Closing the Biquaternion Algebra with Gravity: Updated Construction of the Dirac Framework and the Gravitational Rotor Q_g

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Abstract

This paper is an updated and extended version of my earlier biquaternion (BQ) construction of the Weyl and Dirac matrix environments. The central new result is that the BQ algebra closes once gravity is included through a gravitational rotor $\boldsymbol{Q_g} \in \mathbf{Spin(1,3)}$, whose adjoint action Doppler–twists the Dirac basis and generates the space-time metric directly as $\boldsymbol{g_{\mu\nu} = \langle Q_g\, \beta_\mu\, Q_g^{-1} \beta_\nu \rangle}$. This first-order mechanism reproduces the Painlevé–Gullstrand forms of the Schwarzschild, de Sitter, and Kerr geometries without invoking Christoffel symbols or second-order curvature equations. The same structure yields the mixed tensor $\boldsymbol{M_\mu{}^\nu}$, the gravitational Dirac current, and a minimal Lagrangian that recovers the Einstein–Cartan equations. At the galactic scale, current conservation together with the Bernoulli–Noether condition selects the Constant–Lagrangian (CL) and CL–Hubble (CL–H) flows, producing the observed MOND-type phenomenology without adjustable parameters. The result is a unified, fully closed BQ–Dirac framework in which relativistic quantum mechanics, gravitational geometry, and galactic dynamics emerge from a single first-order rotor calculus.

Keywords: Biquaternions, Dirac Algebra, Gravitational Rotor, Metric Generation, Einstein–Cartan Theory, MOND Phenomenology, Relativistic Quantum Mechanics

Contents

1 Introduction

κ The Lorentz transformation properties of the Dirac matrices have been debated since their introduction. For some authors the gamma matrices are Lorentz-inert objects, even though they are written in a four-vector notation, while for others they transform as ordinary four-vectors. The transformation of Dirac spinors is uncontroversial, but the situation is more ambiguous for the Dirac adjoint, which combines a spinor with the time-like gamma matrix γ^0. The difficulty arises because the gamma matrices connect dynamical four-vectors such as momentum to the Minkowski metric, yet they are not themselves dynamical four-vectors. In standard treatments Lorentz covariance is postulated, and the operator S satisfying $S\gamma^\nu S^{-1} = \Lambda_\mu{}^\nu \gamma^\mu$ is then solved for rather than constructed [1, 2, 3, 4]. Alternative perspectives exist, notably within the geometric algebra community [5, 6], but the structural origin of the Dirac matrices and their adjoint has remained unclear.

In this paper I take a constructive route using a biquaternion (BQ) framework that is morphologically equivalent to Minkowski space-time [7]. Both the Weyl and Dirac matrix environments emerge directly from Pauli-level building blocks. In this representation the Lorentz operator S is expressed as the composition of three elementary operations: a change of representation (Dirac $\longrightarrow$ Weyl), a Lorentz boost in the Weyl basis, and the inverse change (Weyl $\longrightarrow$ Dirac). This removes the usual "assume-and-solve" ambiguity and reveals why, in bilinears of the form $P_\mu \beta^\mu$, one may transform either the coordinates or the basis but not both simultaneously.

Part I develops the Pauli-level BQ formalism and applies it to mechanics and electrodynamics. Although quaternion-based formalisms have a long history [8, 9, 10, 11, 12], the present construction is tailored specifically to recover the Weyl–Dirac structure as a PT-duplex extension of the Pauli basis. This allows spin and space-time to be treated within a single algebraic architecture, and provides a natural setting for Maxwell's equations and the Lorentz force. The Maxwell sector, when written in the BQ language, displays a duality between space-time (minquat) and spin-norm (pauliquat) components, clarifying how the classical vector calculus identities arise.

Part II constructs the full Weyl–Dirac environment and derives the Lorentz transformation properties of the Dirac matrices and spinors from the underlying BQ structure. The gravitational rotor Q_g, an even Clifford element, then extends the adjoint in such a way that the mixed tensor $M_\mu{}^\nu$ and the gravitational Dirac current $J^\nu = u_\mu M^{\mu\nu}$ arise naturally. This resolves the long-standing issue of how to define the Dirac adjoint in curved space. Most importantly, the adjoint action of Q_g provides a first-order mechanism for generating space-time metrics:

$$g_{\mu\nu}(x) = \left\langle Q_g(x)\, \beta_\mu\, Q_g^{-1}(x)\, \beta_\nu \right\rangle . \tag{1}$$

The resulting metric is expressed in Painlevé–Gullstrand (PG) form, a structure well known in gravitational theory [13], but here recovered from a single rapidity field rather than from differential geometric assumptions.

Recent extensions of this framework illustrate that the Q_g rotor acts as an algorithmic bridge between the Dirac algebra and gravitation. In [14], the conservation of the gravitational Dirac current selects the Constant–Lagrangian (CL) and CL–Hubble (CL–H) flows as unique stationary solutions for rotating galactic disks. These flows reproduce the observed phenomenology of flat rotation curves and yield MOND-type acceleration laws without introducing new fields or interpolation functions. Their connection to empirical MOND phenomenology is therefore structural rather than phenomenological, providing a theoretical underpinning to the behaviours catalogued in [15].

In [16], the Q_g formalism was extended to a first-order Lagrangian whose variation reproduces the Einstein–Cartan field equations. Torsion emerges as an algebraically mandatory feature of the mixed tensor $M_\mu{}^\nu$, in line with the classical spin–torsion framework [17], rather than as an additional postulate.

The cosmological applications developed in [18, 19] show that time-dependent rapidity fields generate a linear Friedmann dynamics with a natural resolution of the Hubble tension and a built-in damping mechanism governing density growth, thereby softening the σ_8 discrepancy. The same rotor that generates Schwarzschild, de Sitter, and Painlevé–Gullstrand geometries at local scales produces cosmological expansion at large scales, unifying gravitational physics from the microscopic Dirac level to galactic and cosmological regimes.

Taken together, these developments demonstrate that the BQ–Dirac rotor formalism is not merely an alternative representation of standard relativistic structures, but a unifying mathematical language that links the Dirac algebra, gravitation, torsion, galactic dynamics, and cosmology within a single first-order framework.

2 The Pauli spin level

2.1 A complex quaternion basis for the metric

Quaternions can be represented by the basis $(\hat{1}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$. This basis has the properties $\hat{\mathrm{I}}\hat{\mathrm{I}} = \hat{\mathrm{J}}\hat{\mathrm{J}} = \hat{\mathrm{K}}\hat{\mathrm{K}} = -\hat{1}$ and $\hat{1}\hat{1} = \hat{1}$; $\hat{1}\hat{\mathrm{I}} = \hat{\mathrm{I}}\hat{1} = \hat{\mathrm{I}}$, $\hat{1}\hat{\mathrm{J}} = \hat{\mathrm{J}}\hat{1} = \hat{\mathrm{J}}$ and $\hat{1}\hat{\mathrm{K}} = \hat{\mathrm{K}}\hat{1} = \hat{\mathrm{K}}$; $\hat{\mathrm{I}}\hat{\mathrm{J}} = -\hat{\mathrm{J}}\hat{\mathrm{I}} = \hat{\mathrm{K}}$; $\hat{\mathrm{J}}\hat{\mathrm{K}} = -\hat{\mathrm{K}}\hat{\mathrm{J}} = \hat{\mathrm{I}}$; $\hat{\mathrm{K}}\hat{\mathrm{I}} = -\hat{\mathrm{I}}\hat{\mathrm{K}} = \hat{\mathrm{J}}$. A quaternion number in its summation representation is given by $A = a_0\hat{1} + a_1\hat{\mathrm{I}} + a_2\hat{\mathrm{J}} + a_3\hat{\mathrm{K}}$, in which the a_μ are real numbers . Bi-quaternions or complex quaternions are given by $C = A + iB = c_0\hat{1} + c_1\hat{\mathrm{I}} + c_2\hat{\mathrm{J}} + c_3\hat{\mathrm{K}}$ in which the $c_\mu = a_\mu + ib_\mu$ are complex numbers and the a_μ and b_μ are real numbers.

This standard biquaternion basis $(\hat{1}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$ can be used to provide a basis for relativistic 4-D space-time. One way to do this is by making the time coordinate $c_0 = b_0 i$ complex only and the space coordinates $(c_1, c_2, c_3) = (a_1, a_2, a_3)$ real only, see [20]. Synge called these objects Minkowski quaternions or 'minquats', Silberstein called them 'physical quaternions' [20]. This however produces confusion regarding the time-like complex number as the physics gets more complicated. As Synge put it, *the intrusion of the imaginary element is not trivial* [20]. The main reason is that minquats do not form a closed algebra under addition and multiplication as a subspace inside the wider biquaternion space, due to the multiplication operation. The reason they are used nevertheless is given by Synge: *For the application of quaternions to Lorentz transformations it is essential to introduce Minkowskian quaternions* [20].

The use of minquats produces language conflicts with almost all of modern physics, that is Quantum Mechanics and Special and General Relativity, where the space-time coordinates always are a set of four real numbers. So for several reasons, I choose to insert the time-like complex number of $c_0 = b_0 i$ in the basis instead of in the coordinate. So by using $c_0\hat{1} = b_0 i\hat{1} = b_0\hat{\mathrm{T}}$ the space-time basis is then given by $(\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$. In this way, the coordinates are always a set of real numbers $\in \mathbb{R}$. The space-time basis $(\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$, (a disguised minquat basis) is not closed under multiplications, as already mentioned by Synge.

Remark. In earlier approaches (e.g. Synge's minquats), the imaginary unit is placed in the time coordinate, $c_0 = ib_0$. However, after quaternion multiplication this imaginary factor propagates into the spatial components, so one must continually keep track of where the i has moved in the algebra. By inserting the imaginary unit once and for all into the basis element $\hat{\mathrm{T}} = i\hat{1}$, the coordinates themselves remain real and the metric automatically records the complex character of the time component. This avoids the need for such bookkeeping. A second reason is more intuitive and rooted in physics and in the comparison with general relativity: in GR the coordinates of a four-vector are always real numbers, and consequently the tensors and matrices built from them are real as well. By keeping all coordinates real in the present biquaternion framework, the algebra mirrors this feature of GR, which will later allow a direct comparison between the GR formalism and the present biquaternion language.

A set of four numbers $\in \mathbb{R}$ is given by

$$A^\mu = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

or by $A_\mu = [a_0, a_1, a_2, a_3]$. In this way, the raising or lowering of the index doesn't change any sign. A^μ simply is the transpose of A_μ and vise versa. [1] The biquaternion basis can be given as a set $\mathbf{K}_\mu = (\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$. Then a biquaternion space-time vector can be written as the product

$$A = A_\mu \mathbf{K}^\mu = [a_0, a_1, a_2, a_3] \begin{bmatrix} \hat{\mathrm{T}} \\ \hat{\mathrm{I}} \\ \hat{\mathrm{J}} \\ \hat{\mathrm{K}} \end{bmatrix} = a_0\hat{\mathrm{T}} + a_1\hat{\mathrm{I}} + a_2\hat{\mathrm{J}} + a_3\hat{\mathrm{K}} \tag{2}$$

I apply this to the space-time four vector of relativistic bi-quaternion 4-space R with the four numbers $R_\mu = (r_0, r_1, r_2, r_3) = (ct, r_1, r_2, r_3)$, so with $r_0, r_1, r_2, r_3 \in \mathbb{R}$. Then I have the space-time four-vector as the product of the coordinate set and the basis $R = R_\mu \mathbf{K}^\mu = r_0\hat{\mathrm{T}} + r_1\hat{\mathrm{I}} + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}} = ct\hat{\mathrm{T}} + \mathbf{r}\cdot\mathbf{K}$. I use the three-vector analogue of $R_\mu \mathbf{K}^\mu$ when I write $\mathbf{r}\cdot\mathbf{K}$. In this notation I have $R^T = -r_0\hat{\mathrm{T}} + r_1\hat{\mathrm{I}} + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}} = -r_0\hat{\mathrm{T}} + \mathbf{r}\cdot\mathbf{K}$ for the time reversal operator and $R^P = r_0\hat{\mathrm{T}} - r_1\hat{\mathrm{I}} - r_2\hat{\mathrm{J}} - r_3\hat{\mathrm{K}} = r_0\hat{\mathrm{T}} - \mathbf{r}\cdot\mathbf{K}$ for the space point mirror or parity operator. [2]

Having established the complex quaternion basis $(\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$, it is useful to connect this abstract algebraic framework to an explicit matrix representation. Such a representation allows the biquaternion formalism to be handled with standard linear algebra tools and facilitates comparison with the Pauli spin matrices. In the next subsection I therefore represent the basis elements as 2×2 complex matrices and show how a four-vector acquires a compact matrix form in this setting.

2.2 Matrix representation of the quaternion basis

Quaternions can be represented by 2x2 matrices. Several representations are possible, but most of those choices result in conflict with the standard approach in physics. Given the unit quaternion as $\hat{\mathrm{I}}$, my choice for the space-time four set is

$$\hat{\mathrm{T}} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \hat{\mathrm{I}} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \hat{\mathrm{J}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \hat{\mathrm{K}} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{3}$$

[1] In standard Minkowski space-time with metric signature $(-,+,+,+)$, raising or lowering indices introduces a sign change in the temporal component, e.g. $A_0 = -A^0$. In the present biquaternion framework, however, raising and lowering indices is defined purely algebraically as matrix transposition, so A^μ is simply the transpose of A_μ. The Minkowski metric structure enters later through the multiplication properties of the basis elements, not through index manipulation.

[2] Note that, by direct calculation, the parity operator satisfies $R^P = -R^T$. In this notation, the transpose of a matrix is denoted by the suffix 't', so $R^t_\mu = R^\mu$. The Hermitian (complex) transpose of spinors is indicated by the dagger symbol, as in $\psi^\dagger$, and the complex conjugate by ψ^*. Within this framework, the discrete symmetry operators T and P implement sign changes of temporal and spatial components, in a way that is formally analogous to the index manipulations induced by the metric in general relativity.

I can compare these to the Pauli spin matrices $\boldsymbol{\sigma}_P = (\sigma_x, \sigma_y, \sigma_z)$.

$$\boldsymbol{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \boldsymbol{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \boldsymbol{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

If I exchange the σ_x and the σ_z [3], I get $\mathbf{K} = i\boldsymbol{\sigma}$ and $\mathbf{K}_\mu = i(\hat{1}, \boldsymbol{\sigma})$. So in my use of the Pauli matrices, I use $\boldsymbol{\sigma} \equiv (\sigma_I, \sigma_J, \sigma_K) = (\sigma_z, \sigma_y, \sigma_x)$. So also $\hat{\mathrm{I}} = \hat{\mathrm{T}}\boldsymbol{\sigma}_I, \hat{\mathrm{J}} = \hat{\mathrm{T}}\boldsymbol{\sigma}_J, \hat{\mathrm{K}} = \hat{\mathrm{T}}\boldsymbol{\sigma}_K$ and $\boldsymbol{\sigma}_I = -\hat{\mathrm{T}}\hat{\mathrm{I}}, \boldsymbol{\sigma}_J = -\hat{\mathrm{T}}\hat{\mathrm{J}}, \boldsymbol{\sigma}_K = -\hat{\mathrm{T}}\hat{\mathrm{K}}$.

With this choice of matrices, a four-vector R can be written as

$$R = r_0 \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} + r_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{5}$$

This can be compacted into a matrix representation of R:

$$R = \begin{bmatrix} r_0 i + ir_1 & r_2 + ir_3 \\ -r_2 + ir_3 & r_0 i - ir_1 \end{bmatrix} = \begin{bmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{bmatrix} \tag{6}$$

with the numbers $R_{00}, R_{01}, R_{10}, R_{11} \in \mathbb{C}$. Thus a Minkowski four-vector can be represented as a 2×2 complex matrix, with its algebra inherited from the quaternion basis.

2.3 Multiplication of vectors as matrix multiplication adds pauliquats to minquats

In general, the multiplication of two four-vectors A and B follows matrix multiplication, with $A_{ij}, B_{ij}, C_{ij} \in \mathbb{C}$.

$$AB = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix} = C. \tag{7}$$

So we have

$$C = AB = \begin{bmatrix} A_{00}B_{00} + A_{01}B_{10} & A_{00}B_{01} + A_{01}B_{11} \\ A_{10}B_{00} + A_{11}B_{10} & A_{10}B_{01} + A_{11}B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}. \tag{8}$$

Of course, vectors A, B and C can be expressed with their a_μ, b_μ, c_μ coordinates $\in \mathbb{R}$ and if we use them, after some elementary but elaborate calculations and rearrangements we arrive at the following result of the multiplication expressed in the a_μ, b_μ and c_μ as [4]:

$$c_0 = -a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3$$

[3] This unconventional ordering simplifies the identification with the quaternion basis introduced above.

[4] Here the subscripts K and σ denote the parts in the $\mathbf{K}$ and $\boldsymbol{\sigma}$ directions, respectively.

$$
\begin{aligned}
c_{1K} &= a_2 b_3 - a_3 b_2 \\
c_{2K} &= a_3 b_1 - a_1 b_3 \\
c_{3K} &= a_1 b_2 - a_2 b_1 \\
c_{1\sigma} &= -a_0 b_1 - a_1 b_0 \\
c_{2\sigma} &= -a_0 b_2 - a_2 b_0 \\
c_{3\sigma} &= -a_0 b_3 - a_3 b_0
\end{aligned} \tag{9}
$$

In short, if we use the three-dimensional Euclidean dot and cross products of Euclidean three-vectors in classical physics, this gives for the coordinates

$$
\begin{aligned}
c_0 &= -a_0 b_0 - \mathbf{a} \cdot \mathbf{b} \\
\mathbf{c}_K &= \mathbf{a} \times \mathbf{b} && (10) \\
\mathbf{c}_\sigma &= -a_0 \mathbf{b} - \mathbf{a} b_0 && (11)
\end{aligned}
$$

And in the quaternion notation we get

$$
C = AB = (-a_0 b_0 - \mathbf{a} \cdot \mathbf{b})\hat{1} + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{K} + (-a_0 \mathbf{b} - \mathbf{a} b_0) \cdot \boldsymbol{\sigma} \tag{12}
$$

This matrix multiplication, in which I used $\hat{\mathrm{T}}\hat{\mathrm{T}} = -\hat{1}$ and $\hat{\mathrm{T}}\mathbf{K} = -\boldsymbol{\sigma}$, implies that the space-time basis $(\hat{\mathrm{T}}, \mathbf{K})$ is being duplicated by a spin-norm basis $(\hat{1}, \boldsymbol{\sigma})$ by the multiplication operation.

The relativistically relevant multiplications of two four-vectors are all in the form $C = A^T B$. The difference between AB and $A^T B$ is in the sign of a_0. This results in

$$
C = A^T B = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b})\hat{1} + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{K} + (a_0 \mathbf{b} - \mathbf{a} b_0) \cdot \boldsymbol{\sigma} \tag{13}
$$

Hence the relativistically relevant bilinear is $A^T B$, not AB, and the invariant quadratic is $A^T A$:

$$
C = A^T A = (a_0^2 - \mathbf{a}{\cdot}\mathbf{a})\, \hat{1} = c^2 a_\tau^2\, \hat{1},
$$

The main quadratic form of the metric is $dR^T dR = (c^2 dt^2 - d\mathbf{r}^2)\hat{1} = c^2 d\tau^2 \hat{1} = ds^2 \hat{1}$ with $ds = c d\tau$. The quadratic giving the distance of a point R to the origin of its reference system is given by $R^T R = (c^2 t^2 - \mathbf{r}^2)\hat{1} = c^2 \tau^2 \hat{1} = s^2 \hat{1}$ with $s = c\tau$.

The multiplication of two four vectors can also be arranged as the multiplication of two tensors, a coordinate tensor times a metric tensor using that

$$
(A_\mu \mathbf{K}^\mu)^T B_\nu \mathbf{K}^\nu = A_\mu B^\nu (\mathbf{K}_\mu)^T \mathbf{K}^\nu = C_\mu{}^\nu \mathbf{K}_\mu{}^\nu \tag{14}
$$

with a real coordinate tensor ${C_\mu}^\nu$ [5]. Using mutliplication rules as $\hat{\mathrm{T}}\hat{\mathrm{T}} = -\hat{1}$, $\hat{\mathrm{T}}\hat{\mathrm{I}} = -\sigma_I$, $\hat{\mathrm{I}}\hat{\mathrm{T}} = \sigma_I$ and others, the metric tensor can be expanded as

$$\mathbf{K}_\mu{}^\nu = (\mathbf{K}_\mu)^T\mathbf{K}^\nu = \left[\, -\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}} \,\right] \begin{bmatrix} \hat{\mathrm{T}} \\ \hat{\mathrm{I}} \\ \hat{\mathrm{J}} \\ \hat{\mathrm{K}} \end{bmatrix} = \tag{15}$$

$$\begin{bmatrix} -\hat{\mathrm{T}}\hat{\mathrm{T}} & \hat{\mathrm{I}}\hat{\mathrm{T}} & \hat{\mathrm{J}}\hat{\mathrm{T}} & \hat{\mathrm{K}}\hat{\mathrm{T}} \\ -\hat{\mathrm{T}}\hat{\mathrm{I}} & \hat{\mathrm{I}}\hat{\mathrm{I}} & \hat{\mathrm{J}}\hat{\mathrm{I}} & \hat{\mathrm{K}}\hat{\mathrm{I}} \\ -\hat{\mathrm{T}}\hat{\mathrm{J}} & \hat{\mathrm{I}}\hat{\mathrm{J}} & \hat{\mathrm{J}}\hat{\mathrm{J}} & \hat{\mathrm{K}}\hat{\mathrm{J}} \\ -\hat{\mathrm{T}}\hat{\mathrm{K}} & \hat{\mathrm{I}}\hat{\mathrm{K}} & \hat{\mathrm{J}}\hat{\mathrm{K}} & \hat{\mathrm{K}}\hat{\mathrm{K}} \end{bmatrix} = \begin{bmatrix} \hat{1} & -\sigma_I & -\sigma_J & -\sigma_K \\ \sigma_I & -\hat{1} & -\hat{\mathrm{K}} & \hat{\mathrm{J}} \\ \sigma_J & \hat{\mathrm{K}} & -\hat{1} & -\hat{\mathrm{I}} \\ \sigma_K & -\hat{\mathrm{J}} & \hat{\mathrm{I}} & -\hat{1} \end{bmatrix}. \tag{16}$$

This multiplication product has a norm $\hat{1}$ part, a space $\mathbf{K}$ part and a spin $\boldsymbol{\sigma}$ part. So the multiplication of two four vectors $A^T B = C$ has this multiplication matrix. The multiplication combines the properties of symmetric and anti-symmetric in one product: the scalar part ($\propto \hat{1}$) is symmetric in A, B, the $\mathbf{K}$ part ($\propto \mathbf{a} \times \mathbf{b}$) is antisymmetric, and the $\boldsymbol{\sigma}$ part mixes them through the temporal–spatial coupling.

The inevitable appearance of the spin-norm basis in the multiplication of two Synge minquats or Silberstein physical quaternions explains why the minquats do not form a closed algebra under multiplication [20]. In the present approach, the space-time basis $(\hat{\mathrm{T}}, \mathbf{K})$ likewise does not form a closed algebra by itself: it requires a spin-norm complex dual $(\hat{\mathrm{T}}, \mathbf{K}) = i(\hat{1}, \boldsymbol{\sigma})$ in order to span the full biquaternion space. This extension is obtained under the deliberate convention—a free choice of framework in the Kantian sense—to restrict all coordinates R_μ, P_μ in $R_\mu\mathbf{K}^\mu$ and $P_\mu\boldsymbol{\sigma}^\mu$ to real values. That convention uniquely produces the dual basis.

Interpretive remark. One may view the resulting structure as endowing the physical domain with a dual space-time/spin-norm basis as its natural geometry. Speculatively, this duality might mirror aspects of real physics: electric charges and currents reside in the space-time sector $(\hat{\mathrm{T}}, \mathbf{K})$, whereas hypothetical magnetic monopoles and monopole currents, if they exist, would be associated with the spin-norm sector $(\hat{1}, \boldsymbol{\sigma})$.

In this terminology, Synge's minquats correspond to $R_\mu\mathbf{K}^\mu$ biquaternions, while $P_\mu\boldsymbol{\sigma}^\mu$ may be called pauliquats. Together, minquats and pauliquats span the full biquaternion space. The multiplication of two minquats necessarily produces both

[5] Here the coefficients ${C_\mu}^\nu$ are built purely from the real-number components A_μ and B^ν, while the geometric content of the multiplication resides in the metric basis $(\mathbf{K}_\mu)^T\mathbf{K}^\nu$. This separation mirrors the situation in special and general relativity, where tensors are expressed as coordinate arrays of real numbers contracted with basis elements determined by the metric. In this way, the biquaternion formalism reproduces the same division between numerical components and geometric structure that underlies SR and GR.

a minquat and a pauliquat component. In this picture, electric currents are naturally represented by minquats, magnetic currents (if at all possible) by pauliquats. Intrinsic spin appears as a pauliquat, with its Lorentz dual—intrinsic polarization—as a minquat. This asymmetric pairing of minquats and pauliquats stands in contrast to the electromagnetic supersymmetry sought by some magnetic-monopole approaches.

2.4 The Lorentz transformation

A standard Lorentz transformation between two reference frames connected by a relative velocity v in the $x-$ or Î-direction, with the usual $\gamma = 1/\sqrt{1 - v^2/c^2}$, $\beta = v/c$ and $r_0 = ct$, can be expressed as

$$\begin{bmatrix} r'_0 \\ r'_1 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} \gamma r_0 - \beta\gamma r_1 \\ \gamma r_1 - \beta\gamma r_0 \end{bmatrix}. \tag{17}$$

We want to connect this to our matrix representation of R as in Eq.(6) which gives

$$R'_{00} = ir'_0 + ir'_1 = i\gamma r_0 - i\beta\gamma r_1 + i\gamma r_1 - i\beta\gamma r_0 \tag{18}$$
$$R'_{11} = ir'_0 - ir'_1 = i\gamma r_0 - i\beta\gamma r_1 - i\gamma r_1 + i\beta\gamma r_0. \tag{19}$$

Now we want to introduce rapidity or hyperbolic Special Relativity in order to integrate Lorentz transformations into our matrix metric. In [21] I gave a brief history of rapidity in its relation to the Thomas precession and the geodesic precession. For this paper we only need elementary rapidity definitions. If we use the rapidity ψ as

$$e^{\pm\psi} = \gamma \pm \beta\gamma, \quad \cosh\psi = \gamma, \quad \sinh\psi = \beta\gamma, \quad e^{\pm\psi} = \cosh\psi \pm \sinh\psi, \tag{20}$$

the previous transformations can be rewritten as

$$R'_{00} = ir'_0 + ir'_1 = (\gamma - \beta\gamma)(ir_0 + ir_1) = R_{00}e^{-\psi} \tag{21}$$
$$R'_{11} = ir'_0 - ir'_1 = (\gamma + \beta\gamma)(ir_0 - ir_1) = R_{11}e^{\psi}. \tag{22}$$

As a result we have

$$R^L = \begin{bmatrix} R'_{00} & R'_{01} \\ R'_{10} & R'_{11} \end{bmatrix} = \begin{bmatrix} R_{00}e^{-\psi} & R_{01} \\ R_{10} & R_{11}e^{\psi} \end{bmatrix} = U^{-1}RU^{-1}. \tag{23}$$

The appearance of U^{-1} on both sides reflects that boosts scale the basis elements T̂, Î oppositely; the off-diagonal entries remain unchanged (since r_2, r_3 are invariant under an Î-aligned boost and the factors cancel). In the expression $R^L = U^{-1}RU^{-1}$ we used the matrix U as

$$U = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix}. \tag{24}$$

But this means that we can write the result of a Lorentz transformation on R with a Lorentz velocity in the $\hat{\mathrm{I}}$-direction between the two reference systems as

$$R^L = r_0 \begin{bmatrix} ie^{-\psi} & 0 \\ 0 & ie^{\psi} \end{bmatrix} + r_1 \begin{bmatrix} ie^{-\psi} & 0 \\ 0 & -ie^{\psi} \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \qquad (25)$$

This can be written as

$$R^L = r_0 U^{-1}\hat{\mathrm{T}}U^{-1} + r_1 U^{-1}\hat{\mathrm{I}}U^{-1} + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}} = r_0\hat{\mathrm{T}}^L + r_1\hat{\mathrm{I}}^L + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}}. \quad (26)$$

But because we started with Eq.(17), we now have two equivalent options to express the result of a Lorentz transformation

$$R^L = r'_0\hat{\mathrm{T}} + r'_1\hat{\mathrm{I}} + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}} = r_0\hat{\mathrm{T}}^L + r_1\hat{\mathrm{I}}^L + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}}, \qquad (27)$$

either as a coordinate transformation or as a basis transformation.

This result only works for Lorentz transformation between v_x-, v_1- or $\hat{\mathrm{I}}$-aligned reference systems. Reference systems which do not have their relative Lorentz velocity aligned in the $\hat{\mathrm{I}}$-direction will have to be space rotated into such an alignment before the Lorentz transformation in the form $R^L = U^{-1}RU^{-1}$ is applied. In principle, such a rotation in order to achieve the $\hat{\mathrm{I}}$ alignment of the primary reference frame to a secondary reference frame is always possible as an operation prior to a Lorentz transformation. This unique alignment between two frames of reference S and S', needed to match the physics with the algebra, is analyzed by Synge in [20, p. 41-48] and focuses on the concept of a communal photon. The requirement of reference system alignment is also the reason for the appearance of the Thomas precession and the Thomas-Wigner rotation if the axes are not aligned; the notion that two Lorentz transformations in different directions in space can always be substituted by the subsequent application of one space rotation and one single Lorentz transformation, see [21]. The communal photon of Synge is the one for which the relativistic Doppler shift between S and S' results in $\nu' = \nu e^{\pm\psi}$. The minquat algebra requires inertial observers to align their principal axis along such a communal photon, in my notation the $\hat{\mathrm{I}}$ axis.

The Lorentz transformation of the coordinates $(R^\mu)^L$ can be written as

$$\begin{bmatrix} r'_0 \\ r'_1 \\ r'_2 \\ r'_3 \end{bmatrix} = \Lambda_\nu^{\ \mu} R^\nu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \gamma r_0 - \beta\gamma r_1 \\ \gamma r_1 - \beta\gamma r_0 \\ r_2 \\ r_3 \end{bmatrix}$$

So the Lorentz transformation of $R = R_\mu \mathbf{K}^\mu = \mathbf{K}_\mu R^\mu$ can be presented as

$$R^L = \mathbf{K}_\mu (R^\mu)^L = \mathbf{K}_\mu \Lambda_\nu^{\ \mu} R^\nu = (\mathbf{K}_\mu \Lambda_\nu^{\ \mu}) R^\nu = (\mathbf{K}_\nu)^L R^\nu$$

$$= U^{-1}\mathbf{K}_\nu U^{-1} R^\nu = U^{-1}\mathbf{K}_\nu R^\nu U^{-1} = U^{-1} R U^{-1} \tag{28}$$

Lemma 2.1 (derived). For boosts along $\hat{\text{I}}$, the explicit calculation of the transformed matrix elements in (23)–(28) shows that the biquaternion basis $\mathbf{K}_\mu = (\hat{\text{T}}, \hat{\text{I}}, \hat{\text{J}}, \hat{\text{K}})$ satisfies

$$\boxed{\mathbf{K}_\mu \Lambda_\nu^\mu \ = \ U^{-1}\, \mathbf{K}_\nu\, U^{-1}}.$$

This identity is therefore not an assumption but the algebraic consequence of the biquaternion representation of R and the standard Lorentz transformation of its coordinates. In particular, it provides the constructive link between the coordinate transformation law and the transformation of the basis, so that later the transformation properties of the Dirac equation can be proved from this construction rather than postulated. We can now formulate the lemma more tightly and emphasize its bidirectional use, once its derivation has been established:

Lemma 2.1. *(Basis–component compatibility for $\hat{I}$-aligned boosts) Let Λ_ν^μ be the Lorentz boost in the $\hat{I}$-direction, and $U = \mathrm{diag}(e^{\psi/2}, e^{-\psi/2})$ as in (24). Then the biquaternion basis $\mathbf{K}_\mu = (\hat{T}, \hat{I}, \hat{J}, \hat{K})$ satisfies*

$$\boxed{\mathbf{K}_\mu\, \Lambda_\nu^\mu \ = \ U^{-1}\, \mathbf{K}_\nu\, U^{-1}}, \tag{29}$$

and therefore $R^L = \mathbf{K}_\mu (R^\mu)^L = (U^{-1}\mathbf{K}_\nu U^{-1}) R^\nu = U^{-1} R\, U^{-1}$.

This lemma will serve as the cornerstone for the later derivation of the Lorentz transformation properties of the Dirac equation: because the gamma matrices are built from these basis elements, their transformation laws follow directly from Lemma 1, and thus are proved from construction rather than assumed.

The identity Lemma(2.1) has no analogue for coordinates alone: numerical components cannot satisfy a conjugation relation, whereas the matrix-valued basis does. The matrix representation of the basis is key to this identity, because the relativistic Doppler factor $e^{\pm\psi}$ appears differently attached to the matrix elements. As is the $\hat{\text{I}}$ alignment of the two involved reference frames during the Lorentz transformation. Given that $\mathbf{K}_\mu = i\boldsymbol{\sigma}_\mu$, the identity $\mathbf{K}_\mu \Lambda_\nu^\mu = U^{-1}\mathbf{K}_\nu U^{-1}$ can also be seen as an instruction for the Lorentz transformation of the Pauli spin matrices as a norm-spin four set $\boldsymbol{\sigma}_\mu = (\hat{1}, \boldsymbol{\sigma})$.

The Lorentz transformation of A^T is also interesting, due to the importance of the product $C = A^T B$ and therefore the Lorentz transformation C^L. Given the inverse Lorentz transformation as

$$A^{L^{-1}} \equiv UAU \tag{30}$$

and using $U^T = U$ since U is real diagonal, one can prove

$$\left(A^T\right)^{L^{-1}} = U\left(A^T\right)U = \left(U^{-1}AU^{-1}\right)^T = \left(A^L\right)^T \tag{31}$$

and

$$\left(A^T\right)^L = U^{-1}\left(A^T\right)U^{-1} = (UAU)^T = \left(A^{L^{-1}}\right)^T. \tag{32}$$

The result $\left(A^L\right)^T = UA^TU$ will be used in several important derivations in this paper, when the Lorentz transformation of a product and the possible invariance or Lorentz covariance has to be investigated, as in the next example.

Start with two inertial reference systems S_1 and S_2 connected by a constant relative velocity v, a Lorentz gamma factor $\gamma(v)$ and a rapidity factor $\psi(v)$ defining the Lorentz transformation matrix U. Given A and B in S_1 and their product in S_1 as $C = A^TB$. Then in S_2 one has A^L and B^L and their product $C^L = \left(A^L\right)^T B^L$. We then have

$$\begin{aligned} C^L = \left(A^L\right)^T B^L = \left(A^T\right)^{L^{-1}} B^L = U\left(A^T\right)UU^{-1}BU^{-1} \\ = UA^TBU^{-1} = UCU^{-1}. \end{aligned} \tag{33}$$

As a result, it is easy to prove that the quadratic $A^TA = c^2a_\tau^2\hat{1}$ is Lorentz invariant. We have

$$\begin{aligned} (A^L)^TA^L = (A^T)^{L^{-1}}A^L = UA^TU\,U^{-1}AU^{-1} = U(A^TA)U^{-1} \\ = U(c^2a_\tau^2)\hat{1}U^{-1} = UU^{-1}(c^2a_\tau^2)\hat{1} = c^2a_\tau^2\hat{1} = A^TA. \end{aligned} \tag{34}$$

So both quadratics R^TR and dR^TdR are Lorentz invariant scalars, as has been shown for every quadratic of four-vectors. But they aren't what we consider to be *perfect quadratics*, who we define through the requirement $AA = |A|^2\hat{1}$.

2.5 Adding the dynamic vectors

If I want to apply the previous to relativistic electrodynamics and to quantum physics, I need to further develop the mathematical language, the notation system and the biquaternion elements. I don't claim originality regarding the biquaternion foundations of my notation system. As indicated before, there is a whole subculture around quaternions and biquaternions in physics, see [22], [23], and I have been studying many of those papers. The justification for my paper is to be found in what it adds to this rather large subculture, as part of the more general *plethora of different vector formalisms currently in use [24]*.

But let's return to my project of formulating a pragmatic biquaternion mathematical–physical language through which relativity and quantum can be synthesized. The most relevant dynamic four-vectors must be given a biquaternion representation. The basic definitions I use for that purpose are quite common in the formulations of relativistic dynamics, see for example [25]. I start with a particle with a given three vector velocity as $\mathbf{v}$, a rest mass as m_0 and an inertial mass $m_i = \gamma m_0$, with the usual $\gamma = (\sqrt{1 - v^2/c^2})^{-1}$. Latin suffixes are labels only and never imply summation; Greek suffixes always imply summation over the numbers 0, 1, 2 and 3.. So m_i stands for inertial mass and U_p for potential energy and P_μ stands for a momentum four-vector coordinate row with components $(p_0 = \frac{1}{c}U_i, p_1, p_2, p_3)$, with U_i being the inertial energy (total relativistic energy). The momentum three-vector is written as $\mathbf{p}$ and has components (p_1, p_2, p_3).

I define the coordinate velocity four-vector as

$$V = V_\mu \mathbf{K}^\mu = \frac{d}{dt} R_\mu \mathbf{K}^\mu = c\hat{\mathrm{T}} + \mathbf{v} \cdot \mathbf{K} = v_0 \hat{\mathrm{T}} + \mathbf{v} \cdot \mathbf{K}. \tag{35}$$

The proper velocity four-vector on the other hand will be defined using the proper time $\tau = t_0$, with $t = \gamma t_0 = \gamma\tau$ as

$$U = U_\mu \mathbf{K}^\mu = \frac{d}{d\tau} R_\mu \mathbf{K}^\mu = \frac{d}{\frac{1}{\gamma}dt} R_\mu \mathbf{K}^\mu = \gamma V_\mu \mathbf{K}^\mu = u_0 \hat{\mathrm{T}} + \mathbf{u} \cdot \mathbf{K}. \tag{36}$$

So we have $v_0 = c$ and $u_0 = \gamma c$.

The momentum four-vector will be, at least when we have the symmetry condition $\mathbf{p} = m_i \mathbf{v}$,

$$P = P_\mu \mathbf{K}^\mu = m_i V_\mu \mathbf{K}^\mu = m_i V = m_0 U_\mu \mathbf{K}^\mu = m_0 U. \tag{37}$$

The quadratic of the momentum four-vector $P^T P$ is important in relativity and is at the basis of the Klein-Gordon equation, so we give it here in our formalism:

$$P^T P = (U_i^2/c^2 - \mathbf{p}\cdot\mathbf{p})\, \hat{1} = U_0^2/c^2\, \hat{1} = E^2 \hat{1}, \tag{38}$$

with the shorthand notation $E := U_0/c$. This scalar invariant will reappear in the Weyl and Dirac constructions as the quadratic relation that their linearized equations reproduce.

The four-vector partial derivative $\partial = \partial_\mu \mathbf{K}^\mu$ will be defined using the coordinate four set

$$\partial_\mu = \left[-\frac{1}{c}\partial_t, \nabla_1, \nabla_2, \nabla_3 \right] = [\partial_0, \partial_1, \partial_2, \partial_3]\,. \tag{39}$$

Remark. In standard Minkowski notation the minus sign in $\partial_0 = -\frac{1}{c}\partial_t$ reflects the $(- + ++)$ metric signature. In the present framework, however, this minus sign stems from having shifted the imaginary unit from the coordinate *ict* into the basis element $\hat{\mathrm{T}} = i\hat{\mathrm{1}}$. The minus sign thus arises here as a direct consequence of the basis construction because $\frac{1}{ict}\hat{\mathrm{1}} = -i\frac{1}{ct}\hat{\mathrm{1}} = -\frac{1}{ct}\hat{\mathrm{T}}$. This ensures that $-\partial^T\partial = (-\frac{1}{c^2}\partial_t^2 + \nabla^2)\hat{\mathrm{1}}$, which reproduces the standard Minkowski form of the d'Alembert operator.

The electrodynamic potential four-vector $A = A_\mu \mathbf{K}^\mu$ will be defined by the coordinate four set

$$A_\mu = \left[\frac{1}{c}\phi, A_1, A_2, A_3\right] = [A_0, A_1, A_2, A_3] \tag{40}$$

The electric four current density vector $J = J_\mu \mathbf{K}^\mu$ will be defined by the coordinate four set

$$J_\mu = [c\rho_e, J_1, J_2, J_3] = [J_0, J_1, J_2, J_3]\,, \tag{41}$$

with ρ_e as the electric charge density. The electric four current with a charge q will be also be written as J_μ and the context will indicate which one is used.

Although we defined these four-vectors using the coordinate column notation, we will often use the matrix or summation notation, as for example with $P = P_\mu \mathbf{K}^\mu$, written as

$$\begin{aligned} P &= p_0\hat{\mathrm{T}} + p_1\hat{\mathrm{I}} + p_2\hat{\mathrm{J}} + p_3\hat{\mathrm{K}} = p_0\hat{\mathrm{T}} + \mathbf{p}\cdot\mathbf{K} \\ &= \begin{bmatrix} ip_0 + ip_1 & p_2 + ip_3 \\ -p_2 + ip_3 & ip_0 - ip_1 \end{bmatrix} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}. \end{aligned} \tag{42}$$

The flexibility to use either of these notations is a strength of the mathematical–physical language as developed in this paper. There are cases where one needs to go all the way to the internal scalar matrix notation to solve issues as for example the product rule in calculating a derivative, after which one returns to the more compact notation to evaluate the outcome.

With these definitions, the basic dynamical quantities of relativistic mechanics and electrodynamics are now expressed in the biquaternion formalism, ready for use in subsequent derivations.

2.6 The EM field in our language

If we apply the matrix multiplication rules to the electromagnetic field with four derivative ∂ and four potential A, with $\partial_0 = -\frac{1}{c}\partial_t$ and $A_0 = \frac{1}{c}\phi$, we get $B = \partial^T A$ as

$$B = \partial^T A = (-\frac{1}{c^2}\partial_t\phi - \boldsymbol{\nabla}\cdot\boldsymbol{A})\hat{\mathrm{1}} + (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\mathbf{K} + \frac{1}{c}(-\partial_t\boldsymbol{A} - \boldsymbol{\nabla}\phi)\cdot\boldsymbol{\sigma}. \tag{43}$$

If we apply the Lorenz gauge $\mathbb{B}_0 = -\frac{1}{c^2}\partial_t\phi - \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ and the usual EM definitions of the fields in terms of the potentials we get

$$B = \partial^T A = \boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}. \tag{44}$$

Using $\boldsymbol{\sigma} = -\hat{\mathrm{T}}\mathbf{K} = -i\mathbf{K}$, this can also be written as

$$B = \partial^T A = (\boldsymbol{B} - i\frac{1}{c}\boldsymbol{E}) \cdot \mathbf{K} = \overrightarrow{\mathbb{B}} \cdot \mathbf{K}. \tag{45}$$

The use of $\mathbb{B} = \boldsymbol{B} - i\frac{1}{c}\boldsymbol{E}$ dates back to Minkowski's 1908 treatment of the subject [26]. In my opinion, the flexibility of easy switching between the different modes of notations makes my biquaternion variant suited for unification purposes.

Using $\mathbb{B}$ we can write B as

$$B = \mathbb{B}_1\hat{\mathrm{I}} + \mathbb{B}_2\hat{\mathrm{J}} + \mathbb{B}_3\hat{\mathrm{K}} = \overrightarrow{\mathbb{B}} \cdot \mathbf{K} = \begin{bmatrix} i\mathbb{B}_1 & \mathbb{B}_2 + i\mathbb{B}_3 \\ -\mathbb{B}_2 + i\mathbb{B}_3 & -i\mathbb{B}_1 \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}. \tag{46}$$

Here $\mathbb{B} := \mathbf{B} - i\,\mathbf{E}/c$ is Minkowski's complex field 3-vector; its longitudinal component (parallel to the boost) is invariant, while the transverse pair mixes hyperbolically, as elaborated in the next paragraph.

For the Lorentz transformation of B we can apply the result of the previous section to get $B^L = (\partial^L)^T A^L = (\partial^T)^{L^{-1}} A^L = U(\partial^T)UU^{-1}AU^{-1} = U(\partial^T A)U^{-1} = UBU^{-1}$ for boosts along $\hat{\mathrm{I}}$, so

$$B^L = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} \begin{bmatrix} e^{-\frac{\psi}{2}} & 0 \\ 0 & e^{\frac{\psi}{2}} \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01}e^{\psi} \\ B_{10}e^{-\psi} & B_{11} \end{bmatrix} \tag{47}$$

which, when written out with $\boldsymbol{E}$ and $\boldsymbol{B}$ leads to the usual result for the Lorentz transformation of the EM field with the Lorentz velocity in the x-direction. But it can also be written as a transformation of the basis, while leaving the coordinates invariant:

$$B^L = UBU^{-1} = \mathbb{B}_1 U\hat{\mathrm{I}}U^{-1} + \mathbb{B}_2 U\hat{\mathrm{J}}U^{-1} + \mathbb{B}_3 U\hat{\mathrm{K}}U^{-1} = \tag{48}$$
$$\mathbb{B}_1\hat{\mathrm{I}} + \mathbb{B}_2\hat{\mathrm{J}}^L + \mathbb{B}_3\hat{\mathrm{K}}^L = \mathbb{B}_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \mathbb{B}_2 \begin{bmatrix} 0 & e^{\psi} \\ -e^{-\psi} & 0 \end{bmatrix} + \mathbb{B}_3 \begin{bmatrix} 0 & ie^{\psi} \\ ie^{-\psi} & 0 \end{bmatrix}.$$

(Since U is diagonal, $U\hat{\mathrm{J}}U^{-1}$ and $U\hat{\mathrm{K}}U^{-1}$ acquire the factors $e^{\pm\psi}$, whereas $U\hat{\mathrm{I}}U^{-1} = \hat{\mathrm{I}}$.) The Lorentz transformation of the EM field can be performed by internally twisting the $(\hat{\mathrm{J}}, \hat{\mathrm{K}})$-surface perpendicular to the Lorentz velocity and in the process leaving the EM-coordinates invariant.

That the above equals the usual Lorentz transformation of the EM field can be shown by going back to [26], where he wrote the transformation in a form equivalent to

$$\begin{bmatrix} \mathbb{B}'_1 \\ \mathbb{B}'_2 \\ \mathbb{B}'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & i\beta\gamma \\ 0 & -i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} \mathbb{B}_1 \\ \mathbb{B}_2 \\ \mathbb{B}_3 \end{bmatrix} = \begin{bmatrix} \mathbb{B}_1 \\ \gamma\mathbb{B}_2 + i\beta\gamma\mathbb{B}_3 \\ \gamma\mathbb{B}_3 - i\beta\gamma\mathbb{B}_2 \end{bmatrix} \tag{49}$$

So we have

$$B'_{01} = \mathbb{B}'_2 + i\mathbb{B}'_3 = \gamma\mathbb{B}_2 + i\beta\gamma\mathbb{B}_3 + i\gamma\mathbb{B}_3 + \beta\gamma\mathbb{B}_2 \tag{50}$$

and

$$B'_{10} = -\mathbb{B}'_2 + i\mathbb{B}'_3 = -\gamma\mathbb{B}_2 - i\beta\gamma\mathbb{B}_3 + i\gamma\mathbb{B}_3 + \beta\gamma\mathbb{B}_2. \tag{51}$$

If we use the rapidity ψ as $e^{\psi} = \cosh\psi + \sinh\psi = \gamma + \beta\gamma$, this can be rewritten as

$$B'_{01} = \mathbb{B}'_2 + i\mathbb{B}'_3 = (\gamma + \beta\gamma)(\mathbb{B}_2 + i\mathbb{B}_3) = B_{01}e^{\psi} \tag{52}$$

and

$$B'_{10} = -\mathbb{B}'_2 + i\mathbb{B}'_3 = (\gamma - \beta\gamma)(-\mathbb{B}_2 + i\mathbb{B}_3) = B_{10}e^{-\psi}, \tag{53}$$

which leads to Eqn. (47).

2.7 The Maxwell Equations and the Lorentz force law

We start with presenting the Maxwell equations in the developed formalism and then give the Lorentz force law in the same formalism. The Maxwell equations in our language can be given as $\partial B = \mu_0 J$, using $J = \rho V$. Maxwell's inhomogeneous wave equations can be written as $(-\partial^T\partial)B = -\mu_0\partial^T J$ and with the Lorentz invariant quadratic derivative,

$$-\partial^T\partial = (\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2)\hat{1} \tag{54}$$

we get the homogeneous wave equations of the EM field in free space in the familiar form as

$$(-\partial^T\partial)B = \boldsymbol{\nabla}^2 B - \frac{1}{c^2}\partial_t^2 B = 0. \tag{55}$$

I will look at $\partial B = \mu_0 J$ first. The underlying structure then also applies to the Lorentz Force Law and the inhomogeneous part of the wave equation. I start with

$$B = \partial^T A = \boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}. \tag{56}$$

Then ∂B is given by

$$\partial B = \left(-\frac{1}{c}\partial_t\hat{\mathrm{T}} + \boldsymbol{\nabla}\cdot\mathbf{K}\right)\left(\boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right) =$$

$$-(\boldsymbol{\nabla}\cdot\boldsymbol{B})\hat{1} + \frac{1}{c}(\boldsymbol{\nabla}\cdot\boldsymbol{E})\hat{\mathrm{T}} + (\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{1}{c^2}\partial_t\boldsymbol{E})\cdot\mathbf{K} + \frac{1}{c}(\boldsymbol{\nabla}\times\boldsymbol{E} + \partial_t\boldsymbol{B})\cdot\boldsymbol{\sigma} \tag{57}$$

If we interpret this result using the knowledge regarding the inhomogeneous Maxwell equations, we get an interesting result. First of all, the part of the Maxwell Equation with the dimension of the norm $\hat{1}$ is zero and so is the part in the spin-norm sector ($\boldsymbol{\sigma}$). The space-time parts $\mathbf{K}$ and $\hat{\mathrm{T}}$ equal the space-time parts of the four current density J. In the following, ρ denotes charge density for the current density four-vectors, while q denotes single-particle charge. So we get

$$\partial B = -(\boldsymbol{\nabla} \cdot \boldsymbol{B})\hat{1} + \frac{1}{c}(\boldsymbol{\nabla} \cdot \boldsymbol{E})\hat{\mathrm{T}} + (\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c^2}\partial_t \boldsymbol{E}) \cdot \mathbf{K} + \frac{1}{c}(\boldsymbol{\nabla} \times \boldsymbol{E} + \partial_t \boldsymbol{B}) \cdot \boldsymbol{\sigma} = 0\hat{1} + \mu_0 c\rho\hat{\mathrm{T}} + \mu_0 \boldsymbol{J} \cdot \mathbf{K} + 0\boldsymbol{\sigma} = \mu_0 J. \quad (58)$$

So the spin-norm part of the Maxwell Equations equals zero and the space-time part equals the space-time four current density times μ_0. In the line of this interpretation, magnetic monopoles and the correlated magnetic monopole current should be searched in the pauliquat dimensions of spin-norm, not in the minquat dimensions of space-time.

As for the Lorentz covariance of the Maxwell Equations, this can be demonstrated quite easily. Given the four-vectors ∂, A and J in reference system S_1, with the Maxwell Equations as $\partial(\partial^T A) = \mu_0 J$, then in reference system S_2 we have the four-vectors ∂^L, A^L and J^L and the covariant Maxwell Equations given as $\partial^L(\partial^L)^T A^L = \mu_0 J^L$. In S_2 this can be proven through

$$\partial^L(\partial^L)^T A^L = \partial^L(\partial^T)^{L^{-1}} A^L = U^{-1}\partial U^{-1} U(\partial^T) U U^{-1} A U^{-1} = U^{-1}\partial(\partial^T) A U^{-1} = U^{-1}\partial B U^{-1} = U^{-1}\mu_0 J U^{-1} = \mu_0 J^L. \quad (59)$$

So if we have $\partial B = \mu_0 J$ in one frame of reference, this transforms as $\partial^L B^L = \mu_0 J^L$ in another frame of reference, which means that the equation maintains its form, it is Lorentz covariant. We have form-invariance of the equations.

I will look at $JB = F$ now, with $J = qV$. The underlying structure for the Lorentz Force Law is the same as for the Maxwell equations. So JB is given by

$$JB = \left(cq\hat{\mathrm{T}} + \boldsymbol{J} \cdot \mathbf{K}\right)\left(\boldsymbol{B} \cdot \mathbf{K} + \frac{1}{c}\boldsymbol{E} \cdot \boldsymbol{\sigma}\right) = -(\boldsymbol{J} \cdot \boldsymbol{B})\hat{1} + \frac{1}{c}(\boldsymbol{J} \cdot \boldsymbol{E})\hat{\mathrm{T}} + (\boldsymbol{J} \times \boldsymbol{B} + q\boldsymbol{E}) \cdot \mathbf{K} + (\frac{1}{c}\boldsymbol{J} \times \boldsymbol{E} - cq\boldsymbol{B}) \cdot \boldsymbol{\sigma} \quad (60)$$

If we interpret this result using the knowledge regarding the Lorentz Force Law, we get an interesting result. First of all, the part of the Lorentz force law with the dimension of the norm $\hat{1}$ is zero and so is the part in the spin-norm sector ($\boldsymbol{\sigma}$). The space-time parts $\mathbf{K}$ and $\hat{\mathrm{T}}$ equal the space-time parts of the four force F. Thus we get

$$JB = -(\boldsymbol{J} \cdot \boldsymbol{B})\hat{1} + \frac{1}{c}(\boldsymbol{J} \cdot \boldsymbol{E})\hat{\mathrm{T}} + (\boldsymbol{J} \times \boldsymbol{B} + q\boldsymbol{E}) \cdot \mathbf{K} + (\frac{1}{c}\boldsymbol{J} \times \boldsymbol{E} - cq\boldsymbol{B}) \cdot \boldsymbol{\sigma} = 0\hat{1} + \frac{1}{c}P\hat{\mathrm{T}} + \boldsymbol{F} \cdot \mathbf{K} + 0\boldsymbol{\sigma} = F. \quad (61)$$

So the spin-norm pauliquat part of the Lorentz Force Law equals zero and the space-time minquat part equals the space-time four force.

In both cases, ∂B and BJ, we get a dual spin-norm and space-time product, with the spin-norm equal zero and the non-zero space-time leading to the inhomogeneous four-vectors of current and force. Speculations about magnetic monopoles are connected to these spin-norm parts, the set spanned by pauliquats. In my analysis, if spin-norm is the twin dual of space-time and as such an integral aspect of the metric as foreseen in [27], then searches for magnetic monopoles should focus on this spin-norm aspect of the vacuum.

But from a purely geometric perspective, the product of three four-vectors like in $BJ = \partial^T AJ = F$, we can separate the coordinate four sets ∂_μ, A^ν, and J^μ from the metric basis, as in $BJ = ((\partial_\mu A^\nu)J^\mu)((\mathbf{K}_\mu^T \mathbf{K}^\nu)\mathbf{K}^\mu)$, and focus on the metric product alone. We then get

$$\mathbf{K}_\mu{}^\nu \mathbf{K}^\mu = (\mathbf{K}_\mu^T \mathbf{K}^\nu)\mathbf{K}^\mu = \begin{bmatrix} -\hat{T}\hat{T} & \hat{I}\hat{T} & \hat{J}\hat{T} & \hat{K}\hat{T} \\ -\hat{T}\hat{I} & \hat{I}\hat{I} & \hat{J}\hat{I} & \hat{K}\hat{I} \\ -\hat{T}\hat{J} & \hat{I}\hat{J} & \hat{J}\hat{J} & \hat{K}\hat{J} \\ -\hat{T}\hat{K} & \hat{I}\hat{K} & \hat{J}\hat{K} & \hat{K}\hat{K} \end{bmatrix} \begin{bmatrix} \hat{T} \\ \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix} = \tag{62}$$

$$\begin{bmatrix} -\hat{T}\hat{T}\hat{T} + \hat{I}\hat{T}\hat{I} + \hat{J}\hat{T}\hat{J} + \hat{K}\hat{T}\hat{K} \\ -\hat{T}\hat{I}\hat{T} + \hat{I}\hat{I}\hat{I} + \hat{J}\hat{I}\hat{J} + \hat{K}\hat{I}\hat{K} \\ -\hat{T}\hat{J}\hat{T} + \hat{I}\hat{J}\hat{I} + \hat{J}\hat{J}\hat{J} + \hat{K}\hat{J}\hat{K} \\ -\hat{T}\hat{K}\hat{T} + \hat{I}\hat{K}\hat{I} + \hat{J}\hat{K}\hat{J} + \hat{K}\hat{K}\hat{K} \end{bmatrix} = \begin{bmatrix} \hat{T} - \hat{T} - \hat{T} - \hat{T} \\ \hat{I} - \hat{I} + \hat{I} + \hat{I} \\ \hat{J} + \hat{J} - \hat{J} + \hat{J} \\ \hat{K} + \hat{K} + \hat{K} - \hat{K} \end{bmatrix}, \tag{63}$$

with no norm-spin $(\hat{1}, \boldsymbol{\sigma})$ product in the end result: only pure $(\hat{T}, \mathbf{K})$ components remain. The product of three four-vectors in this metric/geometry environment should produce a space-time four-vector only, as is reflected in the Maxwell equations and the Lorentz Force Law. In other words, the multiplication of three minquats produces a pure minquat, not a pauliquat or a sum of a pauliquat and a minquat. In this formalism, the metric analysis excludes non-zero $(\hat{1}, \boldsymbol{\sigma})$ contributions from these products, which suggests that monopole-like sources cannot be represented within this framework.

3 The Dirac spin level

3.1 The Dirac–Weyl environment and its disconnect from Minkowski space-time

In the 1920s the quadratic relativistic Klein–Gordon wave equation was found inadequate for describing the relativistic electron with intrinsic (Pauli) spin. In his search for a solution, Dirac linearized the Klein–Gordon equation by introducing 4×4 matrices built as duplexes of the 2×2 Pauli matrices. In his two seminal 1928 papers he introduced the Clifford four-set $(\beta, \boldsymbol{\alpha})$ and, using what were later called the gamma matrices, the covariant Clifford four-set $(\beta, \boldsymbol{\gamma})$ [28, 29]. The Pauli matrices were embedded within these structures. Shortly thereafter Weyl introduced an alternative covariant Clifford four-set, related to Dirac's by the distinction between low-velocity and high-velocity relativistic regimes [30]. Weyl also explicitly analyzed parity P and time-reversal T properties of these representations [31, 32].

All such gamma or gamma-like matrices can be represented in terms of 2×2 matrices over the biquaternion Pauli basis $(\hat{\mathrm{I}}, \boldsymbol{\sigma})$. However, using this basis highlights the difficulty of connecting the gamma four-vector—defined in an abstract Dirac spinor space—to the Minkowski four-vectors of special relativity. Once spinor wave functions are introduced to form a Hilbert space, the gap between the abstract representation and Minkowski space-time becomes even more pronounced. The Born rule provides an operational link between spinor amplitudes and experimental probabilities, but it does not in itself restore a direct geometric connection to Minkowski space-time.

Using the results of the previous section, however, we can build gamma-equivalent matrices with an explicit connection to Minkowski space-time. In particular, the biquaternion minquat basis $(\hat{\mathrm{T}}, \mathbf{K})$ offers a direct bridge between the Clifford algebra of relativistic quantum mechanics and the space-time of special relativity. Establishing the Dirac environment on this basis may provide a fruitful and insightful foundation for unifying the two descriptions.

In the next subsection we will construct the gamma matrices explicitly in terms of $(\hat{\mathrm{T}}, \mathbf{K})$, showing how the Dirac equation can be derived directly from this space-time based formalism. A key advantage of this approach is that Lorentz covariance is derived constructively rather than assumed. The transformation properties of the basis elements $(\hat{\mathrm{T}}, \mathbf{K})$, established earlier (Lemma 1), ensure that the gamma-equivalent matrices built from them satisfy the correct Clifford algebra relations and inherit the proper Lorentz behavior. In this way, the Dirac equation and its Lorentz transformation can be connected directly to Minkowski space-time through construction rather than through postulate.

3.2 The Weyl matrices in dual minquat space-time mode as bèta matrices.

In my mathematical–physical language and with a block-doubling construction (analogous to a Möbius transform in form) in mind I can define matrices through the application of parity or point reflection P and time reversal or time reversal T of the energy-momentum four-vector P as

$$\begin{bmatrix} P & P \\ P^P & P^T \end{bmatrix} = \begin{bmatrix} P & P \\ -P^T & P^T \end{bmatrix} =$$

$$p_0 \begin{bmatrix} \hat{\mathrm{T}} & \hat{\mathrm{T}} \\ \hat{\mathrm{T}} & -\hat{\mathrm{T}} \end{bmatrix} + p_1 \begin{bmatrix} \hat{\mathrm{I}} & \hat{\mathrm{I}} \\ -\hat{\mathrm{I}} & \hat{\mathrm{I}} \end{bmatrix} + p_2 \begin{bmatrix} \hat{\mathrm{J}} & \hat{\mathrm{J}} \\ -\hat{\mathrm{J}} & \hat{\mathrm{J}} \end{bmatrix} + p_3 \begin{bmatrix} \hat{\mathrm{K}} & \hat{\mathrm{K}} \\ -\hat{\mathrm{K}} & \hat{\mathrm{K}} \end{bmatrix} =$$

$$p_0 \begin{bmatrix} \hat{T} & \hat{T} \\ \hat{T} & -\hat{T} \end{bmatrix} + \mathbf{p} \cdot \begin{bmatrix} \mathbf{K} & \mathbf{K} \\ -\mathbf{K} & \mathbf{K} \end{bmatrix} \tag{64}$$

The problem with this matrix is that it doesn't represent a Clifford four-set.

I can split the quadruple of P into two duplexes $P_\mu \beta^\mu + P_\mu \xi^\mu$. The bèta's are defined through, constructed as, the parity duplex

$$\not{P} = P_\mu \beta^\mu = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} = p_0 \begin{bmatrix} 0 & \hat{T} \\ \hat{T} & 0 \end{bmatrix} + \mathbf{p} \cdot \begin{bmatrix} 0 & \mathbf{K} \\ -\mathbf{K} & 0 \end{bmatrix} = p_0 \beta_0 + \mathbf{p} \cdot \boldsymbol{\beta} =$$
$$p_0 \begin{bmatrix} 0 & \hat{T} \\ \hat{T} & 0 \end{bmatrix} + p_1 \begin{bmatrix} 0 & \hat{I} \\ -\hat{I} & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & \hat{J} \\ -\hat{J} & 0 \end{bmatrix} + p_3 \begin{bmatrix} 0 & \hat{K} \\ -\hat{K} & 0 \end{bmatrix} \tag{65}$$

with $\not{P} = P_\mu \beta^\mu = p_0\beta_0 + p_1\beta_1 + p_2\beta_2 + p_3\beta_3$.

The xi's are defined through, constructed as, the time reversed duplex

$$P_\mu \xi^\mu = \begin{bmatrix} P & 0 \\ 0 & P^T \end{bmatrix} = p_0 \begin{bmatrix} \hat{T} & 0 \\ 0 & -\hat{T} \end{bmatrix} + \mathbf{p} \cdot \begin{bmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{K} \end{bmatrix} = p_0 \xi_0 + \mathbf{p} \cdot \boldsymbol{\xi} =$$
$$p_0 \begin{bmatrix} \hat{T} & 0 \\ 0 & -\hat{T} \end{bmatrix} + p_1 \begin{bmatrix} \hat{I} & 0 \\ 0 & \hat{I} \end{bmatrix} + p_2 \begin{bmatrix} \hat{J} & 0 \\ 0 & \hat{J} \end{bmatrix} + p_3 \begin{bmatrix} \hat{K} & 0 \\ 0 & \hat{K} \end{bmatrix} \tag{66}$$

with $P_\mu \xi^\mu = p_0\xi_0 + p_1\xi_1 + p_2\xi_2 + p_3\xi_3$.

The relation with the metric of the previous section is direct. In the β_μ case space is mirrored, so the space-time double arises through the parity operation. In the ξ_μ time is reversed, so the space-time double is obtained through the T operation.

Of these two constructions, only the β_μ matrices form a Clifford four-set: their products satisfy $\{\beta_\mu, \beta_\nu\} = 2\eta_{\mu\nu} \mathbb{1}$.[6,7] By contrast, the ξ_μ matrices fail to satisfy the Clifford relations. The β_μ are therefore the natural minquat equivalent of the Weyl γ-matrices, just as those are obtained in the standard approach by doubling the Pauli spin-norm basis. If I use $\hat{T} = i\hat{1}$ and $\mathbf{K} = i\boldsymbol{\sigma}$, the result is

$$\beta_\mu = (\beta_0, \boldsymbol{\beta}) = \left(\begin{bmatrix} 0 & i\hat{1} \\ i\hat{1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{bmatrix} \right) = (i\hat{1}, i\boldsymbol{\gamma}) = i\gamma_\mu \tag{67}$$

which relates the parity dual β_μ to the Weyl gamma representation. This establishes the Weyl representation in the minquat setting. In the next step we will address how the full Dirac representation (obtained in the standard formalism by doubling the Weyl set) can be realized in the minquat environment.

[6] Here, $\mathbb{1}$ is the 4×4 identity matrix, where $\hat{1}$ is the 2×2 identity matrix.

[7] Explicitly, one verifies $\beta_0^2 = \mathbb{1}$, $\beta_i^2 = -\mathbb{1}$ $(i = 1, 2, 3)$, and $\beta_\mu\beta_\nu = -\beta_\nu\beta_\mu$ for $\mu \neq \nu$, so that $\{\beta_\mu, \beta_\nu\} = 2\eta_{\mu\nu} \mathbb{1}$. Thus the β_μ indeed form a Clifford four-set.

3.3 From the Weyl and Dirac equations to the Dirac bèta matrices

The trick in finding Clifford four-sets is connected to the problem of the quadratics and to the problem of formulating equations in the Dirac environment. The quadratics of the energy-momentum four-vectors in the Clifford representation have to be reducible to the Klein Gordon energy condition $P^T P = E^2 \hat{1}$ with $E = \frac{U_0}{c} = m_0 c$. The Weyl bèta representation of $\not{P}$ matches this requirement. The ξ_μ representation doesn't.

In order to linearize the Klein–Gordon condition, we introduce an energy operator in block-matrix form. The scalar relation $P^T P = E^2 \hat{1}$ corresponds in the Weyl setting to the matrix identity

$$\not{P}\not{P} = -E^2 \not{1}.$$

This quadratic can be factorized if we represent the energy through the eigen-time matrix

$$\xi = \begin{bmatrix} \hat{T} & 0 \\ 0 & \hat{T} \end{bmatrix}, \qquad (E\xi)^2 = -E^2 \not{1}.$$

We therefore define

$$\not{E} := E\,\xi, \qquad \not{E}^2 = -E^2 \not{1}.$$

With this construction, the quadratic $\not{P}\not{P} = -E^2 \not{1}$ can be rewritten as $\not{P}\not{P} = \not{E}\not{E}$, which admits the desired linear factorization.

The Weyl or chiral equation stems from the quadratic $\not{P}\not{P} = \not{E}\not{E}$ in the space-time Weyl representation.

$$\not{P}\not{P} = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} = \begin{bmatrix} -PP^T & 0 \\ 0 & -P^T P \end{bmatrix} = \begin{bmatrix} -E^2 \hat{1} & 0 \\ 0 & -E^2 \hat{1} \end{bmatrix} = -E^2 \not{1} = \not{E}\not{E} \tag{68}$$

From the construction we have

$$\not{P}\not{P} - \not{E}\not{E} = 0,$$

which factorizes as

$$(\not{P} - \not{E})(\not{P} + \not{E}) = 0.$$

If these factors are interpreted as operator equations on their own, $(\not{P} \pm \not{E}) = 0$, the only solutions are trivial unless $m_0 = 0$. This explains why the Weyl equation describes only massless particles such as neutrinos.

To obtain nontrivial solutions when $m_0 \neq 0$, one must enlarge the representation space: the operators $\not{P} \pm \not{E}$ act trivially on numbers, but can act nontrivially on spinors, which serve as the carrier space of the linearized equations. We therefore introduce a wave function Ψ, interpreted as a spinor field. In this setting the quadratic relation is imposed on bilinear forms,

$$\Psi^\dagger(\not{P} - \not{E})(\not{P} + \not{E})\Psi = 0,$$

so that it suffices to require

$$(\not{P} + \not{E})\Psi = 0, \qquad \Psi^\dagger(\not{P} - \not{E}) = 0,$$

for nonvanishing Ψ. Here $\Psi^\dagger$ is the Hermitian conjugate row spinor of the column spinor Ψ.

By interpreting spinors as wave-like fields, these equations can be read as eigenvalue conditions for the operators $\not{P} \pm \not{E}$. With $\hat{\not{P}} = -i\hbar\not{\partial}$ we arrive at the Weyl wave equations

$$\Psi^\dagger\hat{\not{P}} = \Psi^\dagger\not{E}, \tag{69}$$

$$\hat{\not{P}}\Psi = -\not{E}\Psi. \tag{70}$$

For zero rest mass ($m_0 = 0$, hence $E = 0$), these reduce to the familiar massless Weyl equations

$$\Psi^\dagger\hat{\not{P}} = 0, \qquad \hat{\not{P}}\Psi = 0.$$

This analysis shows how the Weyl formalism emerges naturally from the factorization of the quadratic condition and why it is restricted to massless particles. To go beyond and accommodate massive fermions, the construction must be extended: by enlarging the representation space and mixing the parity and time–reversal duplexes, one arrives at the Dirac equation in its full form.

Unlike in the Weyl case, where the algebra alone suffices to describe massless fields, the Dirac case necessarily requires the introduction of spinors. Without them, the factorization of the quadratic condition would again reduce to trivial solutions. To describe massive fermions such as the electron, we must therefore enlarge the representation space so that nontrivial solutions exist when $m_0 \neq 0$. This enlargement is achieved by mixing the parity and time–reversal duplexes, leading to the Dirac representation. In this setting the quadratic condition becomes

$$\not{P}\not{P} = (p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta})^2 = -E^2\not{1},$$

with explicit block form

$$\not{P}\not{P} = \begin{bmatrix} p_0\hat{\mathrm{T}} & \mathbf{p}\cdot\mathbf{K} \\ -\mathbf{p}\cdot\mathbf{K} & -p_0\hat{\mathrm{T}} \end{bmatrix} \begin{bmatrix} p_0\hat{\mathrm{T}} & \mathbf{p}\cdot\mathbf{K} \\ -\mathbf{p}\cdot\mathbf{K} & -p_0\hat{\mathrm{T}} \end{bmatrix} =$$

$$\begin{bmatrix} (-p_0^2+\mathbf{p}^2)\hat{1} & 0 \\ 0 & (-p_0^2+\mathbf{p}^2)\hat{1} \end{bmatrix} = -E^2 \mathbb{1}. \tag{71}$$

This leads to the two options for the Dirac equations

$$(\hat{p}_0\beta_0 + \hat{\mathbf{p}}\cdot\boldsymbol{\beta})\Psi = Ei\mathbb{1}\Psi \tag{72}$$

$$\Psi^\dagger(\hat{p}_0\beta_0 + \hat{\mathbf{p}}\cdot\boldsymbol{\beta}) = -E\Psi^\dagger i\mathbb{1} \tag{73}$$

if we use $\hat{P} = -i\hbar\partial$ and a four column spinor Ψ.

From this we can derive the Dirac bèta matrices, i.e. the bèta-matrices in the Dirac representation. The Dirac representation mixes the bèta and the xi representation, so that the temporal part is doubled via ξ, while the spatial parts retain the β structure. This representation thus represents a PT dual, because the Dirac block form uses bèta for the spatial parts and xi for the temporal part. I nevertheless, using the gamma tradition, use the bèta and Feynman slash symbols for both representations in the time-space $(\hat{\mathrm{T}}, \mathbf{K})$ basis. This gives for the Dirac bèta representation

$$\not{P} = P_\mu\beta^\mu = p_0\begin{bmatrix} \hat{\mathrm{T}} & 0 \\ 0 & -\hat{\mathrm{T}} \end{bmatrix} + \mathbf{p}\cdot\begin{bmatrix} 0 & \mathbf{K} \\ -\mathbf{K} & 0 \end{bmatrix} = p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta} =$$
$$p_0\begin{bmatrix} \hat{\mathrm{T}} & 0 \\ 0 & -\hat{\mathrm{T}} \end{bmatrix} + p_1\begin{bmatrix} 0 & \hat{\mathrm{I}} \\ -\hat{\mathrm{I}} & 0 \end{bmatrix} + p_2\begin{bmatrix} 0 & \hat{\mathrm{J}} \\ -\hat{\mathrm{J}} & 0 \end{bmatrix} + p_3\begin{bmatrix} 0 & \hat{\mathrm{K}} \\ -\hat{\mathrm{K}} & 0 \end{bmatrix}. \tag{74}$$

Both the Weyl and Dirac block constructions yield Clifford sets equivalent to the usual gamma matrices, up to the factor of i: $\beta_\mu = i\gamma_\mu$.

So in the space-time representation we have the Weyl $\not{P}$ as

$$\not{P}_w = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \tag{75}$$

and the Dirac $\not{P}$ as

$$\not{P}_d = \begin{bmatrix} p_0\hat{\mathrm{T}} & \mathbf{p}\cdot\mathbf{K} \\ -\mathbf{p}\cdot\mathbf{K} & -p_0\hat{\mathrm{T}} \end{bmatrix} \tag{76}$$

.

3.3.1 The transformation from the Dirac to the Weyl representation and vice versa

A key step in connecting the Weyl and Dirac representations is the similarity transformation S. While this operator is well known in the standard formalism, in the present framework it acquires a central role: it is the explicit bridge that allows us to move between the Weyl and Dirac β-representations constructed above, and it will serve as the core mechanism for establishing Lorentz covariance of the Dirac equation by construction rather than postulate.

Given the Weyl and Dirac β representations of Eqn. (75) and Eqn. (76), the transformation matrix can be written in one of its usual forms as

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{1} & \hat{1} \\ -\hat{1} & \hat{1} \end{bmatrix}. \tag{77}$$

It has the property $\beta_0 S = S^{-1}\beta_0$, and equivalently $S\beta_0 = \beta_0 S^{-1}$.

The switch from the Weyl β_w^ν to the Dirac β_d^ν is then given by $\beta_d^\nu = S\beta_w^\nu S^{-1}$, and the reverse by $\beta_w^\nu = S^{-1}\beta_d^\nu S$. We then also have the transformations $\not{P}_w = S^{-1}\not{P}_d S$ and $\not{P}_d = S\not{P}_w S^{-1}$.

This similarity transformation S will play a central role in the next section, where it provides the constructive mechanism for demonstrating the Lorentz covariance of the Dirac equation within the present framework.

3.4 Lorentz transformations of the vectors in the Dirac and Weyl representation environments

In the Pauli level part of this paper I developed the $(\hat{T}, \mathbf{K})$ relativistic approach. This resulted in the Lorentz transformation of a four-vector $P = (p_0\hat{T}, \boldsymbol{p} \cdot \mathbf{K})$ as $P^L = U^{-1}PU^{-1}$ and the Lorentz transformation of its time reversal P^T as $(P^L)^T = (P^T)^{L^{-1}} = UP^TU$ with U as

$$U = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix} \tag{78}$$

and the rapidity ψ. The quadratic P^TP is a Lorentz invariant scalar $\frac{U_0^2}{c^2}\hat{1} = E^2\hat{1}$ with the dimension of the norm $\hat{1}$. If in the space-time minquat β_μ representation we have the Weyl $\not{P}$ in a reference system S as

$$\not{P} = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \tag{79}$$

then in reference system S' we have P^L and so also the Weyl $\not{P}^L$ as

$$\not{P}^L = \begin{bmatrix} 0 & P^L \\ -(P^L)^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & U^{-1}PU^{-1} \\ -UP^TU & 0 \end{bmatrix} \tag{80}$$

The question then is which matrix can generate this result. The obvious answer is

$$\not{P}_w^L = \Lambda_W \not{P}_w \Lambda_W^{-1} = \tag{81}$$

$$\begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & U^{-1} \end{bmatrix} = \begin{bmatrix} 0 & U^{-1}PU^{-1} \\ -UP^TU & 0 \end{bmatrix} \tag{82}$$

with the Lorentz transformation matrix

$$\Lambda_{\mathrm{W}} = \begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix} \tag{83}$$

and its inverse $\Lambda_{\mathrm{W}}^{-1}$.

As for the generator of Λ_{W}, we have

$$\Lambda_{\mathrm{W}} = \begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} e^{-\frac{\psi}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\psi}{2}} & 0 & 0 \\ 0 & 0 & e^{\frac{\psi}{2}} & 0 \\ 0 & 0 & 0 & e^{-\frac{\psi}{2}} \end{bmatrix} =$$

$$\begin{bmatrix} \cosh\left(\frac{\psi}{2}\right) & 0 & 0 & 0 \\ 0 & \cosh\left(\frac{\psi}{2}\right) & 0 & 0 \\ 0 & 0 & \cosh\left(\frac{\psi}{2}\right) & 0 \\ 0 & 0 & 0 & \cosh\left(\frac{\psi}{2}\right) \end{bmatrix} +$$

$$\begin{bmatrix} -\sinh\left(\frac{\psi}{2}\right) & 0 & 0 & 0 \\ 0 & \sinh\left(\frac{\psi}{2}\right) & 0 & 0 \\ 0 & 0 & \sinh\left(\frac{\psi}{2}\right) & 0 \\ 0 & 0 & 0 & -\sinh\left(\frac{\psi}{2}\right) \end{bmatrix} =$$

$$\cosh\left(\frac{\psi}{2}\right) \begin{bmatrix} \hat{1} & 0 \\ 0 & \hat{1} \end{bmatrix} + \sinh\left(\frac{\psi}{2}\right) \begin{bmatrix} -\sigma_I & 0 \\ 0 & \sigma_I \end{bmatrix} =$$

$$\not{1} \cosh\left(\frac{\psi}{2}\right) + \alpha_I \sinh\left(\frac{\psi}{2}\right) = \not{1} e^{\alpha_I \left(\frac{\psi}{2}\right)}. \tag{84}$$

The α_I is defined in the Weyl presentation as:

$$\alpha_I = \begin{bmatrix} -\sigma_I & 0 \\ 0 & \sigma_I \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{85}$$

and this ensures $\alpha_I^2 = \not{1}$. The inverse of Λ_{W} is then obviously given by $\Lambda_{\mathrm{W}}^{-1} = \not{1} e^{-\alpha_I \left(\frac{\psi}{2}\right)}$. These expressions hold for boosts aligned with $\hat{\mathrm{I}}$ (our x-axis). Generic boosts are obtained by a spatial rotation to align with $\hat{\mathrm{I}}$, apply the formulas, then rotate back.

The Klein Gordon energy-momentum condition's Lorentz invariance or covariance depends on the product $\not{p}^L\not{p}^L$. Using the previous result, we have for the Lorentz transformation of the product $\not{p}\not{p}$ in the Weyl representation

$$\not{p}^L\not{p}^L = \Lambda_W\not{p}\Lambda_W^{-1}\Lambda_W\not{p}\Lambda_W^{-1} = \Lambda_W\not{p}\not{p}\Lambda_W^{-1} = \\ \Lambda_W(-E^2\not{1})\Lambda_W^{-1} = -E^2\not{1}\Lambda_W\Lambda_W^{-1} = -E^2\not{1} = \not{p}\not{p}, \tag{86}$$

so a Lorentz invariant product. This ensures the Lorentz invariance of the Klein Gordon energy-momentum condition $\not{p}\not{p} = \not{E}\not{E}$ in the Weyl representation.

In the Dirac version, where $\not{p} = p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta}$, things get more complicated. We have to start with the Dirac $\not{p}_d$ in the primary reference system and we want to end up with $\not{p}_d^L$ in the secondary reference system. We know how to transform between the Dirac and the Weyl representations and we know how to Lorentz transform the Weyl $\not{p}_w$. This means we have to go from Dirac to Weyl in the primary reference system, then Lorentz transform the Weyl four-vector to the secondary reference system and then transform back from the Weyl to the Dirac representation, three operations in total. The total result gives

$$\not{p}_d^L = S\Lambda_W S^{-1}\not{p}_d S\Lambda_W^{-1}S^{-1}. \tag{87}$$

In the Dirac representation,

$$\not{p}_d^L = S\Lambda_W S^{-1}\,\not{p}_d\,S\Lambda_W^{-1}S^{-1}, \qquad \Lambda_D := S\Lambda_W S^{-1}, \qquad \Lambda_D^{-1} := S\Lambda_W^{-1}S^{-1}.$$

Then

$$\not{p}_d^L\not{p}_d^L = \Lambda_D\not{p}_d\Lambda_D^{-1}\,\Lambda_D\not{p}_d\Lambda_D^{-1} = \Lambda_D(\not{p}_d^2)\Lambda_D^{-1} = \Lambda_D(-E^2\not{1})\Lambda_D^{-1} = -E^2\not{1},$$

so the Klein–Gordon condition remains Lorentz invariant.

In details, with rapidity ψ, the operator $\Lambda_D = S\Lambda S^{-1}$ is given as

$$\Lambda_D = \begin{bmatrix} \cosh(\frac{\psi}{2})\hat{1} & \sinh(\frac{\psi}{2})\sigma_I \\ \sinh(\frac{\psi}{2})\sigma_I & \cosh(\frac{\psi}{2})\hat{1} \end{bmatrix} = \not{1}\cosh(\frac{\psi}{2}) + \alpha_I\sinh(\frac{\psi}{2}) = \not{1}e^{\left(\alpha_I\frac{\psi}{2}\right)}, \tag{88}$$

with $\not{1}e^{\left(\alpha_I\frac{\psi}{2}\right)}$ as the generator of the Lorentz boost. The α_I in the Dirac representation is defined as:

$$\alpha_I = \begin{bmatrix} 0 & \sigma_I \\ \sigma_I & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{89}$$

The operator $\Lambda_{\mathrm{D}}^{-1} = S\Lambda_{\mathrm{W}}^{-1}S^{-1}$ is given as

$$\Lambda_{\mathrm{D}}^{-1} = \begin{bmatrix} \cosh(\frac{\psi}{2})\hat{1} & -\sinh(\frac{\psi}{2})\sigma_I \\ -\sinh(\frac{\psi}{2})\sigma_I & \cosh(\frac{\psi}{2})\hat{1} \end{bmatrix} =$$

$$\not{1}\cosh(\frac{\psi}{2}) - \alpha_I \sinh(\frac{\psi}{2}) = \not{1} e^{-\left(\alpha_I \frac{\psi}{2}\right)}. \tag{90}$$

In the transformation of the four-vector we have $\not{P}_d = P_\mu \beta^\mu$. Because the operators only work on the matrix aspect of each of the elements of β^μ, the Lorentz transformation can also be written as

$$\not{P}^L = e^{\left(\alpha_I \frac{\psi}{2}\right)} \not{P} e^{-\left(\alpha_I \frac{\psi}{2}\right)} = \Lambda_{\mathrm{D}} P_\mu \beta^\mu \Lambda_{\mathrm{D}}^{-1} = P_\mu \Lambda_{\mathrm{D}} \beta^\mu \Lambda_{\mathrm{D}}^{-1} \tag{91}$$

and we can focus on

$$(\beta^\mu)^L = \Lambda_{\mathrm{D}} \beta^\mu \Lambda_{\mathrm{D}}^{-1} \tag{92}$$

thus interpreting the Lorentz transformation as a boost of the dual minquat metric.

Using the Lorentz transformation expression of the operator combinations Λ_{D} and $\Lambda_{\mathrm{D}}^{-1}$ in terms of the rapidity and the hyperbolic trigonometric expressions, we can calculate the result on the bèta matrices of the Λ_{D} and $\Lambda_{\mathrm{D}}^{-1}$ operators. After some calculations this results in

$$\boxed{\Lambda_{\mathrm{D}} \beta^\mu \Lambda_{\mathrm{D}}^{-1} = \Lambda_\nu{}^\mu \beta^\nu} \tag{93}$$

with, given the usual Lorentz boost $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$,

$$(\beta^\mu)^L = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}^L = \Lambda_\nu^\mu \beta^\nu =$$

$$\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \gamma\beta_0 - \beta\gamma\beta_1 \\ \gamma\beta_1 - \beta\gamma\beta_0 \\ \beta_2 \\ \beta_3 \end{bmatrix}. \tag{94}$$

Independently from the previous direct calculations based on known Λ_{D}, S and $\Lambda_\nu{}^\mu$, the Lorentz transformation of P can also be presented as a transformation of the coordinates P_μ with a fixed metric K^μ, see the results of Eqn.(28). For the bèta dual this applies as well. Thus one either transforms the coordinates P_μ or one transforms the metric in β^μ, but not both. The end result will be the same. For the first we get

$$\not{P}^L = (P_\mu \beta^\mu)^L = (P_\mu)^L \beta^\mu = (P_\nu \Lambda_\mu{}^\nu)\beta^\mu = P'_\mu \beta^\mu. \tag{95}$$

But this can also be written as

$$\not{P}^L = (P_\nu \Lambda_\mu{}^\nu)\beta^\mu = P_\nu(\Lambda_\mu{}^\nu \beta^\mu) = P_\nu \beta'^\nu. \tag{96}$$

So we have $(\beta^\nu)^L = \Lambda_\mu{}^\nu \beta^\mu$. This is the precise sense in which the β^ν transform as a 4-vector basis under boosts, complementing the dual picture where one holds the basis fixed and transforms P_μ. And we have, in the Dirac representation, $(\beta^\nu)^L = \Lambda_\mathrm{D}\beta^\nu\Lambda_\mathrm{D}^{-1}$, leading to

$$\Lambda_\mu{}^\nu \beta^\mu = \Lambda_\mathrm{D}\beta^\nu\Lambda_\mathrm{D}^{-1}. \tag{97}$$

In the space-time Weyl representation the results are the same, giving

$$\Lambda_\mathrm{W}\beta^\mu\Lambda_\mathrm{W}^{-1} = \Lambda_\mu{}^\nu \beta^\mu. \tag{98}$$

A remark is necessary: one has to keep track of the representation one is in, Weyl or Dirac, because the same $\Lambda_\mu{}^\nu$ and β^μ symbols are used but they aren't equal in the respective representations.

The ease of the Lorentz transformation and the proving of Lorentz covariance or invariance in the developed math-phys environment can be contrasted with the usual approach as critically analyzed and alternatively presented in [33]. The relation $\Lambda_\mathrm{D}\beta^\mu\Lambda_\mathrm{D}^{-1} = \Lambda_\mu{}^\nu\beta^\mu$ for the Dirac matrices in this paper has been constructed using already known matrices. I do not use this relationship as a starting point in the process of finding the operator Λ_D, as is done in the literature. As I mentioned in the introduction, in [2, p. 147, Eqn. 5.102], the Λ_D is a black box, whereas in my approach I opened the box and found $\Lambda_\mathrm{D} = S\Lambda_\mathrm{W}S^{-1}$, a relation that I constructed and then used to prove $\Lambda_\mathrm{D}\beta^\mu\Lambda_\mathrm{D}^{-1} = \Lambda_\mu{}^\nu\beta^\mu$ instead of assuming it first and solving it later. I do not assume and solve, I construct and prove instead. This was possible because of its connection to the Lorentz transformation approach in the biquaternion representation of the Pauli level physics. Thus the covariance of the Dirac equation is not assumed but emerges from the geometric structure established at the Pauli level.

My approach confirms the claim that the bèta matrices can transform like a 'regular' four-vector, but it also confirms the approach that the bèta matrices remain fixed during a Lorentz transformation. One has to realize that in the Feynman $\not{P} = P_\mu\beta^\mu$ notation, the Lorentz transformation is either performed on P_μ with fixed β^μ or on β^μ with fixed P_μ: one either transforms the coordinates or one transforms the metric, but not both. Either the metric aspect of $\not{P}$ is Doppler twisted or the dynamic variables are, but not both.

3.5 Lorentz transformations of the spinors in the Weyl representation environments

The Lorentz transformation of the spinors is known to be half the Lorentz transformation of a four-vector. In case of the Weyl bèta representation we have $\not{p}^L = \Lambda\not{p}\Lambda^{-1}$, so we expect to have either $\Psi^L = \Lambda\Psi$ or $\Psi^L = \Lambda^{-1}\Psi$.

Now, in physics, the Lorentz transformation of EM-waves represents a relativistic Doppler boost, represented by the factor $e^{\psi} = \gamma + \gamma\beta$ shifting the frequency and wavelength to the red or to the blue. Matter waves and the associated phenomena are duly called so because they exhibit wave phenomena as refraction and interference. So matter waves should have wave-fronts and crests and troughs and as such undergo the equivalent of Doppler shifts when observed from $v\hat{I}$-boosted reference systems. But measurements always involve intensities, never pure waves, so the intensities should exhibit quantum Doppler shifts.

We further know that if the spinor Ψ represents a matter wave, then $\Theta = \not{p}\Psi$ also represents a matter wave and both should Lorentz transform identically. From this we can infer that $\Psi^L = \Lambda\Psi$, because then

$$\Theta^L = (\not{p}\Psi)^L = \not{p}^L\Psi^L = \Lambda\not{p}\Lambda^{-1}\Psi^L = \Lambda\not{p}\Lambda^{-1}\Lambda\Psi = \Lambda\not{p}\Psi = \Lambda\Theta \tag{99}$$

From $\Psi^L = \Lambda\Psi$ we can derive the relation

$$(\Psi^L)^\dagger = (\Lambda\Psi)^\dagger = \Psi^\dagger\Lambda, \tag{100}$$

due to the fact that Λ is diagonal real and thus equal to its conjugate transpose.

We then get for Lorentz transformation of the intensity $\Psi^\dagger\Psi$ of the matter wave Ψ

$$\begin{aligned}(\Psi^\dagger\Psi)^L = (\Psi^L)^\dagger(\Psi^L) = (\Lambda\Psi)^\dagger(\Lambda\Psi) = \Psi^\dagger\Lambda\Lambda\Psi = \Psi^\dagger\Lambda^2\Psi = \Psi^\dagger e^{\alpha_I\psi}\Psi = \\ \Psi^\dagger\Psi\cosh(\psi) + \Psi^\dagger\alpha_I\Psi\sinh(\psi) = \Psi^\dagger\Psi\gamma + \Psi^\dagger\alpha_I\Psi\gamma\beta,\end{aligned} \tag{101}$$

with alpha matrix α_I, Lorentz boost $\gamma = \cosh(\psi)$, $\gamma\beta = \sinh(\psi)$. This is the quantum equivalent of a Doppler boost with rapidity ψ, as should be expected for a wave phenomenon when observed from a moving reference system.

In the Weyl representation, boosting the probability density doesn't mix the spinors because we have a diagonal matrix in the Lorentz boost operator, as

$$\begin{aligned}(\Psi^\dagger\Psi)^L = \begin{bmatrix}\Psi_1^* & \Psi_2^* & \Psi_3^* & \Psi_4^*\end{bmatrix}\begin{bmatrix}\gamma-\gamma\beta & 0 & 0 & 0 \\ 0 & \gamma+\gamma\beta & 0 & 0 \\ 0 & 0 & \gamma+\gamma\beta & 0 \\ 0 & 0 & 0 & \gamma-\gamma\beta\end{bmatrix}\begin{bmatrix}\Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4\end{bmatrix} \\ = \gamma\Psi_1^*\Psi_1 - \gamma\beta\Psi_1^*\Psi_1 + \gamma\Psi_2^*\Psi_2 + \gamma\beta\Psi_2^*\Psi_2 \\ +\gamma\Psi_3^*\Psi_3 + \gamma\beta\Psi_3^*\Psi_3 + \gamma\Psi_4^*\Psi_4 - \gamma\beta\Psi_4^*\Psi_4 =\end{aligned}$$

$$\Psi_1^*\Psi_1 e^{-\psi} + \Psi_2^*\Psi_2 e^{\psi} + \Psi_3^*\Psi_3 e^{\psi} + \Psi_4^*\Psi_4 e^{-\psi}. \qquad (102)$$

The factor $\gamma \pm \gamma\beta = e^{\pm\psi}$ represents a relativistic wavelength/frequency Doppler shift of the intensities.

In Wave Mechanics, the equations are wave equations and the Lagrangians are the intensities of those waves. The Klein-Gordon energy-momentum condition is Lorentz Invariant, the linearized Dirac equation transforms like a wave Ψ and the Lagrangian wave intensity derived from that equation transforms Doppler like with a factor $e^{\alpha_I \psi}$.

The conditon $\Psi^L = \Lambda\Psi$ gives

$$\Psi_w^L = \Lambda\Psi_w = \begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} \Psi_w^1 \\ \Psi_w^2 \end{bmatrix} = \begin{bmatrix} U^{-1}\Psi_w^1 \\ U\Psi_w^2 \end{bmatrix}. \qquad (103)$$

In this Weyl space–time representation the two bispinors Ψ_w^1 and Ψ_w^2 transform independently: they are rescaled but not mixed by a Lorentz boost. Their intensities acquire multiplicative factors $e^{\pm\psi}$, which are the relativistic Doppler factors for red- and blue-shifting of matter waves.

This result is significant: it shows that in the Weyl representation the relativistic transformation of spinors — normally introduced axiomatically in quantum field theory via the abstract operator $S(\Lambda)$ — arises here as a direct consequence of the biquaternion formalism. The Lorentz action is both diagonal and Doppler-like, linking the algebraic structure to the physical wave nature of spinor fields.

3.6 Lorentz transformations of the spinors in the Dirac representation environments

The same line of reasoning will give us the Lorentz transformation rules for the spinors in the space-time Dirac representation, respectively

$$\Psi_d^L = \Lambda_D\Psi_d = \Lambda_D\Psi_d \qquad (104)$$

and

$$(\Psi_d^L)^\dagger = (\Psi_d^\dagger)\Lambda_D = (\Psi_d^\dagger)\Lambda_D. \qquad (105)$$

For the intensities we then get

$$(\Psi_d^\dagger\Psi_d)^L = (\Psi_d^L)^\dagger\Psi_d^L = (\Psi_d^\dagger)\Lambda_D\Lambda_D\Psi_d = (\Psi_d^\dagger)\Lambda_D^2\Psi_d. \qquad (106)$$

In the Dirac representation, we have to calculate Λ_D^2 in order to be able to evaluate the result. In detail, with rapidity ψ, the operator Λ_D^2 is given as

$$\Lambda_D^2 = \begin{bmatrix} \cosh(\psi)\hat{1} & \sinh(\psi)\sigma_I \\ \sinh(\psi)\sigma_I & \cosh(\psi)\hat{1} \end{bmatrix} = \mathbb{1}\cosh(\psi) + \alpha_I \sinh(\psi) = \mathbb{1} e^{(\alpha_I\psi)}, \quad (107)$$

with $\mathbb{1}e^{(\alpha_I\psi)}$ as the generator of the Lorentz boost delivered Doppler shift of the probability/field density, as

$$(\Psi^\dagger\Psi)^L = \Psi^\dagger e^{(\alpha_I\psi)}\Psi = \Psi^\dagger\Psi\cosh(\psi) + \Psi^\dagger\alpha_I\Psi\sinh(\psi). \tag{108}$$

The operator $\Lambda_D^2 = S\Lambda_W^2 S^{-1}$ coincides with the original Dirac-spinor boost operator identified by Darwin (1928), confirming that the construction reproduces the standard result [34]. The intensity $\Psi^\dagger\Psi$ is not an invariant, but transforms covariantly with a Doppler factor, exactly as expected for a wave intensity.

Zooming in further and using $\cosh(\psi) = \gamma$ and $\sinh(\psi) = \gamma\beta$, we get for the Dirac representation

$$\begin{aligned}(\Psi^\dagger\Psi)^L &= \begin{bmatrix}\Psi_1^* & \Psi_2^* & \Psi_3^* & \Psi_4^*\end{bmatrix}\begin{bmatrix}\gamma & 0 & \gamma\beta & 0\\ 0 & \gamma & 0 & -\gamma\beta\\ \gamma\beta & 0 & \gamma & 0\\ 0 & -\gamma\beta & 0 & \gamma\end{bmatrix}\begin{bmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{bmatrix}\\ &= \gamma\Psi_1^*\Psi_1 + \gamma\beta\Psi_1^*\Psi_3 + \gamma\Psi_2^*\Psi_2 - \gamma\beta\Psi_2^*\Psi_4\\ &\quad+\gamma\Psi_3^*\Psi_3 + \gamma\beta\Psi_3^*\Psi_1 + \gamma\Psi_4^*\Psi_4 - \gamma\beta\Psi_4^*\Psi_2.\end{aligned} \tag{109}$$

We see that in the Dirac representation, boosting the probability density mixes the spinors and thus the particles and the anti-particles, the electrons and the positrons. This mixing is gives the Dirac representation its capacity to describe both particle and antiparticle solutions.

The structure of these transformations look familiar. If we define $\gamma' = \cosh(\frac{\psi}{2})$ and $\gamma'\beta' = \sinh(\frac{\psi}{2})$, we get the Lorentz transformation of Ψ as

$$\Psi^L = \begin{bmatrix}\gamma'\hat{1} & \gamma'\beta'\sigma_I\\ \gamma'\beta'\sigma_I & \gamma'\hat{1}\end{bmatrix}\begin{bmatrix}\Psi^1\\ \Psi^2\end{bmatrix} = \begin{bmatrix}\gamma'\hat{1}\Psi^1 + \gamma'\beta'\sigma_I\Psi^2\\ \gamma'\hat{1}\Psi^2 + \gamma'\beta'\sigma_I\Psi^1\end{bmatrix}. \tag{110}$$

In the hyperbolic formulation, the details of the Lorentz transformation of Ψ gives

$$\begin{aligned}\Psi^L = \begin{bmatrix}(\Psi^1)^L\\ (\Psi^2)^L\end{bmatrix} &= \begin{bmatrix}\cosh(\frac{\psi}{2})\hat{1} & \sinh(\frac{\psi}{2})\sigma_I\\ \sinh(\frac{\psi}{2})\sigma_I & \cosh(\frac{\psi}{2})\hat{1}\end{bmatrix}\begin{bmatrix}\Psi^1\\ \Psi^2\end{bmatrix} =\\ &\begin{bmatrix}\cosh(\frac{\psi}{2})\hat{1}\Psi^1 + \sinh(\frac{\psi}{2})\sigma_I\Psi^2\\ \cosh(\frac{\psi}{2})\hat{1}\Psi^2 + \sinh(\frac{\psi}{2})\sigma_I\Psi^1\end{bmatrix}.\end{aligned} \tag{111}$$

What we see here is that the Lorentz transformation of the Dirac spinor necessarily mixes the two twin Pauli spinors Ψ^1 and Ψ^2. As a consequence, one cannot Lorentz transform a single Pauli spinor in the Dirac representation: a Lorentz transformation of the Pauli equation without its full Dirac twin is impossible. This shows that the Pauli equation on its own cannot possibly be relativistic – not because of the

Pauli matrices themselves, as is often suggested in the literature [35], but because a two-component spinor does not furnish a complete Lorentz representation.

While this fact is well known in relativistic quantum mechanics, the present construction provides a direct algebraic proof of it: by explicitly exhibiting how boosts act on the biquaternion/β-matrix formalism, we see the unavoidable mixing of the two Pauli spinors. This constructive demonstration complements the usual argument (based on representation theory) and highlights the structural role of spinor doubling in the emergence of relativity at the Dirac level.

3.7 Connecting the beta-matrices to the gamma-matrices and to the Dirac alpha and spin matrices

My reversed order of the Pauli spin matrices, with $\sigma_I = \sigma_z$, $\sigma_J = \sigma_y$, $\sigma_K = \sigma_x$ and $\boldsymbol{\sigma} = (\sigma_I, \sigma_J, \sigma_K)$ implies that the usual (x, y, z) order of the gamma matrices are reversed correspondingly, with $\gamma_1 = \gamma_I = \gamma_z$, $\gamma_2 = \gamma_J = \gamma_y$, $\gamma_3 = \gamma_K = \gamma_x$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = (\gamma_I, \gamma_J, \gamma_K))$. The connection between the usual gamma's and the bèta's that I use is simply $\beta = i\gamma$.

The set of gamma matrices in the Dirac representation, $\gamma_\mu = (\beta, \boldsymbol{\gamma}) = (\gamma_0, \boldsymbol{\gamma})$, can then be defined as

$$\gamma_\mu = (\beta, \boldsymbol{\gamma}) = (\gamma_0, \boldsymbol{\gamma}) = \left(\begin{bmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} \right) \tag{112}$$

The set of gamma matrices in the Weyl representation, $\gamma_\mu = (\gamma_0, \boldsymbol{\gamma})$, can be defined as

$$\gamma_\mu = (\gamma_0, \boldsymbol{\gamma}) = \left(\begin{bmatrix} 0 & \hat{1} \\ \hat{1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} \right) \tag{113}$$

In my bèta based $(\hat{1}, \boldsymbol{\sigma})$ norm-spin basis the Dirac set $\alpha_\mu = (\mathbb{1}, \boldsymbol{\alpha})$ can be represented as

$$\alpha_\mu = (\mathbb{1}, \boldsymbol{\alpha}) = \left(\begin{bmatrix} \hat{1} & 0 \\ 0 & \hat{1} \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} \right). \tag{114}$$

The most straightforward doubling of the Pauli level norm-spin set $(\hat{1}, \boldsymbol{\sigma})$ is the Dirac level norm-spin set $\Sigma_\mu = (\mathbb{1}, \boldsymbol{\Sigma})$ defined as

$$\Sigma_\mu = (\mathbb{1}, \boldsymbol{\Sigma}) = \left(\begin{bmatrix} \hat{1} & 0 \\ 0 & \hat{1} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} \right). \tag{115}$$

To go back to the tensor $\beta_\mu \beta^\nu = i\gamma_\mu i\gamma^\nu = -\gamma_\mu \gamma^\nu$ is given by

$$\beta_\mu \beta^\nu = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} =$$

$$\begin{bmatrix} \beta_0\beta_0 & \beta_1\beta_0 & \beta_2\beta_0 & \beta_3\beta_0 \\ \beta_0\beta_1 & \beta_1\beta_1 & \beta_2\beta_1 & \beta_3\beta_1 \\ \beta_0\beta_2 & \beta_1\beta_2 & \beta_2\beta_2 & \beta_3\beta_2 \\ \beta_0\beta_3 & \beta_1\beta_3 & \beta_2\beta_3 & \beta_3\beta_3 \end{bmatrix} = \begin{bmatrix} -\not{1} & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & \not{1} & i\Sigma_{21} & -i\Sigma_{31} \\ -\alpha_2 & -i\Sigma_{12} & \not{1} & i\Sigma_{32} \\ -\alpha_3 & i\Sigma_{13} & -i\Sigma_{23} & \not{1} \end{bmatrix}. \tag{116}$$

Thus, the product $\beta_\mu\beta^\nu$ firmly connects the minquat domain to the pauliquat domain on the Dirac level. The product of two Dirac level duplex minquats produces a mixture of a duplex minquat and a a duplex pauliquat, as was the case on the Pauli level.

3.8 General vector multiplication in the Weyl-Dirac environment

If we multiply two vectors in the Pauli environment we get sensible products with $C = A^T B$. In the Weyl- and Dirac environment, such a product is already implied in the definition of the vectors when $\not{A}$ and $\not{B}$ are multiplied to give $\not{C}$. We need this product frequently in the remainder of the paper, so we will work it out here. There are several ways to write down the product $\not{C} = \not{A}\not{B}$, but we well start from the definitions onward, beginning in the Weyl representation:

$$\not{C} = \not{A}\not{B} = -(c\not{1} + i\mathbf{d}\cdot\boldsymbol{\Sigma} + \mathbf{e}\cdot\boldsymbol{\alpha}) \tag{117}$$

with $c = a_0b_0 - \boldsymbol{a}\cdot\boldsymbol{b}$, $\boldsymbol{d} = \boldsymbol{a}\times\boldsymbol{b}$ and $\boldsymbol{e} = \boldsymbol{a}b_0 - a_0\boldsymbol{b}$. These results are the same in the Weyl and in the Dirac representations. It is worthwile to write the product as matrices

$$\not{C} = \not{A}\not{B} = \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix}\begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix} = \begin{bmatrix} -AB^T & 0 \\ 0 & -A^T B \end{bmatrix} =$$

$$-\begin{bmatrix} (a_0b_0 - \mathbf{a}\cdot\mathbf{b})\hat{1} & 0 \\ 0 & (a_0b_0 - \mathbf{a}\cdot\mathbf{b})\hat{1} \end{bmatrix} - \begin{bmatrix} (\mathbf{a}\times\mathbf{b})\cdot\mathbf{K} & 0 \\ 0 & (\mathbf{a}\times\mathbf{b})\cdot\mathbf{K} \end{bmatrix}$$

$$-\begin{bmatrix} -(a_0\mathbf{b} - \mathbf{a}b_0)\cdot\boldsymbol{\sigma} & 0 \\ 0 & (a_0\mathbf{b} - \mathbf{a}b_0)\cdot\boldsymbol{\sigma} \end{bmatrix} =$$

$$-\left(c\begin{bmatrix} \hat{1} & 0 \\ 0 & \hat{1} \end{bmatrix} + \mathbf{d}\cdot\begin{bmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{K} \end{bmatrix} + \mathbf{e}\cdot\begin{bmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix}\right) = -\left(c\not{1} + \mathbf{d}\cdot\boldsymbol{\Sigma} + \mathbf{e}\cdot\boldsymbol{\alpha}\right) \tag{118}$$

Because the difference between $A^T B$ and AB^T is the sign switching of a_0 and b_0, this only impacts the vector $\mathbf{e}$ in $\boldsymbol{\alpha}$: the $\mathbf{e}$next to the upper left $\boldsymbol{\sigma}$ turns negative and the lower right doesn't. This change is absorbed in the definition of $\boldsymbol{\alpha}$, such that $\mathbf{e}$ can still be taken out of the matrix. On the Dirac level, working out the multiplication is a bit more work, but with the use of S one can always transform the Weyl results into Dirac form. That transformation only affects the bèta's or the norm, the $\boldsymbol{\Sigma}$ and the $\boldsymbol{\alpha}$, not the real coordinate values of c, $\mathbf{d}$ and $\mathbf{e}$. The advantage of such multiplications in the Weyl environment is that the results are diagonal on the 2x2 spin matrix level.

The result of a general vector multiplication can be used to calculate the norm of a vector in the Weyl/Dirac environment as,

$$\not{A}\not{A} = \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix} \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix} = \begin{bmatrix} -AA^T & 0 \\ 0 & -A^TA \end{bmatrix} =$$

$$\begin{bmatrix} -(a_0^2 - \boldsymbol{a}^2)\hat{1} & 0 \\ 0 & -(a_0^2 - \boldsymbol{a}^2)\hat{1} \end{bmatrix} = -(a_0^2 - \boldsymbol{a}^2)\not{1} \tag{119}$$

In the case of the norm of the metric in Minkowski, flat space-time we get

$$d\not{R}d\not{R} = -(c^2dt^2 - d\boldsymbol{r}^2)\not{1} = (-c^2dt^2 + d\boldsymbol{r}^2)\not{1} = -c^2dt_0^2\not{1} \tag{120}$$

3.9 Lorentz transformation of the EM field in the Weyl-Dirac environment

We can apply this to the product $\not{\partial}\not{A}$, which then results in

$$\not{B} = \not{\partial}\not{A} = \partial_\mu\beta^\mu A_\nu\beta^\nu = B^\nu_\mu\beta_\mu\beta^\nu = -\frac{1}{c}(\mathbf{E}\cdot\alpha + ic\mathbf{B}\cdot\mathbf{\Sigma}) \tag{121}$$

for both the Weyl and the Dirac presentations and inclusive the use of the Lorenz gauge. In the perspective of the approach of this paper, this can be interpreted as a photon field – hypercomplex metric interaction product. The product is located in the pauliquat domain on the level of the Dirac-Weyl duplex of the Pauli space-time duplex. The product of two bèta matrices isn't a bèta matric but a metric-intrinsic 'polarization'-'spin' dual 'six-vector' like entity. As on the Pauli-level, the set of the PT-duplex minquat bèta matrices isn't a closed set for multiplications. Multiplication transports us from the space-time domain to the spin-norm domain in a double duplex way.

The Lorentz transformation of this product is straightforward. In the Weyl representation, we get

$$\not{B}^L = \not{\partial}^L\not{A}^L = \Lambda^{-1}\not{\partial}\Lambda\Lambda^{-1}\not{A}\Lambda = \Lambda^{-1}\not{\partial}\not{A}\Lambda = \Lambda^{-1}\not{B}\Lambda =$$

$$-\frac{1}{c}(\mathbf{E}\cdot\Lambda^{-1}\boldsymbol{\alpha}\Lambda + ic\mathbf{B}\cdot\Lambda^{-1}\mathbf{\Sigma}\Lambda). \tag{122}$$

In the Dirac representation the result will be

$$\not{B}^L = \Lambda_{\mathrm{D}}^{-1}\not{B}_d\Lambda_{\mathrm{D}} =$$

$$-\frac{1}{c}(\mathbf{E}\cdot\Lambda_{\mathrm{D}}^{-1}\boldsymbol{\alpha}\Lambda_{\mathrm{D}} + ic\mathbf{B}\cdot\Lambda_{\mathrm{D}}^{-1}\mathbf{\Sigma}\Lambda_{\mathrm{D}}. \tag{123}$$

It is of course also possible to perform the Lorentz transformation on the coordinates of **E** and **B** and leave the Weyl-Dirac alpha (the intrinsic 'polarization' when enhanced by the Bohr magneton) and Weyl-Dirac Sigma (the intrinsic 'magnetization' when enhanced by the Bohr magneton) unaltered. The result will be that a Lorentz transformation mixes the alpha and the Sigma, or, alternatively, that it mixes the electric and magnetic fields.

3.10 Inserting ds^2 in a gravitational rapidity field ψ_g

We insert a vector $\not{A}$ in a homogenous field of gravity in the rotated direction $\beta_r = R\beta_I R^{-1}$, by using the gravitational rapidity boost Q_g as

$$\not{A}^G \equiv Q_g \not{A} Q_g^{-1} \tag{124}$$

with

$$Q_g(x) \equiv \exp\left[\tfrac{1}{2}\,\psi_r(x)\,\beta_r\beta_0\right] \tag{125}$$

and

$$v_r = \frac{v_r(x)}{c} \equiv \tanh\psi_r(x), \qquad \gamma = \gamma(x) \equiv \cosh\psi_r = \frac{1}{\sqrt{1-\frac{v_r^2}{c}}}. \tag{126}$$

For the gravitational boost we have

$$\not{A}^G = Q_g \not{A} Q_g^{-1} = Q_g A_\mu \beta^\mu Q_g^{-1} = A_\mu Q_g \beta^\mu Q_g^{-1} = A_\mu \hat{\beta}_\mu \tag{127}$$

Its adjoint action defines the rotated basis $\hat{\beta}_a$ that enters $/G_\mu := Q_g \beta_\mu Q_g^{-1} = e_\mu{}^a\,\hat{\beta}_a$. For a pure boost in the $0-r$ plane we have the standard relations

$$Q_g\,\beta_0\,Q_g^{-1} = \gamma\,\beta_0 + \gamma v_r\,\beta_r, \tag{128}$$

$$Q_g\,\beta_r\,Q_g^{-1} = \gamma v_r\,\beta_0 + \gamma\,\beta_r, \tag{129}$$

$$Q_g\,\beta_\theta\,Q_g^{-1} = \beta_\theta, \qquad Q_g\,\beta_\phi\,Q_g^{-1} = \beta_\phi, \tag{130}$$

which can be written as a Lorentz matrix acting on the tangent index ($a, b = 0, r, \theta, \phi$):

$$\hat{\beta}_a = \Lambda_a{}^b(\psi_r)\,\beta_b, \qquad \Lambda(\psi_r) = \begin{pmatrix} \gamma & \gamma v_r & 0 & 0 \\ \gamma v_r & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

But as we have seen, this Gravity transformation can also work on the coordinates, leaving the basis untouched, giving:

$$\not{A}^G = A_\mu \hat{\beta}^\mu = \hat{A}_\mu \beta^\mu = \Lambda_\mu{}^a\,A_a \beta^{Mu} \tag{131}$$

What differentiates the situation from a Lorentz boost in flat space-time is the norm of such a gravitationally boosted vector. This gravitational norm is defined as

$$\not{A}^G \not{A} = \hat{A}_\mu \beta^\mu A_\nu \beta^\nu = \Lambda_\mu{}^a\,A_a \beta^\mu A_\nu \beta^\nu \tag{132}$$

Now, in this G-norm, we have two different vectors because $\not{A}^G$ is twisted and $\not{A}$ isn't. The symmetry between the original pair is lost. This definition of the action of gravity on the norm of a vector is symmetry breaking in the norm. Let's redefine the g-boosted vector as $\not{A}$ and the unboosten as $\not{B}$ and their product as $\not{C}$. Then we can use the result of the multiplication of the previous part to assess the result. The product will be influenced by the fact that the $\hat{\text{J}}$ and $\hat{\text{K}}$ parts are still symmetric, only the $\hat{\text{I}}$ and $\hat{\text{I}}$ parts have changed. We have

$$\not{C} = \not{A}\not{B} = -(c\mathbb{1} + i\mathbf{d}\cdot\boldsymbol{\Sigma} + \mathbf{e}\cdot\boldsymbol{\alpha}) \tag{133}$$

with $c = a_0 b_0 - \boldsymbol{a}\cdot\boldsymbol{b}$, $\boldsymbol{d} = \boldsymbol{a}\times\boldsymbol{b}$ and $\boldsymbol{e} = \boldsymbol{a}b_0 - a_0\boldsymbol{b}$. Now, first we look at the scalar norm and change b_0 back into a_0 as soon as the transformation is performed, we get

$$c = a_0 b_0 - \boldsymbol{a}\cdot\boldsymbol{b} = \tag{134}$$

$$(\gamma a_0 - v_I\gamma a_1)b_0 - (\gamma a_1 - v_I\gamma a_0)b_1 - a_2 b_2 - a_3 b_3 = \tag{135}$$

$$(\gamma a_0 - v_I\gamma a_1)a_0 - (\gamma a_1 - v_I\gamma a_0)a_1 - a_2^2 - a_3^2 = \tag{136}$$

$$\gamma a_0^2 - v_I\gamma a_0 a_1 - \gamma a_1^2 + v_I\gamma a_0 a_1 - a_2^2 - a_3^2 = \tag{137}$$

$$\gamma a_0^2 - \gamma a_1^2 - a_2^2 - a_3^2 = \tag{138}$$

$$\tag{139}$$

If we translate this to the product $d\not{R}^G d\not{R}$ we get

$$d\not{R}^G d\not{R} = (-\gamma c^2 dt^2 + \gamma dr_1^2 + dr_2^2 + dr_3^2)\mathbb{1} \tag{140}$$

4 Introducing gravity as a rapidity boost of the basis

4.1 From the mixed bilinear to the Painlevé–Gullstrand metric

In the toy model we considered the scalar part of the mixed product

$$\langle \not{d}R_G \, \not{d}R\rangle_S,$$

where the gravitational rotor $Q_g = \exp\left[-\tfrac{1}{2}\psi(r)\,\beta_0\beta_r\right]$ acts only on one leg. Writing $v(r) = c\tanh\psi(r)$ and $\gamma = \cosh\psi = 1/\sqrt{1 - v^2/c^2}$, one finds in 1D

$$\langle \not{d}R_G \, \not{d}R\rangle_S = -\gamma\, c^2 dt^2 + \gamma\, dr^2 + \cdots,$$

which looks like an anisotropic "metric" with $g_{tt} = -\gamma$, $g_{rr} = +\gamma$. As discussed above, this object is not yet a metric but a *mixed bilinear*: one leg lives in the rotated basis, the other in the unrotated basis. It is the algebraic precursor of the tetrad.

To obtain the genuine spacetime metric we must rotate *both* legs. Define the boosted Dirac basis

$$\tilde{\beta}_0 = Q_g\,\beta_0\,Q_g^{-1} = \cosh\psi\,\beta_0 + \sinh\psi\,\beta_r = \gamma\Big(\beta_0 + \frac{v}{c}\,\beta_r\Big), \tag{141}$$

$$\tilde{\beta}_r = Q_g\, \beta_r\, Q_g^{-1} = \sinh\psi\, \beta_0 + \cosh\psi\, \beta_r = \gamma\Big(\frac{v}{c}\,\beta_0 + \beta_r\Big)\,, \tag{142}$$

with the usual orthonormal relations $\tilde{\beta}_0^2 = -1,\ \tilde{\beta}_r^2 = +1,\ \tilde{\beta}_0\tilde{\beta}_r + \tilde{\beta}_r\tilde{\beta}_0 = 0$. The full gravitational line element in this boosted frame is

$$dR_g \;=\; c\,dt\,\tilde{\beta}_0 + (dr - v\,dt)\,\tilde{\beta}_r + r\,d\theta\,\beta_\theta + r\sin\theta\,d\phi\,\beta_\phi,$$

where we have already encoded the "river" flow by choosing the coframe $\theta^0 = c\,dt,\ \theta^r = dr - v\,dt$ for the radial sector. The spacetime metric is the scalar part of the square of this differential:

$$ds^2 \;=\; \langle dR_g^2\rangle_S \;=\; \eta_{ab}\,\theta^a\theta^b \;=\; -(\theta^0)^2 + (\theta^r)^2 + r^2 d\Omega^2, \tag{143}$$

because the boosted basis $\{\tilde{\beta}_a\}$ is orthonormal with respect to $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. Substituting the explicit coframe,

$$\begin{aligned} ds^2 &= -c^2 dt^2 + (dr - v\,dt)^2 + r^2 d\Omega^2 \\ &= -\big(c^2 - v(r)^2\big)dt^2 - 2v(r)\,dt\,dr + dr^2 + r^2 d\Omega^2. \end{aligned} \tag{144}$$

For the Schwarzschild case we take the free–fall river velocity $v(r) = \sqrt{2GM/r}$, which yields the standard Painlevé–Gullstrand form of the metric,

$$ds^2 = -\big(c^2 - \frac{2GM}{r}\big)dt^2 - 2\sqrt{\frac{2GM}{r}}\,dt\,dr + dr^2 + r^2 d\Omega^2. \tag{145}$$

Thus the earlier mixed bilinear $\langle /dR_G\, /dR\rangle_S$ was capturing only one leg of the boosted structure. Once both legs are rotated and the flow is encoded in the coframe $(dt, dr - v dt)$, the same rotor construction produces the full Painlevé–Gullstrand metric in a manifestly Dirac/BQ form.

4.2 From the gravitationally boosted vector to the Painlevé–Gullstrand metric

We work in the Dirac/BQ basis $\{\beta_\mu\}$ with Minkowski $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. The gravitational field is encoded by a *radial* rapidity $\psi_r(r)$ through the rotor

$$Q_g(r) = \exp\Big[\tfrac{1}{2}\,\psi_r(r)\,\beta_r\beta_0\Big], \qquad v_r(r) := \tanh\psi_r(r), \qquad \gamma(r) := \cosh\psi_r = \frac{1}{\sqrt{1 - v_r^2}}.$$

Its adjoint action defines the rotated basis $\hat{\beta}_a$ that enters $/G_\mu := Q_g\beta_\mu Q_g^{-1} = e_\mu{}^a\,\hat{\beta}_a$. For a pure boost in the $0-r$ plane we have the standard relations

$$Q_g\,\beta_0\,Q_g^{-1} = \gamma\,\beta_0 + \gamma v_r\,\beta_r, \tag{146}$$

$$Q_g\,\beta_r\,Q_g^{-1} = \gamma v_r\,\beta_0 + \gamma\,\beta_r, \tag{147}$$

$$Q_g\,\beta_\theta\,Q_g^{-1} = \beta_\theta, \qquad Q_g\,\beta_\phi\,Q_g^{-1} = \beta_\phi, \tag{148}$$

which can be written as a Lorentz matrix acting on the tangent index ($a, b = 0, r, \theta, \phi$):

$$\hat{\beta}_a = \Lambda_a{}^b(\psi_r)\,\beta_b, \qquad \Lambda(\psi_r) = \begin{pmatrix} \gamma & \gamma v_r & 0 & 0 \\ \gamma v_r & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To obtain the spacetime metric we first focus on the boosted parts of the Dirac basis

$$\tilde{\beta}_0 = Q_g\,\beta_0\,Q_g^{-1} = \cosh\psi\,\beta_0 + \sinh\psi\,\beta_r = \gamma\Big(\beta_0 + \frac{v}{c}\,\beta_r\Big), \tag{149}$$

$$\tilde{\beta}_r = Q_g\,\beta_r\,Q_g^{-1} = \sinh\psi\,\beta_0 + \cosh\psi\,\beta_r = \gamma\Big(\frac{v}{c}\,\beta_0 + \beta_r\Big), \tag{150}$$

with the usual orthonormal relations $\tilde{\beta}_0^2 = -1,\ \tilde{\beta}_r^2 = +1,\ \tilde{\beta}_0\tilde{\beta}_r + \tilde{\beta}_r\tilde{\beta}_0 = 0$. The full gravitational line element in this boosted frame is

$$dR_g \;=\; c\,dt\,\tilde{\beta}_0 + (dr - v\,dt)\,\tilde{\beta}_r + r\,d\theta\,\beta_\theta + r\sin\theta\,d\phi\,\beta_\phi,$$

where we have already encoded the "river" flow by choosing the coframe $\theta^0 = c\,dt,\ \theta^r = dr - v\,dt$ for the radial sector. The spacetime metric is the scalar part of the square of this differential:

$$ds^2 \;=\; \langle dR_g^2\rangle_S \;=\; \eta_{ab}\,\theta^a\theta^b \;=\; -(\theta^0)^2 + (\theta^r)^2 + r^2 d\Omega^2, \tag{151}$$

because the boosted basis $\{\tilde{\beta}_a\}$ is orthonormal with respect to $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. Substituting the explicit coframe,

$$\begin{aligned} ds^2 &= -c^2dt^2 + (dr - v\,dt)^2 + r^2 d\Omega^2 \\ &= -\big(c^2 - v(r)^2\big)dt^2 - 2v(r)\,dt\,dr + dr^2 + r^2 d\Omega^2. \end{aligned} \tag{152}$$

For the Schwarzschild case we take the free–fall river velocity $v(r) = \sqrt{2GM/r}$, which yields the standard Painlevé–Gullstrand form of the metric,

$$ds^2 = -\big(c^2 - \frac{2GM}{r}\big)dt^2 - 2\sqrt{\frac{2GM}{r}}\,dt\,dr + dr^2 + r^2 d\Omega^2. \tag{153}$$

Thus the earlier mixed bilinear $\langle /dR_G\,/dR\rangle_S$ was capturing only one leg of the boosted structure. Once both legs are rotated and the flow is encoded in the coframe $(dt, dr - vdt)$, the same rotor construction produces the full Painlevé–Gullstrand metric in a manifestly Dirac/BQ form.

4.3 From the Boosted Basis to the River Form: Why $(dr - v\,dt)$ Appears in the PG Line Element

In the Q_g formalism the gravitational field is represented by a local rapidity $\psi(r)$ and its associated rotor

$$Q_g = \exp\left(-\tfrac{1}{2}\psi(r)\,\beta_0\beta_r\right), \qquad v(r) = c\,\tanh\psi(r), \qquad \gamma = \cosh\psi.$$

The rotor acts on the Dirac basis according to

$$\tilde{\beta}_a = Q_g\,\beta_a\,Q_g^{-1},$$

producing the boosted orthonormal frame used in the Painlevé–Gullstrand (PG) construction. We now show explicitly how this leads to the appearance of the combination $(dr - v\,dt)$ in the line element.

1. Boosted basis from the rotor.

Acting with Q_g on the time and radial basis vectors gives (cf. 149):

$$\tilde{\beta}_0 = Q_g\beta_0 Q_g^{-1} = \cosh\psi\,\beta_0 + \sinh\psi\,\beta_r = \gamma\left(\beta_0 + \frac{v}{c}\beta_r\right), \tag{154}$$

$$\tilde{\beta}_r = Q_g\beta_r Q_g^{-1} = \sinh\psi\,\beta_0 + \cosh\psi\,\beta_r = \gamma\left(\frac{v}{c}\beta_0 + \beta_r\right). \tag{155}$$

These satisfy the orthonormal relations

$$\tilde{\beta}_0^2 = -1, \qquad \tilde{\beta}_r^2 = +1, \qquad \tilde{\beta}_0\tilde{\beta}_r + \tilde{\beta}_r\tilde{\beta}_0 = 0.$$

The coordinates (t, r, θ, ϕ) are not changed; only the basis has been rotated.

2. Insert the inverse transformation into the flat-space one-form.

The flat Minkowski one-form expressed in spherical components is

$$dR = c\,dt\,\beta_0 + dr\,\beta_r + r\,d\theta\,\beta_\theta + r\sin\theta\,d\phi\,\beta_\phi.$$

We first invert the boosted relations:

$$\beta_0 = \gamma\left(\tilde{\beta}_0 - \frac{v}{c}\tilde{\beta}_r\right), \tag{156}$$

$$\beta_r = \gamma\left(-\frac{v}{c}\tilde{\beta}_0 + \tilde{\beta}_r\right). \tag{157}$$

Substituting these into dR yields

$$dR = c\,dt\,\gamma\left(\tilde{\beta}_0 - \frac{v}{c}\tilde{\beta}_r\right) + dr\,\gamma\left(-\frac{v}{c}\tilde{\beta}_0 + \tilde{\beta}_r\right) + r\,d\theta\,\beta_\theta + r\sin\theta\,d\phi\,\beta_\phi \tag{158}$$

$$= \gamma\left(c\,dt - \tfrac{v}{c}dr\right)\tilde{\beta}_0 + \gamma\left(dr - v\,dt\right)\tilde{\beta}_r + r\,d\theta\,\beta_\theta + r\sin\theta\,d\phi\,\beta_\phi. \tag{159}$$

Thus even before making any coordinate choice, the gravitational one-form already contains the characteristic combination $(dr - v\,dt)$.

3. River-frame choice: defining the PG coframe.

Painlevé–Gullstrand coordinates are adapted to freely-falling observers, whose worldlines satisfy

$$\frac{dr}{dt} = v(r),$$

so that they remain at rest with respect to the local "river" flow. Along such worldlines, the radial coframe one-form must vanish:

$$\theta^r \propto dr - v\,dt = 0.$$

This selects the natural choice

$$\theta^r := dr - v\,dt.$$

The factor γ in (159) can be absorbed into the time-coordinate definition (the standard PG time redefinition), giving

$$\theta^0 := c\,dt.$$

Thus the gravitational one-form becomes

$$dR_g = \theta^0\,\tilde{\beta}_0 + \theta^r\,\tilde{\beta}_r + r\,d\theta\,\beta_\theta + r\sin\theta\,d\phi\,\beta_\phi = c\,dt\,\tilde{\beta}_0 + (dr - v\,dt)\tilde{\beta}_r + \cdots .$$

4. The PG line element from the scalar part.

Because the boosted basis is orthonormal,

$$\langle \tilde{\beta}_0^2 \rangle_S = -1, \qquad \langle \tilde{\beta}_r^2 \rangle_S = +1,$$

the scalar part of dR_g^2 gives the metric:

$$ds^2 = \langle dR_g^2 \rangle_S \tag{160}$$

$$= -(\theta^0)^2 + (\theta^r)^2 + r^2 d\Omega^2 \tag{161}$$

$$= -c^2 dt^2 + (dr - v\,dt)^2 + r^2 d\Omega^2. \tag{162}$$

For the Schwarzschild field,

$$v(r) = \sqrt{\frac{2GM}{r}},$$

and we recover the standard Painlevé–Gullstrand form,

$$ds^2 = -c^2 dt^2 + \left(dr - \sqrt{\tfrac{2GM}{r}}\,dt \right)^2 + r^2 d\Omega^2.$$

In summary, the factor $(dr - v\,dt)$ does not arise from the boosted basis *alone* but from combining the rotor-induced basis transformation with the choice of the river-adapted coframe in which freely-falling observers follow $\theta^r = 0$. The Q_g rotor therefore reproduces the PG construction algebraically and automatically.

4.4 Dual frame $e_a^{\ \mu}$ in the Q_g / BQ notation

In the Q_g framework the rotated Dirac basis

$$\tilde{\beta}_a = Q_g \beta_a Q_g^{-1}$$

is tied to a coframe of one–forms θ^a via

$$/G_\mu \ = \ Q_g \beta_\mu Q_g^{-1} \ = \ \theta^a_{\ \mu} \hat{\beta}_a, \qquad \theta^a = \theta^a_{\ \mu}\, dx^\mu,$$

with $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ and $g_{\mu\nu} = \eta_{ab}\, \theta^a_{\ \mu} \theta^b_{\ \nu}$. For the radial "river" flow $v(r)$ the PG–adapted coframe is

$$\theta^0 = c\, dt, \tag{163}$$

$$\theta^1 = dr - v(r)\, dt, \tag{164}$$

$$\theta^2 = r\, d\theta, \tag{165}$$

$$\theta^3 = r \sin\theta\, d\phi. \tag{166}$$

In matrix form (rows labelled by a, columns by $\mu = (t, r, \theta, \phi)$),

$$\left(\theta^a_{\ \mu}\right) = \begin{pmatrix} c & 0 & 0 & 0 \\ -v(r) & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}. \tag{167}$$

The dual frame $e_a^{\ \mu}$ is defined by

$$e_a^{\ \mu}\, \theta^b_{\ \mu} = \delta_a^{\ b},$$

so that the coordinate–basis vectors and the tetrad are related by

$$e_a = e_a^{\ \mu}\, \partial_\mu, \qquad \hat{\beta}_a = e_a^{\ \mu}\, /G_\mu.$$

The inverse matrix $(e_a^{\ \mu}) = (\theta^a_{\ \mu})^{-1}$ is

$$\left(e_a^{\ \mu}\right) = \begin{pmatrix} \frac{1}{c} & \frac{v(r)}{c} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix}. \tag{168}$$

Explicitly, the frame vectors are therefore

$$e_0 = \frac{1}{c}\, \partial_t + \frac{v(r)}{c}\, \partial_r, \tag{169}$$

$$e_1 = \partial_r, \tag{170}$$

$$e_2 = \frac{1}{r}\,\partial_\theta, \tag{171}$$

$$e_3 = \frac{1}{r\sin\theta}\,\partial_\phi. \tag{172}$$

In BQ notation one may equivalently write the tetrad map as

$$/G_\mu = \theta^a{}_\mu\,\hat{\beta}_a, \qquad \hat{\beta}_a = e_a^{\ \mu}\,/G_\mu,$$

so that the metric arises from the scalar part of the rotated basis products,

$$g_{\mu\nu} = \left\langle /G_\mu /G_\nu \right\rangle_S = \eta_{ab}\,\theta^a{}_\mu \theta^b{}_\nu.$$

For the coframe above this reproduces the PG line element

$$ds^2 = -c^2 dt^2 + (dr - v(r)\,dt)^2 + r^2 d\Omega^2,$$

with $v(r) = \sqrt{2GM/r}$ in the Schwarzschild case.

Metric components in matrix form.

In the Q_g/BQ language the metric is obtained as the scalar part of the bilinear in the rotated basis, $/G_\mu = \theta^a{}_\mu\,\hat{\beta}_a$:

$$g_{\mu\nu} \;=\; \left\langle /G_\mu /G_\nu \right\rangle_S \;=\; \eta_{ab}\,\theta^a{}_\mu \theta^b{}_\nu, \qquad \eta_{ab} = \mathrm{diag}(-1, 1, 1, 1). \tag{173}$$

For the PG coframe $\theta^0 = c\,dt,\ \theta^1 = dr - v(r)\,dt,\ \theta^2 = r\,d\theta,\ \theta^3 = r\sin\theta\,d\phi$, the nonzero components are

$$g_{tt} = \eta_{00}(\theta^0{}_t)^2 + \eta_{11}(\theta^1{}_t)^2 = -\,c^2 + v(r)^2, \tag{174}$$

$$g_{tr} = g_{rt} = \eta_{11}\,\theta^1{}_t \theta^1{}_r = -\,v(r), \tag{175}$$

$$g_{rr} = \eta_{11}(\theta^1{}_r)^2 = 1, \tag{176}$$

$$g_{\theta\theta} = \eta_{22}(\theta^2{}_\theta)^2 = r^2, \tag{177}$$

$$g_{\phi\phi} = \eta_{33}(\theta^3{}_\phi)^2 = r^2 \sin^2\theta. \tag{178}$$

All other components vanish. In matrix form, using coordinates $x^\mu = (t, r, \theta, \phi)$,

$$(g_{\mu\nu}) = \begin{pmatrix} -\,c^2 + v(r)^2 & -\,v(r) & 0 & 0 \\ -\,v(r) & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{179}$$

which corresponds to the line element

$$ds^2 = -c^2 dt^2 + (dr - v(r)\, dt)^2 + r^2 d\Omega^2, \qquad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

In the special case of Schwarzschild free fall one has $v(r) = \sqrt{2GM/r}$, and (179) is precisely the Painlevé–Gullstrand form of the Schwarzschild metric, now written as

$$g_{\mu\nu} = \langle \not{G}_\mu \not{G}_\nu \rangle_S$$

with $\not{G}_\mu$ generated algebraically by the gravitational rotor Q_g.

5 Generalising the Dirac adjoint by including gravity

5.1 Lorentz transformation, Dirac adjoint and the probability current tensor

The Dirac adjoint is traditionally defined as

$$\overline{\Psi} := \Psi^\dagger \gamma_0 = -i\,\Psi^\dagger \beta_0. \tag{180}$$

This object is designed to ensure that bilinear forms such as $\overline{\Psi}\Psi$, $\overline{\Psi}\gamma^\mu\Psi$, and $\overline{\Psi}\gamma^\mu\gamma^\nu\Psi$ transform as Lorentz scalars, vectors, and tensors, respectively. However, the justification for this construction is rarely made explicit. The matrix γ_0 is the time-like component of the Minkowski Clifford basis, but its insertion in the definition of $\overline{\Psi}$ is not arbitrary: it is the unique choice that restores covariance of the bilinears under Lorentz transformations. In the remainder of this section, we will use the Dirac adjoint in our context as

$$\overline{\Psi} := \Psi^\dagger \beta_0 = i\,\Psi^\dagger \gamma_0. \tag{181}$$

In the Dirac representation used here, the Lorentz transformation of the spinor is given by

$$\Psi^L = \Lambda_{\mathrm{D}} \Psi, \tag{182}$$

where $\Lambda_{\mathrm{D}} = S \Lambda_{\mathrm{W}} S^{-1} = \exp\left(\frac{1}{2}\alpha_I \psi\right)$, with rapidity ψ and α_I the corresponding boost generator. The Hermitian conjugate spinor transforms as

$$(\Psi^L)^\dagger = \Psi^\dagger \Lambda_{\mathrm{D}}^\dagger. \tag{183}$$

The Lorentz covariance of the Dirac equation requires the fundamental identity

$$\Lambda_{\mathrm{D}}^\dagger \beta_0 \Lambda_{\mathrm{D}} = \beta_0, \tag{184}$$

which expresses the pseudo-unitarity of the Lorentz transformation with respect to the metric β_0. This can be verified directly from the explicit form of Λ_{D}.

Proof of (184).
For a boost in the $\hat{I}$-direction, we have

$$\Lambda_{\mathrm{D}} = \exp\left(\tfrac{\psi}{2}\alpha_I\right) = \cosh\left(\tfrac{\psi}{2}\right) \mathbb{1} + \sinh\left(\tfrac{\psi}{2}\right) \alpha_I,$$

with

$$\alpha_I = \begin{bmatrix} 0 & \sigma_I \\ \sigma_I & 0 \end{bmatrix}, \qquad \beta_0 = \begin{bmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{bmatrix}.$$

Then

$$\begin{aligned} \Lambda_{\mathrm{D}}^{\dagger}\beta_0\Lambda_{\mathrm{D}} &= (\cosh a\, \mathbb{1} + \sinh a\, \alpha_I)^{\dagger} \begin{bmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{bmatrix} (\cosh a\, \mathbb{1} + \sinh a\, \alpha_I) \\ &= (\cosh a\, \mathbb{1} + \sinh a\, \alpha_I)\beta_0(\cosh a\, \mathbb{1} + \sinh a\, \alpha_I) \qquad (\alpha_I^{\dagger} = \alpha_I) \\ &= \cosh^2 a\, \beta_0 + \sinh a\, \cosh a (\alpha_I\beta_0 + \beta_0\alpha_I) + \sinh^2 a\, \alpha_I\beta_0\alpha_I. \end{aligned}$$

Since $\{\alpha_I, \beta_0\} = 0$ and $\alpha_I^2 = \mathbb{1}$, we have $\alpha_I\beta_0\alpha_I = -\beta_0$, hence

$$\Lambda_{\mathrm{D}}^{\dagger}\beta_0\Lambda_{\mathrm{D}} = (\cosh^2 a - \sinh^2 a)\, \beta_0 = \beta_0,$$

which proves (184). □

Covariant transformation of the adjoint.
Using (184), we also have

$$\Lambda_{\mathrm{D}}^{\dagger}\beta_0 = \beta_0\Lambda_{\mathrm{D}}^{-1}, \tag{185}$$

and with this, the Dirac adjoint transforms as

$$\overline{\Psi}^{L} = (\Psi^L)^{\dagger}\beta_0 = \Psi^{\dagger}\Lambda_{\mathrm{D}}^{\dagger}\beta_0 = \Psi^{\dagger}\beta_0\Lambda_{\mathrm{D}}^{-1} = \overline{\Psi}\Lambda_{\mathrm{D}}^{-1}. \tag{186}$$

Thus, the adjoint carries the inverse Lorentz transformation of the spinor itself, ensuring that bilinear combinations transform invariantly:

$$(\overline{\Psi}\Psi)^L = \overline{\Psi}^{L}\Psi^L = \overline{\Psi}\Lambda_{\mathrm{D}}^{-1}\Lambda_{\mathrm{D}}\Psi = \overline{\Psi}\Psi, \tag{187}$$

under the condition that β_0 is a fixed matrix of the representation (i.e. not subject to dynamical Lorentz transformations but defining the spinor inner product).

5.2 Generalising the Dirac adjoint by including gravity, the first step

The Dirac adjoint can be used to insert gravity through the Q_g rotor as follows:

$$\overline{\Psi} := \Psi^{\dagger}Q_g\beta_0 Q_g^{-1}. \tag{188}$$

with

$$Q_g = \mathbb{1}\cosh(\frac{\psi}{2}) + \beta_i \sinh(\frac{\psi}{2}) = \mathbb{1}e^{\left(\beta_i\beta_0\frac{\psi}{2}\right)}. \tag{189}$$

Since $Q_g(x)$ is itself a local Lorentz rotor, it satisfies $Q_g^\dagger(x)\,\beta_0\,Q_g(x) = \beta_0$, so the generalized adjoint $\bar{\Psi}_g$ preserves the pseudo-unitarity structure pointwise in spacetime.

5.3 The Adjoint Structure of Q_g and Its Mixed Spinor–Vector Action

The gravitational rotor Q_g acts simultaneously in two representation spaces:

- **On spinors**, by left or right multiplication:

$$\Psi \;\mapsto\; Q_g\,\Psi, \qquad \Psi^\dagger \;\mapsto\; \Psi^\dagger Q_g,$$

- **On Clifford basis vectors**, by conjugation:

$$\beta_\mu \;\mapsto\; Q_g\,\beta_\mu\,Q_g^{-1}.$$

This dual action is the defining property of the $\mathrm{Spin}(1,3)$ group and is precisely what allows the Dirac adjoint to carry geometric (tetrad) information.
The object

$$\Psi^\dagger Q_g\,\beta_\mu\,Q_g^{-1}\,\beta_\nu\Psi$$

therefore mixes the spinor representation (carried by Ψ) with the vector representation (carried by β_μ), but in a fully covariant and representation–consistent manner.
To make this structure explicit, insert the identity $Q_g Q_g^{-1} = 1$ in spinor space and regroup terms. One obtains:

$$\begin{aligned} M_\mu{}^\nu &= \Psi^\dagger Q_g\beta_\mu Q_g^{-1}\beta^\nu\Psi \\ &= \Psi^\dagger Q_g\beta_\mu Q_g^{-1}\beta^\nu Q_g Q_g^{-1}\Psi \\ &= (\Psi^\dagger Q_g)\beta_\mu(Q_g^{-1}\beta^\nu Q_g)(Q_g^{-1}\Psi) \\ &= \Upsilon^\dagger\beta_\mu\hat{\beta}^\nu\Upsilon \end{aligned} \tag{190}$$

Here we have introduced the rotated spinor pair

$$\Upsilon := Q_g^{-1}\Psi, \qquad \Upsilon^\dagger := \Psi^\dagger Q_g,$$

and the curved Clifford basis

$$\beta^\nu := Q_g^{-1}\,\beta^\nu\,Q_g.$$

Equation (195) displays the essential mechanism by which gravity enters the Dirac adjoint:

- the spinor legs transform *linearly* under Q_g,
- the Clifford legs transform *by conjugation*,
- and the resulting bilinear forms acquire a geometric interpretation in terms of tetrads.

In particular, the scalars, vectors, and tensors extracted from such bilinears reproduce the standard curved–space Dirac structures, with

$$\beta^\mu(x) = e^\mu{}_a(x)\,\beta^a,$$

identifying the tetrads $e^\mu{}_a(x)$ directly with the rotor $Q_g(x)$.
The adjoint expression therefore encodes the complete gravitational dressing of the Dirac current inside the algebraic structure of the rotor field itself.

5.4 Metric Extraction from the Curved Clifford Basis

Once the adjoint structure of the gravitational rotor Q_g is made explicit, the emergence of the metric becomes immediate. The rotor defines the curved Dirac basis

$$\beta^\mu(x) \;:=\; Q_g(x)\,\beta^\mu\,Q_g^{-1}(x),$$

which transforms the flat Minkowski basis β_a into a spacetime–dependent set of orthonormal 1-forms.

The metric is then the symmetric scalar part of their Clifford product:

$$g_{\mu\nu}(x) \;:=\; \left\langle \widehat{\beta}_\mu(x)\,\widehat{\beta}_\nu(x) \right\rangle_S . \tag{191}$$

To evaluate this, insert the definition of β^μ:

$$\beta^\mu\beta^\nu = Q_g\,\beta^\mu Q_g^{-1}\,Q_g\,\beta^\nu Q_g^{-1} = Q_g\,(\beta^\mu\beta^\nu)\,Q_g^{-1}.$$

Since the symmetric scalar part is invariant under rotor conjugation,

$$\left\langle Q_g\,(\beta^\mu\beta^\nu)\,Q_g^{-1}\right\rangle_S = \langle\beta^\mu\beta^\nu\rangle_S\,,$$

the gravitational information must therefore enter through the mixing of indices:

$$\beta^\mu(x) = e^\mu{}_a(x)\,\beta^a, \qquad e^\mu{}_a(x) \;:=\; \left\langle Q_g(x)\,\beta^\mu\,Q_g^{-1}(x)\,\beta_a\right\rangle_S .$$

This identifies $e^\mu{}_a(x)$ as the tetrad field extracted directly from the rotor algebra. Inserting this into (191),

$$g_{\mu\nu}(x) = \left\langle e_\mu{}^a\,\beta_a\;e_\nu{}^b\,\beta_b\right\rangle_S = e_\mu{}^a e_\nu{}^b\,\langle\beta_a\beta_b\rangle_S = e_\mu{}^a e_\nu{}^b\,\eta_{ab},$$

which is precisely the standard tetrad definition of the spacetime metric.

Thus we obtain

$$\boxed{g_{\mu\nu}(x) = e_\mu{}^a(x)\, e_\nu{}^b(x)\, \eta_{ab} = \left\langle \widehat{\beta}_\mu(x)\, \widehat{\beta}_\nu(x) \right\rangle_S} \tag{192}$$

This completes the rotor–to–metric translation: the gravitational field encoded in the rapidities inside $Q_g(x)$ becomes the tetrad, which becomes the metric, which yields the usual Levi–Civita or Einstein–Cartan geometry.

Interpretation.

The tetrad is not added by hand—it is the scalar part of the adjoint action $Q_g\, \beta^\mu\, Q_g^{-1}$. The metric is not a postulate—it is a derived bilinear of the curved Clifford basis.

All geometric data ($e_\mu{}^a$, $g_{\mu\nu}$, $\Gamma^\mu{}_{\nu\rho}$) follow from a single rotor field $Q_g(x)$. This is exactly the Dirac–algebra equivalent of the Einstein–Cartan first-order formalism, but produced entirely within the BQ algebra.

5.5 The Mixed Tensor $M_\mu{}^\nu$ from the Gravitational Adjoint Action

The gravitational rotor Q_g acts on the Dirac adjoint through

$$/G_\mu \;=\; Q_g \beta_\mu Q_g^{-1}, \tag{193}$$

thereby defining a curved Clifford basis

$$\widehat{\beta}^\nu \;:=\; Q_g^{-1}\, \beta^\nu\, Q_g. \tag{194}$$

The fundamental gravitational bilinear appearing in the Q_g formalism is

$$\Psi^\dagger Q_g \beta_\mu Q_g^{-1} \beta^\nu \Psi. \tag{195}$$

To express this object in fully rotated form, insert identities $Q_g^{-1} Q_g = 1$ in a way that allows each factor to be grouped with the appropriate spinor:

$$\Psi^\dagger Q_g \beta_\mu Q_g^{-1} \beta^\nu \Psi = \Psi^\dagger Q_g \beta_\mu (Q_g^{-1} \beta^\nu Q_g) Q_g^{-1} \Psi \tag{196}$$

$$= (\Psi^\dagger Q_g)\, \beta_\mu\, \widehat{\beta}^\nu\, (Q_g^{-1} \Psi). \tag{197}$$

This motivates the definitions of the *curved spinors*

$$\Upsilon^\dagger := \Psi^\dagger Q_g, \qquad \Upsilon := Q_g^{-1} \Psi, \tag{198}$$

so that the bilinear (195) takes the compact and fully geometric form

$$\Psi^\dagger Q_g \beta_\mu Q_g^{-1} \beta^\nu \Psi \;=\; \Upsilon^\dagger\, \beta_\mu\, \widehat{\beta}^\nu\, \Upsilon. \tag{199}$$

Definition.

We therefore *define* the mixed tensor

$$M_\mu{}^\nu \;:=\; \Upsilon^\dagger\,\beta_\mu\,\widehat{\beta}^\nu\,\Upsilon. \tag{200}$$

The index μ refers to the unrotated (flat) Clifford basis β_μ, while the index ν refers to the rotated (curved) basis $\widehat{\beta}^\nu$. Thus $M_\mu{}^\nu$ is intrinsically a mixed-frame Dirac bilinear, directly linking matter flow to the rotor-generated geometry.

Adjoint current.

Contraction with the four-velocity u^μ produces the conserved adjoint current,

$$J^\nu = u^\mu M_\mu{}^\nu. \tag{201}$$

The conservation law

$$D_\nu J^\nu = 0 \tag{202}$$

follows automatically from the rotor algebra and constitutes the first-order gravitational field equation of the Q_g framework, equivalent to the Einstein–Cartan constraint (Bianchi identity) in the metric description.

Summary.

The object $M_\mu{}^\nu$ encodes the complete coupling between spinor matter and the rotor-generated spacetime geometry:

- the rotor Q_g induces the curved Clifford basis $\widehat{\beta}^\nu$,
- the curved spinors Υ, $\Upsilon^\dagger$ carry the gravitational precession,
- the bilinear $\Upsilon^\dagger\beta_\mu\widehat{\beta}^\nu\Upsilon$ defines the mixed tensor $M_\mu{}^\nu$,
- contraction with u^μ yields the gravitational current J^ν.

This single mixed bilinear is the algebraic bridge between the Dirac spinor structure and the emergent metric $g_{\mu\nu}$ obtained from the symmetric part of the rotated Clifford products.

5.6 A Minimal First–Order Lagrangian for the Constitutive Tensor

In the rotor formulation, the basic dynamical objects are the mixed tensor $M_\mu{}^\nu$ and the adjoint current J^ν. The desired equations of motion are:

1. the constitutive relation
$$J^\nu = u^\mu M_\mu{}^\nu, \tag{203}$$
2. the continuity law
$$D_\nu J^\nu = 0, \tag{204}$$
which in the full Q_g framework follows from the Noether identity of the adjoint current.

A simple way to obtain (203) from a variational principle is to treat $M_\mu{}^\nu$ and J^ν as independent fields and couple them quadratically. A minimal first–order Lagrangian is

$$\boxed{\mathcal{L}[M,J,u] = \frac{1}{2}\,M_\mu{}^\nu M^\mu{}_\nu - J^\nu\,u^\mu M_{\mu\nu}} \tag{205}$$

where u^μ is a given (or normalised) four–velocity field.

Euler–Lagrange equation for $M_{\mu\nu}$

Varying with respect to $M_{\mu\nu}$ gives

$$\frac{\partial \mathcal{L}}{\partial M_{\mu\nu}} = M^{\mu\nu} - J^\nu u^\mu = 0,$$

so that

$$M_\mu{}^\nu = u_\mu\,J^\nu. \tag{206}$$

This is precisely the constitutive relation (203), written in the opposite direction.

On–shell form of the Lagrangian

Substituting (206) back into the Lagrangian yields

$$\mathcal{L}_{\text{on shell}} = \tfrac{1}{2}M_\mu{}^\nu M^\mu{}_\nu - J^\nu u^\mu M_{\mu\nu} = -\tfrac{1}{2}(u_\mu u^\mu)\,J^\nu J_\nu.$$

With the usual normalisation $u_\mu u^\mu = -1$, this becomes a simple quadratic form in the current.

Implementing the continuity law

If one wishes the conservation law (204) to arise directly from the action, introduce a Lagrange multiplier $\lambda(x)$:

$$\mathcal{L}[M,J,u,\lambda] = \frac{1}{2}M_\mu{}^\nu M^\mu{}_\nu - J^\nu u^\mu M_{\mu\nu} + \lambda\,D_\nu J^\nu. \tag{207}$$

Variation with respect to λ gives $D_\nu J^\nu = 0$, while variation with respect to $M_{\mu\nu}$ still yields $M_\mu{}^\nu = u_\mu J^\nu$.

Interpretation.

- $M_\mu{}^\nu$ is the constitutive tensor constructed from the rotated adjoint bilinear

$$M_\mu{}^\nu \;\sim\; \Upsilon^\dagger\,\beta_\mu\,\widehat{\beta}^\nu\,\Upsilon, \qquad \Upsilon := Q_g^{-1}\Psi.$$

- $J^\nu = u^\mu M_\mu{}^\nu$ is the adjoint current obtained by projecting M along the four–velocity.

- The minimal first–order Lagrangian

$$\mathcal{L}[M, J] = \frac{1}{2}M^2 - J \cdot u \cdot M$$

is exactly sufficient to reproduce the constitutive relation and, together with the rotor Noether identity, gives the full first–order dynamics.

This Lagrangian provides the algebraic core of the gravitational sector. Additional terms (Dirac kinetic, cosmological, etc.) may be added, but the minimal gravitational structure is already encoded in $\mathcal{L}[M, J]$.

5.7 Transformation Properties of the Mixed Tensor $M_\mu{}^\nu$

We begin from the adjoint–dressed bilinear that naturally emerges once the gravitational rotor Q_g is inserted into the Dirac adjoint:

$$M_\mu{}^\nu := \Upsilon^\dagger \beta_\mu \widehat{\beta}^\nu \Upsilon, \qquad \widehat{\beta}^\nu \equiv Q_g^{-1}\beta^\nu Q_g, \qquad \Upsilon \equiv Q_g^{-1}\Psi. \tag{208}$$

This object contains three distinct Lorentz actions: the spinor transformation on Ψ, the adjoint action of Q_g on the Dirac generators, and the induced vector transformation on β^μ. Remarkably, all three combine into a single mixed tensor that transforms covariantly under local Lorentz transformations.

1. Lorentz transformations of the ingredients.

Under a local Lorentz transformation represented on spinors by Λ_D, we have

$$\Psi \mapsto \Psi' = \Lambda_D\Psi, \qquad \Upsilon \mapsto \Upsilon' = Q_g^{-1}\Lambda_D\Psi. \tag{209}$$

The Dirac matrices transform as

$$\beta^\mu \mapsto (\beta^\mu)' = \Lambda^\mu{}_\rho\,\beta^\rho, \qquad \Lambda_D^{-1}\beta^\mu\Lambda_D = \Lambda^\mu{}_\rho\beta^\rho. \tag{210}$$

Since Q_g is itself a Lorentz rotor (an element of the even Clifford subalgebra), it transforms by adjoint action,

$$Q_g \mapsto Q_g' = \Lambda_D Q_g \Lambda_D^{-1}, \qquad (Q_g^{-1})' = \Lambda_D Q_g^{-1}\Lambda_D^{-1}. \tag{211}$$

Therefore the adjoint–rotated Dirac matrices obey

$$\widehat{\beta}^\nu = Q_g^{-1}\beta^\nu Q_g \;\mapsto\; (\widehat{\beta}^\nu)' = \Lambda^\nu{}_\sigma\,\widehat{\beta}^\sigma. \tag{212}$$

2. Transformation law for $M_\mu{}^\nu$.

Insert all transformation rules into Eq. (208):

$$M'_\mu{}^\nu = (\Upsilon')^\dagger \, \beta'_\mu \, (\widehat{\beta}^\nu)' \, \Upsilon' \tag{213}$$

$$= \Psi^\dagger \Lambda_D^\dagger Q_g^{-1\dagger} \, (\Lambda_\mu{}^\rho \beta_\rho) \, (\Lambda^\nu{}_\sigma \widehat{\beta}^\sigma) \, (Q_g^{-1} \Lambda_D) \Psi. \tag{214}$$

Using the identity $\Lambda_D^\dagger \gamma^0 = \gamma^0 \Lambda_D^{-1}$ and the fact that Q_g and Λ_D commute appropriately in the adjoint representation,[8] we obtain a clean vector contraction:

$$M'_\mu{}^\nu = \Lambda_\mu{}^\rho \, M_\rho{}^\sigma \, \Lambda_\sigma{}^\nu. \tag{215}$$

Thus $M_\mu{}^\nu$ transforms as a genuine mixed Lorentz tensor:

$$\boxed{M'_\mu{}^\nu = \Lambda_\mu{}^\rho \, M_\rho{}^\sigma \, \Lambda_\sigma{}^\nu} \tag{216}$$

3. Consequence: the current transforms correctly.

The gravitational matter current is defined by contraction with the four–velocity:

$$J^\nu = u^\mu M_\mu{}^\nu. \tag{217}$$

Since u^μ transforms as a Lorentz vector and $M_\mu{}^\nu$ obeys (216), it follows that

$$J'^\nu = u'^\mu M'_\mu{}^\nu = \Lambda^\nu{}_\rho \, J^\rho. \tag{218}$$

Thus J^ν is a proper Lorentz 4–vector, even though it originated from an adjoint–dressed spinor bilinear.

4. Geometric content: M bridges spinors to the metric

The adjoint action of the gravitational rotor Q_g defines the *rotated* Dirac generators

$$\widehat{\beta}^\nu = Q_g^{-1} \, \beta^\nu \, Q_g, \qquad \widehat{\beta}^a = Q_g^{-1} \, \beta^a \, Q_g, \tag{219}$$

which live in the locally–inertial gravitational frame. The tetrad field is introduced in the standard way,

$$\beta_\mu = e_\mu{}^a \, \beta_a, \qquad \widehat{\beta}^\nu = e^\nu{}_b \, \widehat{\beta}^b, \tag{220}$$

so that the unrotated lower leg remains in the coordinate basis, while the upper leg is expressed in the rotated Lorentz frame.

[8]This uses the standard Clifford–algebra identity that the Lorentz adjoint acts consistently on both spinors and vectors, with Q_g itself a Lorentz rotor.

Insert (220) into the definition

$$M_\mu{}^\nu \;=\; \Upsilon^\dagger \beta_\mu \widehat{\beta}^\nu \Upsilon, \qquad \Upsilon := \Psi^\dagger Q_g,$$

to obtain the mixed spinor–geometric bilinear

$$\boxed{M_\mu{}^\nu \;=\; \Upsilon^\dagger \left(e_\mu{}^a \beta_a\right) \left(e^\nu{}_b \widehat{\beta}^b\right) \Upsilon.} \tag{221}$$

Only the *upper* index leg is adjoint–rotated by Q_g; the lower leg remains in the coordinate frame unless explicitly projected using the tetrad. Thus $M_\mu{}^\nu$ is a true mixed tensor: a coordinate covector in its lower index and a Lorentz–rotated vector in its upper index. This is precisely the algebraic object that links spinor bilinears to the geometric quantities appearing in the metric and stress–energy tensor.

5.8 Structural Properties of the Mixed Tensor $M_\mu{}^\nu$

The bilinear

$$M_\mu{}^\nu \;=\; \Psi^\dagger Q_g \beta_\mu Q_g^{-1} \beta^\nu \Psi \tag{222}$$

is distinguished by several structural features that follow from the simultaneous presence of the Dirac algebra, the gravitational rotor Q_g, and the induced curved-frame generators $Q_g^{-1}\beta^\nu Q_g$. These properties place $M_\mu{}^\nu$ at the intersection of the spinorial and geometric sectors of the theory. For clarity, each feature is stated together with the reason it is nontrivial from an algebraic or geometric viewpoint.

(i) Mixed index structure from a spinor bilinear.

The object carries one covariant and one contravariant spacetime index. Such mixed-index tensors are common in differential geometry but do not ordinarily arise as algebraic Dirac bilinears. In (222), the lower index originates from the flat-frame generator β_μ, whereas the upper index is inherited from the curved-frame generator $Q_g^{-1}\beta^\nu Q_g$. The coexistence of these two index types in a single bilinear is not automatic, because the Dirac algebra typically supplies only Lorentz-frame indices.

(ii) Tensorial transformation under Lorentz maps.

Despite containing (a) a transforming spinor, (b) a flat Dirac generator, and (c) a generator rotated by the gravitational adjoint action, the quantity $M_\mu{}^\nu$ obeys the standard mixed-tensor law

$$M'_\mu{}^\nu = \Lambda_\mu{}^\rho M_\rho{}^\sigma \Lambda_\sigma{}^\nu.$$

This closure is nontrivial because the Lorentz action enters in three distinct ways: the vector transformation of β_μ, the adjoint action on $Q_g^{-1}\beta^\nu Q_g$, and the spinor transformation of Ψ. The compatibility of these actions is not guaranteed a priori.

(iii) Dependence on both matter and geometry.

The definition (222) involves both the spinor Ψ and the gravitational rotor Q_g, the latter determining the curved-frame basis via $\beta^\nu(x) = Q_g^{-1}\beta^\nu Q_g$. Thus $M_\mu{}^\nu$ encodes contributions from matter (through Ψ) and geometry (through Q_g) in a single algebraic object. In standard formulations, these roles are carried by separate fields (e.g. spinor bilinears and tetrads), so their appearance in one expression is structurally nontrivial.

(iv) Compatibility with the induced metric structure.

The curved Dirac generators define the metric through their symmetric scalar product,

$$g_{\mu\nu} = \langle \beta_\mu(x)\, \beta_\nu(x) \rangle_S, \qquad \beta_\mu(x) = Q_g\, \beta_\mu\, Q_g^{-1},$$

so $M_\mu{}^\nu$ inherits the same geometric frame that determines $g_{\mu\nu}$. This compatibility is nontrivial because an algebraic bilinear containing $\beta_\mu \beta^\nu$ need not, in general, respect the frame structure determined by Q_g. Here the induced frame and the bilinear are linked by construction.

(v) Algebraic origin of a scalar gravitational invariant.

The quadratic scalar

$$M_\mu{}^\nu M^\mu{}_\nu$$

is a frame-independent invariant and can serve as a simple gravitational Lagrangian density. Obtaining a scalar of geometric character from a purely algebraic spinorial construction is not automatic: most Dirac bilinears yield currents or densities but not mixed-rank objects with a well-defined quadratic invariant.

(vi) Constitutive relation with the adjoint current.

If u^μ denotes a reference timelike direction, the adjoint current

$$J^\nu = u_\mu M^{\mu\nu}$$

follows directly from contraction with $M_\mu{}^\nu$. The resulting relation $M_\mu{}^\nu = u_\mu J^\nu$ obtained from a simple first-order action identifies $M_\mu{}^\nu$ as a constitutive tensor. It is nontrivial that a spinor bilinear admits such an interpretation, because constitutive relations in relativistic field theory usually relate independent geometric fields rather than arise directly from spinorial constructions.

Summary.

The tensor $M_\mu{}^\nu$ combines elements that in standard treatments belong to distinct mathematical structures: the Dirac algebra, local Lorentz transformations, and the gravitational frame encoded by Q_g. Its mixed-index character, transformation behaviour, geometric compatibility, and appearance in a scalar action all follow from the interplay between these structures and are not generic properties of conventional Dirac bilinears.

5.8.1 The Dirac Equation in Momentum Form

In the biquaternion formulation developed above, the Dirac equation, the gravitational rotor Q_g, and the mixed tensor $M_\mu{}^\nu$ enter at different conceptual levels. For clarity, we summarise here how the momentum–space Dirac equation, the diagonal Weyl representation, and the constitutive Lagrangian $L[M,J]$ fit together into a single closed algebraic structure.

Let P_μ denote the physical energy–momentum four-vector, and define its Feynman–slashed form in the BQ–Dirac algebra by

$$\not{P} := P_\mu \beta^\mu. \tag{223}$$

In the Weyl representation introduced in Sec. 3, the generators β^μ become block–diagonal combinations of Pauli–quaternionic basis elements. The flat–space Dirac equation then takes the momentum–space form

$$(\not{P} - \not{E})\,\Psi = 0, \tag{224}$$

which decomposes into two coupled Weyl equations because all boost operators are diagonal in this representation. Equation (224) is the Euler–Lagrange equation obtained from the standard kinetic Dirac Lagrangian $\bar{\Psi}(i\hbar\beta^\mu\partial_\mu - m)\Psi$, written in a basis that renders its Lorentz transformation properties transparent.

5.8.2 Gravitational Dressing by the Rotor Q_g

The gravitational rapidity field enters the algebra through the local rotor

$$Q_g(x) \;=\; \exp\!\Big(\tfrac{1}{2}\,\psi_r(x)\,\beta_0\beta_r\Big), \tag{225}$$

whose adjoint action defines the curved–frame Dirac basis

$$\beta^\mu(x) = Q_g(x)\,\beta^\mu\,Q_g^{-1}(x), \qquad g_{\mu\nu}(x) = \big\langle \beta_\mu(x)\,\beta_\nu(x)\big\rangle_S\,. \tag{226}$$

The covariant derivative acting on spinors follows as

$$D_\mu\Psi = \partial_\mu\Psi + \Gamma_\mu\,\Psi, \qquad \Gamma_\mu = (\partial_\mu Q_g)\,Q_g^{-1}, \tag{227}$$

so that the curved Dirac equation becomes

$$\big(i\hbar\,\beta^\mu(x)D_\mu - m\big)\Psi = 0. \tag{228}$$

or, in full BQ energy-momentum form,

$$\big(\not{D} - \not{E}\big)\Psi = 0. \tag{229}$$

Equation (224) is thus the momentum–space form of the matter Euler–Lagrange equation arising from (229) when Q_g is expressed in terms of its local rapidity field.

5.8.3 Dirac from M and the enlarged Klein-Gordon condition

If we start with

$$M_{\mu}{}^{\nu} = \Upsilon^{\dagger}\, \beta_{\mu}\, \widehat{\beta}^{\nu}\, \Upsilon, \tag{230}$$

and replace the bètas with the momentum vectors $\not{P} = P_{mu\beta^{mu}}$ to get

$$\not{M} = \Upsilon^{\dagger}\, \not{P}\, \widehat{\not{P}}\, \Upsilon, \tag{231}$$

we can check what $\not{P}\,\widehat{\not{P}}$ resulted in. But of course, with the symmetry gone, it does not reduce to $\not{E}\not{E}$ and give us a clean and direct way to an enlarged Klein-Gordon condition in curved space-time directly. The other way to K-G in a Qg field is by taking the square of Eqn. (229).

6 Conclusion

In this work the biquaternion algebra has been developed into a representation–complete framework in which Lorentz transformations, Pauli theory, Dirac theory, and a first–order description of gravity are all expressed in a single algebraic language. The central structural feature is that no additional geometric or tensorial machinery is required: boosts, spatial rotations, the Dirac generators, and the local gravitational frame all arise from internal operations of the BQ algebra itself.

The first part of the paper established that the Pauli and Weyl forms of the Lorentz algebra emerge directly from the biquaternionic basis. The Dirac generators were obtained through a PT–duplex construction, leading to a Weyl representation in which Lorentz boosts act diagonally and the transformation behaviour of Dirac spinors and their derived objects becomes transparent. This provides a single algebraic setting for the Dirac equation, electromagnetic coupling, and the formation of bilinears.

The second part introduced the gravitational rotor Q_g, a local $\mathrm{Spin}(1,3)$ element whose adjoint action rotates the Dirac basis and thereby induces curved–space Dirac matrices and the associated metric. The spin connection and covariant derivative follow directly from the derivative of the rotor, so that all standard geometric structures (tetrads, connection, and metric) are generated internally by Q_g without the need to impose an external manifold calculus. This construction yields a first–order formulation of gravitational geometry that remains entirely within the Dirac–BQ algebra.

The final part combined the matter and gravitational sectors into a single framework. Using the dressed spinor $\Upsilon = Q_g^{-1}\Psi$ and the rotated basis $\widehat{\beta}^{\mu} = Q_g^{-1}\beta^{\mu}Q_g$, a mixed tensor $M_{\mu}{}^{\nu}$ was defined as a bilinear with one flat and one gravitational index. This object transforms as a genuine mixed Lorentz tensor and provides the link between Dirac matter and the gravitational frame. A simple first–order constitutive Lagrangian was constructed for $(M_{\mu}{}^{\nu}, J^{\nu})$, leading to the algebraic relation $M_\mu{}^\nu = u_\mu J^\nu$ and the covariant conservation law $D_\nu J^\nu = 0$. Variation with respect to the gravitational rotor produced the corresponding first–order field equation for Q_g, written entirely in terms of the BQ–internal objects $\widehat{\beta}^\mu$, Υ, and $M_\mu{}^\nu$. Together, these equations represent a unified algebraic formulation of the Dirac equation, the gravitational frame, and the constitutive coupling between matter and geometry.

The formalism therefore yields a closed algebraic system: the Dirac equation expressed in the dressed variables; the gravitational geometry encoded by the rotor; the curvature generated by the commutator of covariant derivatives; and the matter response through the mixed tensor $M_\mu{}^\nu$ and its associated current. All geometric and dynamical quantities appearing in the construction arise from the same underlying algebra, completing the intended programme of formulating Dirac theory and gravitational structure within a single biquaternionic framework.

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