H^2 r + \frac{v_{\text{flat}}^2}{r},$$

we can formally associate an *effective mass function*

$$M_{\text{eff}}(r) = M - \frac{2Hr^{3/2}}{G}\sqrt{2GM} + \frac{H^2 r^3}{G} + \frac{r\, v_{\text{flat}}^2}{G}.$$

Each term in M_{eff} describes a distinct dynamical contribution to the observed rotation field:

1. **Baryonic core.** The Newtonian component M represents the luminous matter, dominating the bulge region.
2. **Transition coupling.** The mixed term proportional to $H\sqrt{GM}\, r^{3/2}$ expresses the dynamical feedback between local mass and the background expansion.
3. **Cosmic component.** The $H^2 r^3$ term acts as an algebraic representation of the cosmological background density that connects the galactic flow to the Hubble field.

4. **Rim constant.** The additional boundary term rv_{flat}^2/G retains the memory of the bulge–disk transition at $r = R$. It fixes the flat velocity plateau and must therefore be considered an intrinsic part of the galactic effective mass distribution.

This interpretation differs from MOND, in which the baryonic mass alone determines the rotation amplitude through a fixed acceleration scale a_0. In the CLH and Q_g framework, the baryonic and cosmological terms are algebraically linked, and the rim constant provides the closure condition that connects the Newtonian and asymptotic regimes. The result is a self–consistent definition of an effective mass that already encodes the observed mass discrepancy without introducing dark matter as a separate component.

Summary. The baryonic mass M in the Q_g framework thus represents the Newtonian anchor of a fourfold flow equilibrium. The total effective mass $M_{\text{eff}}(r)$ combines the luminous content with the dynamical and cosmological corrections that arise naturally from the rotor coupling, ensuring continuity between bulge, disk, and cosmic domains.

6.4 Covariance and cosmological embedding of MOND within the CLH and Q_g framework

The CLH acceleration law reproduces MOND phenomenology while maintaining explicit covariance within the Q_g rotor formalism. Its terms derive from first–order flow equations rather than from a modified second–order metric dynamics. The resulting equations are covariant under local Lorentz transformations of the spinor basis, ensuring that both matter flow and gravitational field originate from a single algebraic adjoint current.

1. Covariant structure. In the Q_g language, the cosmic expansion is described by the rapidity field ψ_H through $\dot{\psi}_H = -4\pi G\rho_\Psi + \Lambda$. The MOND acceleration scale a_0 corresponds directly to the Hubble rapidity, $a_0 = cH$, and therefore transforms as a scalar under coordinate changes. No external modification of inertia or gravity is required: the MOND law follows from the covariant evolution of the flow field itself.

2. Natural cosmological embedding. Because H and ρ_Ψ enter the same first–order relation, the galactic flow and the cosmological background are two aspects of a single field. The rim constant v_{flat}^2/r acts as the algebraic boundary through which the local disk couples to the global Hubble expansion, completing the continuity between galactic and cosmic regimes. At large radii the H^2r term of Eq. (21) becomes dominant, producing the correct asymptotic transition to de–Sitter behaviour.

3. Relation to MOND covariance. While relativistic extensions of MOND such as TeVeS introduce separate tensor and scalar fields to restore covariance, the Q_g formalism achieves the same result within the Dirac algebra itself. Covariance is maintained automatically because the gravitational rotor transforms as a spinor operator rather than as a metric tensor.

Summary. The CLH acceleration law therefore provides both the empirical success of MOND and a natural cosmological embedding within a fully covariant algebraic theory. The identification $a_0 = cH$ arises directly from the first–order flow law, and the rim constant supplies the boundary condition that links local and global dynamics without additional fields or parameters.

7 Conclusion

In this paper we developed the explicit calculus of the gravitational rotor Q_g and demonstrated how a wide class of general–relativistic metrics, galactic flow fields, and MOND–type dynamical laws emerge directly from its first–order adjoint action.

(1) Metrics from rapidity fields

Beginning with the Dirac/BQ basis $\{\beta_\mu\}$, we showed that a single rapidity field ψ_r generates the exact Schwarzschild tetrad via the adjoint action $Q_g\beta_\mu Q_g^{-1}$, and that the corresponding metric appears immediately in Painlevé–Gullstrand (PG) form. A time–dependent Hubble rapidity ψ_H similarly yields the spatially–flat FRW metric, and the two can be superposed (being collinear boosts) to produce a Schwarzschild–de Sitter (or McVittie–type) PG geometry. The extension to an azimuthal flow v_ϕ produces the general PG form with $g_{t\phi}$ cross–terms, recovering the Kerr and Kerr–de Sitter structures when Ω matches the GR frame–dragging law.

Across all these cases, the tetrads and metrics arise algebraically from rapidity fields, without introducing Christoffel symbols or second–order curvature equations: the geometry is generated entirely by the rotor.

(2) The CL–H flow from the Qg conservation law

The core of the paper established the Constant–Lagrangian with Hubble boundary (CL–H) flow as the unique axisymmetric, stationary solution of the first–order Qg conservation law

$$D_\mu J^\mu = 0, \qquad J_\nu = u^\mu M_{\mu\nu},$$

supplemented by the Bernoulli–Noether invariant. In the PG frame the BNC reduces, in the low–velocity regime, to the conserved combination

$$v_\phi^2 + (v_r - w)^2 = \text{const},$$

where $w = \sqrt{2GM/r} - Hr$ is the mixed Schwarzschild–Hubble river velocity. This invariant enforces a fixed quadratic partition between azimuthal support and relative radial flow.

The boundary condition at the bulge edge $r = R$,

$$v_\phi^2(R) = \tfrac{1}{2}w^2(R),$$

selects the integration constant and therefore fixes the Lagrangian flow velocity v_L uniquely. The resulting CL–H law

$$v_\phi^2(r) = \frac{3}{2}\, w^2(R) - w^2(r)$$

determines the entire disk azimuthal profile, the pitch–angle relation $\tan\alpha = w^2/v_\phi^2$, and the free–fall behaviour in the Qg/CEP sense. Thus, the flow is not freely chosen: it is uniquely determined by the conservation law and a single physical matching condition.

(3) Consistency with Einstein's field constraints

By construction, the tetrad $e^a{}_\mu = (Qg\,\beta_\mu Q_g^{-1})^a$ satisfies the contracted Bianchi identities identically. Because the Qg current obeys $D_\mu J^\mu = 0$, the induced metric $g_{\mu\nu} = \langle /G_\mu\, /G_\nu\rangle_S$ automatically satisfies the Einstein field constraints. The CL–H flow therefore gives a legitimate Einstein–admissible geometry without solving the Einstein equations explicitly: the first–order Qg equation replaces the GR constraint system, while remaining fully covariant.

(4) MOND–type acceleration laws

Using the CL–H rotational law, we derived the acceleration field

$$a(r) = \frac{GM}{r^2} - \frac{2H}{r}\sqrt{\frac{2GM}{r}} + \frac{v_{\text{flat}}^2}{r} + H^2 r.$$

The four terms reproduce, respectively:

1. the Newtonian inner region,
2. the MOND–type $\sqrt{a_N a_0}$ transition term with $a_0 = cH$,
3. the flat velocity plateau set by the rim boundary,
4. the cosmological de Sitter tail at large radii.

No interpolation function or new scale parameter is introduced: the MOND behaviour appears as a consequence of the Qg flow and its cosmological boundary condition. The result provides a covariant and cosmologically embedded MOND–type acceleration law derived from a single rapidity–based gravitational rotor.

(5) Outlook

The results demonstrate that the rotor formalism supplies a complete first–order calculus for constructing tetrads, metrics, and galactic flow fields directly from rapidity variables. The CL–H solution arises from the same algebraic structure and yields rotation curves that naturally interpolate between the Newtonian, MOND, and cosmological regimes.

In summary, the gravitational rotor Q_g provides a unified algebraic mechanism for deriving relativistic metrics, flow dynamics, and observational galactic laws within a single first–order framework, with no need for modified gravity or additional fields.

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