On the impossibility of divergent sequences in The Collatz conjecture

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Abstract

This paper proves that it is impossible to have an infinite sequence of numbers, built according to the rules of the Collatz Conjecture. This is due to the accumulation of unfulfillable requirements for the first member of this sequence when the number of steps becomes infinite.

I. Theorem on the Change of the Minimal starting value

1.1. Statement

In any chain of two or more consecutive operations in a Collatz sequence of the "up" type (odd step 3n+1, followed by a single division by 2), the value, which defines the minimum possible starting value (n) of the sequence, necessarily changes with respect to the current denominator.

1.2. Proof

Let us consider the function f(n) = (3n+1) / 2^a, which transforms an odd number into the next odd number in the trajectory. In a series of two consecutive "steepest" climbs (where the division by 2 is minimal, i.e. a=1), the minimal solution for the initial term n undergoes the following transformation:

Step 1:

The expression is f1(n) = (3n+1) / 2.

For the result to be an integer, it is necessary:

3n + 1 is divisible by 2 => n is odd (n = 1, 3, 5...)

Minimal solution (Base r1): 1

Step 2:

We apply a second consecutive operation on the result of the first:

f2(n) = (3 * ((3n+1) / 2) + 1) / 2 = (9n + 3 + 2) / 4 = (9n + 5) / 4

For f2(n) to be an integer, it is necessary that the numerator is divisible by 4:

(9n + 5) / 4 = integer

So n + 1 must be divisible by 4 => n = 3, 7, 11...

New minimal basis (r2): 3

Proved: r2 > r1

II.Proof

If we consider the operations in a Collatz sequence not as generators of the next number, but as generators of a fraction equivalent to the next number, we can see how increasing the number of steps leads to increasingly restrictive requirements for the first number from which the sequence began. The accumulation of restrictions, if too long (beyond the possibilities of the first number to cover all the criteria), would lead to a critical point at which the first number itself would no longer meet the restrictions.

This would clearly prove that an "infinite Collatz sequence" is not possible.

Let us present an exemplary and real Collatz trajectory to understand the idea. We take as a starting point an odd number that is a multiple of 3, since this type of number can be considered an odd beginning of any Collatz sequence. They have no "predecessors" that are odd, but only an infinite number of even predecessors are possible, all of which reach the first odd one.

Let's consider the sequence that starts with 3.

It is:

3, 10, 5, 16, 8, 4, 2, 1

Next comes a repetition of the trivial cycle 4, 2, 1.

Let's denote the beginning 3 by n and consider the sequence as fractions:

n 3n+1 (3n+1)/2 (9n+5)/2 (9n+5)/4 (9n+5)/8 (9n+5)/16 (9n+5)/32

Let's now consider what is the minimum n in each fraction, so that it itself is an integer:

In the first two numbers (3 and 10) there are no restrictions - n can be any positive integer and the value of the expression will be an integer.

In the third expression, a requirement already arises. For (3n+1)/2 to be an integer, n must be odd. The same requirement remains for the fourth step.

However, an exact value arises in the fifth expression. The expression (9n+5)/4 can be an integer only if n = 3 or other suitable larger numbers (7, 11, 15, etc.). The difference is the size of the denominator and an arithmetic progression is formed.

In the next step, the list of suitable numbers becomes 3, 11, 19, etc.

In the seventh step, the list is 3, 19, 35, etc.

In the last step, it is 3, 35, 67, etc.

Obviously, all sets have a common first term (3), and this allows the series itself with a similar sequence of steps to exist with this very beginning.

But this is not always the case. Consecutive terms of a series of fractions can create non-identical sets with completely different minimum values.

Let's examine another example - starting from n=9.

The sequence is as follows:

9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1

We represent the sequence by fractions:

n   3n+1   (3n+1)/2   (3n+1)/4   (9n+7)/4   (9n+7)/8 …

These are the first 6 numbers. Let's trace the requirements for n to be integers.

Obviously, at the third step, there is a requirement that n be odd.

At the fourth step, there is a requirement that n = 5 or other suitable ones (9, 13, 17, etc.).

At the fifth step, the possible values are 1, 5, 9, 13, 17, etc.

Obviously, we have a smaller available value (1), but it cannot be used because it is not present in the possible values for the previous step. Thus, there can be no "reduction" of the requirements here, since it is limited by the minimum of each previous step (otherwise the series simply cannot "pass" through them).

This is a very important point in the present analysis, because it proves that in any Collatz series it is not possible to exist at any stage a "relaxation" of the conditions, even if for some particular fraction smaller solutions are possible. The general solution will always require searching for the "smallest common value satisfying all steps".

Thus here - at the fifth step the minimum requirement remains that of the fourth n=5.

We continue with the sixth step.

The expression is: (9n+7)/8

The possible values of n for which it is an integer are 1, 9, 17, 25, etc.

This set of possible values already excludes 5. We cannot choose 1 either, because 5 is already fixed as the "smallest possible value of the sets so far". So we have to move on to the next value, which is present in all the sets so far. This is obviously n=9.

In the following fractions, the minimum requirement in the sets of possible solutions at each step does not change and remains the same until the end (9).

So far we have considered two real sequences starting from the first multiples of 3 – these are 3 and 9 itself.

But now we consider an imaginary sequence in which the seventh term is not an "upward movement" as in the indicated sequence (starting at 9), but is, for example, another "downward movement". Then the seventh term would have the form:

(9n+7)/16

The numbers satisfying this expression are 1, 17, 33, 49, etc.

The smallest number that can satisfy all the previous steps is 17. It would generate the steps:

17, 52, 26, 13, 40, 20, 10 … etc. down to 5 and 4,2,1

The first steps are the same:

n   3n+1   (3n+1)/2   (3n+1)/4   (9n+7)/4   (9n+7)/8

The seventh step is 10, etc.

Again, we have an increase in the possible first number.

But in fact, it will be even larger, because 17 is not a multiple of 3, i.e. it cannot be an "absolute beginning of a chain" (or a "leaf" in classical terminology).

If we only look for a complete chain starting from a multiple of 3, then the minimum value of this step becomes 33.

But this number is absent from the admissible set for the 5th step. Thus, the smallest that is a multiple of 3 and is present in all admissible sets becomes 81.

In this case, we see how changing just one step drastically changes the minimum size of the first number.

Obviously, in no real Collatz series can a situation arise in which a fraction creates a condition that is larger than the very first number in the series. This is because no number can generate a sufficient number of operations so that the accumulated requirements exceed the requirement for its own size. All numbers reach some cycle, after which the fractions begin to repeat and generate the same requirements, and there is no subsequent expansion of the requirements that would require a new choice.

If we take an "imaginary" step as we experimented above, then the requirements for the first term automatically change and a possible series is shifted as a start into a zone of larger numbers.

Thus we come to the key question of the present study - is a hypothetical "infinite series" with Collatz operations possible.

The obvious answer is "No". With an infinite number of steps, the conditions for the first number will increase infinitely and at some stage will surpass the number itself. It will also have to become infinitely large. That is, there cannot be an infinite Collatz sequence that has a fixed starting number.

III. The Impossible Counterexample

However, is it possible to have a unique trajectory in which the conditions do not grow? And so the same conditions are repeated indefinitely? Or at least to have a single number that then appears as a possible solution in all sets to infinity and thus the growth is stopped?

This is not possible.

The minimum possible value that satisfies the requirement that every fraction be an integer is essentially an odd number between 1 and the denominator. This number cannot be constant, because as we proved in the **Theorem** at the beginning, every two consecutive moves "up" by Collatz lead to a change in this number. Consecutive moves "up" are mandatory when searching for an infinite trajectory, because otherwise the growth of the numbers would not be possible.

On the other hand, at each step, the sets of possible solutions represent arithmetic progressions with a difference of exactly the denominator. The first number is incomplete – it is part of the size of the denominator between 1 and its size. The following are arithmetic progressions from the "first" plus the denominator, then the "second" plus the denominator, etc. In order to find terms that repeat as a size in all progressions simultaneously, we will have to look for only those that differ from each other by the largest difference reached (the highest power of 2). This will constantly shift the minimum possible term "upward".

That is, if in some range we "fix" some basis as "eternal" and claim that it will not grow, then in the next series of 2 consecutive movements "upward", this basis will change. And since any basis that we have accepted is a minimum, it will change "upward". That is, will go to any of the following terms of the arithmetic progression starting from it and with a difference of the corresponding power of 2.

This excludes the possibility of a fixed "anchor" as a starting number that does not change to infinity. The requirements for the first number will increase, which means that there is no such first number that would start an infinite Collatz series.

Therefore, an infinite Collatz series is impossible.

Conclusion

This paper answers one of the two subcases for the possible rejection of the Collatz Conjecture – namely – a hypothetical "divergent series". The possibility of an alternative cycle beyond 4,2,1 remains open.

This paper can be seen as a simplified version of the previous paper, entitled "On the impossibility of an infinite chain in The Collatz Conjecture"[1].

References

[1] Bozhilov, D. (2025). On the impossibility of an infinite chain in The Collatz Conjecture. Zenodo. https://doi.org/10.5281/zenodo.17599182

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