On the impossibility of an infinite chain in The Collatz Conjecture

Author: Dobri Bozhilov

Sofia, Bulgaria, AI Laboratory

zadrugata2019@gmail.com

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Abstract

The Collatz Conjecture defines a number sequence constructed as follows:

If X_1 is odd, then X_2 = 3X_1 + 1.

If X_1 is even, then X_2 = X_1 / 2.

The conjecture itself asserts that every such sequence, starting from any number, eventually reaches the number 1 (or, more precisely, the cycle 4-2-1).

There are two hypothetical exceptions - either another cycle other than 4-2-1, or an infinite divergent sequence. In this paper, we will prove that the second option is impossible.

Proof

Let us consider the consecutive odd terms of this sequence, defined by the function f(X) = (3X + 1)/2 for an odd integer X, and determine whether they can grow indefinitely without interruption. That is, whether there can never be more than one "division by 2" between two consecutive odd terms.

To achieve this, every term X must satisfy the condition X = 4k + 3, where k is any integer. In other words, X must be an odd number that, when divided by 4, leaves a remainder of 3. Only then can the condition of "no more than one division by 2 between two odd terms" be met.

For example, the numbers 7, 11, 15, etc., are "good" terms.

In contrast, the numbers 9, 13, 17, etc., are "bad" terms.

Clearly, (7 * 3 + 1)/2 = 11, which is again odd.

However, (9 * 3 + 1)/2 = 14, which is even and will require at least one additional division by 2, causing the next odd term to be smaller than the previous one.

Thus, we see that half of the odd numbers are "good," while the other half are "bad." The "good" ones are those that, when divided by 4, leave a remainder of 3, i.e., satisfy X = 4k + 3.

The question is: how many times can we repeat the operation (3X + 1)/2 before reaching a term that is even?

The thesis to be proven is defined as the following theorem:

Every sequence of odd numbers formed by the rule $X_n = (X_{n-1} * 3 + 1)/2$ is finite, regardless of the size of the first term. That is, an infinite sequence of odd numbers formed in this way is impossible.

Let us examine the sequence of iterations:

The first number is:

$X_1 = 4k + 3$, where k is an arbitrary integer.

Then:

$X_2 = ((4k + 3) * 3 + 1)/2 = 6k + 5$, which is an integer and odd.

$X_3 = 9k + 8$, which is an integer and may be even or odd depending on k. Here, we see that additional conditions on k begin to emerge, narrowing the initial definition of "any integer."

$X_4 = (27k + 25)/2$, which is an integer only if k is odd, since $27k + 25$ must be even. Thus, the entire set of even numbers is eliminated as possible values for k.

$X_5 = (81k + 77)/4$, which is an integer if $k = 4k_1 + 3$, where k_1 is any integer, since $81k + 77$ must be divisible by 4.

Thus, we now have an even stricter condition on k: it must itself leave a remainder of 3 when divided by 4.

If we substitute this new k into the initial expression, we find that the first number in the sequence must satisfy:

$X_1 = 16k_1 + 15$, where k_1 is an arbitrary integer.

This means the definition of X has been narrowed, and it clearly cannot be less than 15.

However, this implies that if we choose an initial $X_1 < 15$ and assume an infinite number of iterations, after a certain number of steps, a requirement will arise that the initial number must be greater than the one chosen.

This is a contradiction.

Therefore, regardless of the initial number chosen, it is impossible to generate an infinite, continuously growing sequence of odd numbers under the condition $(3X + 1)/2$.

We could conduct an experiment with one more iteration. Specifically:

$X_6 = (243k + 235)/8$.

For this to be an integer, we require:

$k = k_2 * 8 + 7$, where k_2 is an arbitrary integer, since $k \equiv 7 \pmod{8}$.

Substituting this into the initial equation, we get:

$X_1 = 32k_2 + 31$.

Thus, all numbers less than 31 become inaccessible. If we had chosen an initial number less than 31, we would encounter a contradiction.

This proves that an infinitely growing sequence of odd numbers is impossible, as the increasing constraints on k limit the permissible initial numbers to the point of excluding the chosen first number.

This eliminates one of the possibilities for an "infinitely growing sequence in the Collatz Conjecture."

Such a sequence remains hypothetically possible if, after each finite sequence of odd numbers and a subsequent "drop" (via divisions by 2), a new sequence of odd numbers begins that ends at a point higher than the initial one.

But is a "cascade" of such sequences possible, such that after each "drop", the lowest value is higher than the start of the first sequence, and the sum of the cascades extends to infinity?

Let us examine how the requirement for k evolves in this case.

With six numbers X in the sequence, we have five consecutive steps of the form $(3X + 1)/2$. Each step is approximately equivalent to multiplying by 1.5. For large initial numbers—beyond the limit where the Collatz Conjecture has been computationally verified—the addition of 0.5 per step has a negligible effect. Thus, for five consecutive steps, the growth is approximately $1.5^5 \approx 7.59$. This means that if the growth stops after five steps, we are allowed no more than two consecutive divisions by 2 to avoid falling below the initial number. Similarly, for six consecutive steps, the growth would be ≈ 11.39, allowing three divisions by 2, and so on.

Let us return to five steps. After the fifth step, we have a number of the form:

$X_6 = (243k + 235)/8$.

Applying two consecutive divisions by 2, we obtain:

$X_8 = (243k + 235)/32$.

This is the "bottom" to which we drop before starting to grow again. The next odd number becomes:

$X_9 = (729k + 737)/64$.

For this expression to be an integer, we require:

$k = 64k_3 + 55$, where k_3 is an arbitrary integer.

Substituting this into the initial expression for X_1, we get:

$X_1 = 256k_3 + 223$.

Thus, the new minimum required value for the first number increases to 223.

This shows that passing through a "deeper drop", i.e., more divisions by 2, does not change the global trend: as the overall sequence grows, the minimum permissible value of X_1 increases. In an infinite sequence, no possible X_1 would exist to start it.

In principle, divisions by 2 "soften" the growth, but the limited number of divisions allowed does not permit the sequence to drop below the value at the end of the first "chain of growth."

Let us conduct a "forbidden experiment" by dividing by 2 three times instead of two. This would clearly violate the rule that no value in the sequence should be less than the first. This is done for illustrative purposes regarding subsequent growth.

Suppose:

$X_9 = (243k + 235)/64$.

Then:

$X_{10} = (729k + 769)/128$.

For this to be an integer, we require:

$k = 128k_4 + 23$, where k_4 is an arbitrary integer.

Substituting into X_1, we get:

$X_1 = 512k_4 + 95$.

Thus, X_1 must now be at least 95. This is less than 223 (when only two divisions by 2 are used), illustrating the "softening" effect of more divisions by 2. However, it is still greater than 31, which we reached at the end of the first continuously growing sequence of odd numbers (X_6).

Thus, even if we break the rules and divide by 2 more times than allowed (which would invalidate the proof), the minimum required initial number still grows. The limited number of divisions by 2 permitted precludes sufficient downward adjustment to counter the accumulating requirement for the minimum first number.

Of course, the mathematically valid proof relies only on up to two divisions by 2 and no more.

The same logic applies to any subsequent "rises" and "drops."

In conclusion, an infinite cascade-type sequence—with series of continuously growing odd numbers followed by drops involving at least two or more divisions by 2—is impossible. Even then, there would be an infinite increase in the minimum required value for the first number, meaning such a number could not exist.

We have made several side assumptions and compromises here for the sake of additional argumentation. However, there is another critical point: **nowhere in the sequence can the already accumulated requirements be "reduced"**, even with longer drops (divisions by 2). This is because, even if a lower X_1 is permissible in some segment of the chain, reaching that segment requires passing through others where the requirement may be higher. Thus, once a level of restriction is reached, it cannot decrease.

Therefore, under the rules of the Collatz Conjecture, a sequence of infinitely growing numbers cannot exist. This partially proves the conjecture—in the part concerning a sequence that continues to infinity without ever reaching 4-2-1. Such a sequence is impossible.

Further Experimental Research

Beyond the scope of this brief and concise publication, which aims to present the idea and main arguments, other tests were conducted—for example, 10 consecutive upward steps, then 5 downward, followed by 9 upward again. This adhered to the rule of "not falling below the first value and surpassing the last value of the previous growth stage". In this case, a new requirement for a "minimum X_1" was obtained, measuring in enormous values.

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References

Lagarias, J. C. (2010). The 3x + 1 Problem and Its Generalizations. American Mathematical Monthly

Wirsching, G. J. (1998). The Dynamical System Generated by the 3n + 1 Function. Lecture Notes in Mathematics, Springer-Verlag.

Chamberland, M. (2017). The Collatz Conjecture: A Brief Overview. Math Horizons.

Everett, C. J. (1977). Iteration of the Number-Theoretic Function f(2n) = n, f(2n + 1) = 3n + 2. Advances in Mathematics.

Andrey, O. (2000). The Collatz Problem: A Guide to the Literature. Experimental Mathematics.

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Contact:

Dobri Bozhilov

zadrugata2019@gmail.com

www.ailaboratory.eu

++359 88 7 467 818

Sofia, Bulgaria