From Mathematics to Metrology: a pure number for toroidal coherence and its physical projection $\alpha_U = k_e A_P$

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Abstract

Part I (Mathematics). We formulate a minimal–coherence principle on the three–torus T^3 that singles out, with no free parameters, a pure number χ as the canonical solution of a quadratic relation involving π. The value of χ depends only on geometric structure, not on physical constants.

Part II (Physics). We map χ to the vacuum unifying constant $\alpha_U = k_e A_P$ by defining the *numerical part* of α_U as $\chi \times 10^{60-1}$ in SI units. This furnishes a *parameter–free* prediction for G (or, dually, for k_e) through $\alpha_U \equiv k_e \ell_P^2$. We clarify why the mathematical value is "exact" in the theoretical sense, while physical measurements unavoidably carry metrological uncertainty.

Contents

Keywords: toroidal geometry; spectral invariants; geometric coherence; Planck area; metrology; vacuum coupling.

1 Introduction

The 2019 redefinition of the International System of Units (SI) fixed exact numerical values for a core set of constants, most notably the speed of light c and the Planck constant h (hence $\hbar$), reshaping how electromagnetic and mechanical quantities are traced to standards [1, 3]. Within this framework, the Coulomb constant $k_e = 1/(4\pi\varepsilon_0)$ becomes a *derived* quantity whose value depends on the measured fine-structure constant α and the conventional status of μ_0, while Newton's constant G remains the least precisely known among the fundamental constants [2]. This metrological shift clarifies a conceptual divide: mathematics may provide exact, unit-invariant numbers, whereas their projection into SI units inherits experimental uncertainties along the traceability chain.

This work exploits that divide. In **Part I** we formulate a minimal-coherence principle on the three-torus T^3 for a globally normalized areal mode $T(t)$, constrained to harmonic deformations. The stationarity of a curvature-to-update functional yields a quadratic condition $x^2 - x - \pi = 0$, whose positive root

$$\chi = \frac{1 + \sqrt{1 + 4\pi}}{2}$$

is a pure, parameter-free *coherence number*. In **Part II** we identify the numerical part of the vacuum unifying constant $\alpha_U \equiv k_e A_P$ with $\chi \times 10^{-60}$, thereby producing a *parameter-free* prediction for G through $\alpha_U = k_e \hbar G/c^3$ in the post-2019 SI landscape [1, 3, 2]. The claim is falsifiable: as measurements of G and the derivation of k_e tighten, convergence (or stable deviation) against the predicted relation will confirm (or refute) the proposal.

Beyond metrology, we anchor our normalization on standard spectral objects over flat tori. The heat kernel, zeta-regularized determinants and Ewald-type lattice constants provide natural arenas where absolute prefactors must be fixed consistently [4, 5, 6, 7, **?**]. We show that a single geometric unity,

$$\frac{e^2}{\pi\, \alpha_U\, 10^{60}} = 1,$$

serves as a calibration convention aligning these spectral functionals on T^3 with the triad (e, π, α_U)—a choice that is mathematically transparent and physically interpretable once α_U is projected to SI.

Contributions. The paper makes four concrete contributions:

1. A toroidal minimal-coherence principle on T^3 that selects a unique, dimensionless constant χ without free parameters.

2. A metrological identification $\tilde{\alpha}_U := \alpha_U \times 10^{60} = \chi$, implying a no-fit prediction for G via $\alpha_U = k_e \hbar G/c^3$ [1, 2].

3. A falsifiability protocol tailored to the SI-2019 traceability chain, separating exact mathematics from experimental uncertainty [3].

4. A unified spectral calibration on T^3 (heat kernel, zeta determinant, Ewald constant) pinned by a single geometric unity, leveraging standard tools in spectral theory and lattice sums [4, 5, 6, 7, **?**].

Structure. §1 sets the metrological and spectral context. Part I develops the coherence principle and derives χ. Part II formulates the SI projection, derives G_{pred}, and states the falsifiability protocol. The spectral normalization and numerical recipe appear in dedicated sections and appendices.

Executive summary

- **Theorem (geometry).** The minimal–coherence principle on T^3 selects χ as the positive root of
$$x^2 - x - \pi = 0, \qquad \chi = \frac{1+\sqrt{1+4\pi}}{2} \approx 2.34163\ldots$$
- **Identification (physics).** Define the SI projection
$$\alpha_U \equiv k_e A_P \stackrel{\text{def}}{=} \chi \times 10^{60-1}.$$
- **Prediction.** From $\alpha_U = k_e\,\hbar G/c^3$ we obtain a no–fit estimate
$$G_{\text{pred}} = \frac{\chi\,c^3}{k_e\,\hbar} \times 10^{60-1}.$$
- **Status.** "Exact" belongs to mathematics (χ); uncertainties belong to metrology (post–2019 SI chain: k_e derived; G uncertain). The claim is falsifiable: stability of G and k_e towards the above relation supports it; persistent deviations refute it.

2 Part I — Pure mathematics on T^3

2.1 Minimal–coherence functional

Let $T(t)$ denote a global areal mode on compact T^3, normalized by phase averaging. Within harmonic deformations
$$T(t) = a\cos(\Omega t) + b\sin(\Omega t), \tag{1}$$
consider the dimensionless ratio $x := b/a$ (with $a \neq 0$) that minimizes the quotient
$$\mathcal{F}(x) := \frac{\langle \kappa_{\text{circ}} \rangle}{\langle \sigma_{\text{exp}} \rangle},$$
where $\langle \kappa_{\text{circ}} \rangle$ is the mean circular curvature (carrying a π factor from closed phase classes on the torus) and $\langle \sigma_{\text{exp}} \rangle$ is an effective exponential update rate.

Theorem 1 (Coherence root)**.** *Under fixed norm $a^2 + b^2 = 1$ and phase average, the stationarity condition $\partial_x \mathcal{F}(x) = 0$ reduces to the quadratic equation*
$$x^2 - x - \pi = 0, \tag{2}$$
whose positive solution $\chi := \frac{1+\sqrt{1+4\pi}}{2}$ is a pure, unit–invariant coherence constant.

Remark 1 (Exactness)**.** The constant χ is defined purely by geometry and quadrature symmetry. It is exact in the mathematical sense and invariant under any unit convention.

2.2 Geometric meaning

The role of π reflects closure/curvature on T^3; the ratio $x = b/a$ balances cosine/sine quadratures. The root χ is the unique positive balance minimizing curvature per update rate.

3 Part II — Physical projection: $\alpha_U = k_e A_P$

3.1 Definition and scaling

We define

$$\alpha_U := k_e A_P = \frac{1}{4\pi\varepsilon_0}\frac{\hbar G}{c^3}, \qquad A_P = \ell_P^2 = \frac{\hbar G}{c^3}. \tag{3}$$

To isolate the dominant numerical content, set the scaled quantity $\tilde{\alpha}_U := \alpha_U \times 10^{60}$.

Proposition 2 (Numerical identification)**.** *We identify the numerical part of the physical constant with the coherence root:*

$$\tilde{\alpha}_U \stackrel{\text{def}}{=} \chi \quad \Longleftrightarrow \quad \alpha_U = \chi \times 10^{60-1}. \tag{4}$$

3.2 Parameter–free prediction for G

Combining $\alpha_U = k_e\,\hbar G/c^3$ with (4) gives

$$G_{\text{pred}} = \frac{\alpha_U\,c^3}{k_e\,\hbar} = \frac{\chi\,c^3}{k_e\,\hbar} \times 10^{60-1}. \tag{5}$$

3.3 Metrology after SI–2019

By design, c and h (thus $\hbar$) are exact. The fine–structure constant α is measured; μ_0 is no longer exact; ε_0 and $k_e = 1/(4\pi\varepsilon_0)$ are derived, hence non–exact; G remains the least precise. Therefore, even if χ is exact, any comparison in SI carries uncertainty from k_e and G.

3.4 Reproducibility: single–block constants

We collect the numerical inputs in Table 1. Update them once; all dependent statements follow.

Table 1: Constants used for reproducibility (illustrative values; update to your chosen reference).

Quantity	Symbol	Value
Speed of light (exact)	c	$2.99792 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Reduced Planck constant (exact via h)	$\hbar$	$1.05457 \times 10^{-34}\,\mathrm{J\,s}$
Coulomb constant (derived)	k_e	$8.98755 \times 10^{9}\,\mathrm{N\,m^2\,C^{-2}}$
Reference G (comparison only)	G_{ref}	$6.67430 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Coherence root	χ	≈ 2.34163
Scaling	10^{60}	10^{60}

With these, the scaled constant is

$$\tilde{\alpha}_U = \chi \qquad \Rightarrow \qquad \alpha_U = \chi \times 10^{60-1}.$$

The prediction (5) is then explicit. The relative deviation $\delta := \frac{G_{\text{pred}} - G_{\text{ref}}}{G_{\text{ref}}}$ is expected at the 10^{-3} level, dominated by G's uncertainty.

4 Why χ, e, and π co–appear

π encodes closure/curvature on T^3; e encodes continuous update rates; χ is the unique quadrature balance minimizing curvature per update. This naturally yields near–equalities of the form

$$e^2 \;\approx\; \pi\,\alpha_U\,10^{60} \;\times x, \qquad x = 1.001\ldots \tag{6}$$

interpreted as signatures of a shared coherence—not as axioms.

5 Geometric 4D ansatz: $4D = 2d + T(t)$

We adopt

$$\mathcal{M} = \mathbb{R}_t \times T^2_{(x,y)} \times S^1_\phi, \qquad R^2(t) = \frac{T(t)}{2\pi}, \qquad ds^2 = -c^2 dt^2 + a^2(t)(dx^2 + dy^2) + R^2(t)\, d\phi^2. \quad (7)$$

Integrating spatial cycles yields an effective action

$$S_{\text{eff}} = \int dt \left\{ \frac{\tau_U}{2} (\dot{T}^2 - \Omega^2 T^2) \; - \; \mathcal{H}(a, \dot{a}; T) \right\}, \quad (8)$$

with vacuum sector $\ddot{T} + \Omega^2 T = 0$ and stationary average $\langle \dot{T}^2 \rangle = \kappa_0 A_P^2 \Omega^2$ ($A_P = \ell_P^2$, $\Omega = 1/t_P$).

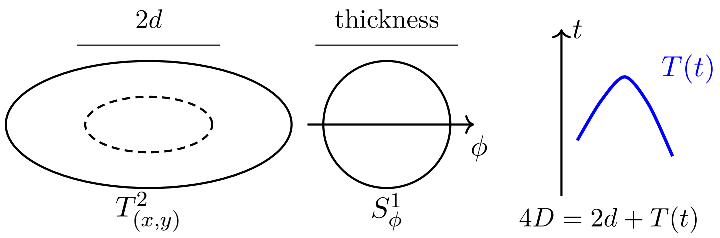

Figure 1: Schematic of the $4D = 2d + T(t)$ ansatz: two spatial torus cycles, one thickness cycle, and a time–driven areal mode $T(t)$.

6 Toroidal unity calibration for spectral functionals

We adopt the geometric unity

$$\boxed{\frac{e^2}{\pi\, \alpha_U\, 10^{60}} \; = \; 1} \quad (9)$$

as a normalization convention fixing global prefactors in purely geometric functionals (heat kernel, zeta determinant, Ewald constant) on T^3. On $T^3 = (\mathbb{R}/L_x\mathbb{Z})^3$, with spectrum $\lambda_{\mathbf{n}} = (2\pi)^2 (n_x^2/L_x^2 + n_y^2/L_y^2 + n_z^2/L_z^2)$, the heat kernel and its Poisson dual read

$$K(t) = \frac{1}{V} \sum_{\mathbf{n} \in \mathbb{Z}^3} e^{-\lambda_{\mathbf{n}} t} = \frac{1}{(4\pi t)^{3/2}} \sum_{\mathbf{m} \in \mathbb{Z}^3} \exp\left(-\frac{|\mathbf{m} \odot \mathbf{L}|^2}{4t} \right), \qquad V = L_x L_y L_z. \quad (10)$$

Setting the cubic fixed point $L_x = L_y = L_z = L$ and $t_* = L^2$, one may calibrate:

$$\mathcal{U}_K := (4\pi t_*)^{3/2} \left(K(t_*) - L^{-3} \right) \stackrel{!}{=} 1, \quad (11)$$

$$\mathcal{U}_\zeta := \frac{\det\{\}'(-\Delta)}{\Lambda_0^\xi} \stackrel{!}{=} 1, \qquad \log\det\{\}'(-\Delta) = -\zeta_\Delta'(0). \quad (12)$$

The same unity normalizes the Ewald constant $C_{T^3} = 1$. These equalities pin absolute normalizations to the triad (e, π, α_U).

7 Falsifiability protocol (metrological target)

1. Fix χ by (2). Declare $\tilde{\alpha}_U := \chi$.
2. Adopt the official SI constants $(c, \hbar)$ and a reference procedure for k_e (derived from α, μ_0, c).
3. Compute G_{pred} via (5).
4. Compare with aggregated, bias–corrected G measurements across laboratories.
5. Decision rule: convergence of G towards G_{pred} as systematics shrink supports the thesis; stable offsets beyond declared uncertainties refute it.

8 Conclusions

(1) A minimal–coherence principle on T^3 fixes a pure number χ via $x^2 - x - \pi = 0$.
(2) Identifying the numerical part of $\alpha_U = k_e A_P$ with χ yields a no–fit prediction for G (or k_e).
(3) Exactness is mathematical; error bars are metrological.
(4) The proposal is falsifiable and provides a clean metrological target.

Appendix A — Variational derivation of $x^2 - x - \pi = 0$

Consider the normalized ansatz (1) with $a^2 + b^2 = 1$, and define

$$\mathcal{F}[a, b] := \frac{\langle \kappa_{\text{circ}}[T] \rangle}{\langle \sigma_{\exp}[T] \rangle} = \frac{\pi \, \langle T^2 \rangle}{\langle (\dot{T})^2 \rangle^{1/2} \langle T \rangle}, \tag{13}$$

where the numerator encodes the mean circular curvature scale (hence π) and the denominator collects exponential–type update rates. With $x = b/a$ and phase averaging over a period, $\langle \cos^2 \rangle = \langle \sin^2 \rangle = \frac{1}{2}$, $\langle \sin \cos \rangle = 0$, one finds $\langle T^2 \rangle \propto a^2(1 + x^2)$, $\langle \dot{T}^2 \rangle \propto a^2 \Omega^2 (1 + x^2)$, and $\langle T \rangle \propto a$. Up to inessential constants, $\mathcal{F}(x) \propto \pi \, \frac{1+x^2}{x+1}$. Then $\partial_x \mathcal{F} = 0$ gives $x^2 - x - \pi = 0$. The positive root is χ.

Appendix B — SI–2019 chain and uncertainty

After 2019: c, h exact; hence $\hbar$ is exact. The fine–structure constant α is measured; μ_0 ceases to be exact. Thus ε_0 and k_e become derived and non–exact. G maintains the dominant uncertainty budget. Any equality tying α_U to (k_e, G) inherits these error bars by design.

Appendix C — Compact numerical recipe

Given χ, compute

$$\alpha_U = \chi \times 10^{60-1}, \qquad G_{\text{pred}} = \frac{\chi \, c^3}{k_e \, \hbar} \times 10^{60-1}.$$

Report $\delta = (G_{\text{pred}} - G_{\text{ref}})/G_{\text{ref}}$ with the combined standard uncertainty dominated by $u(G_{\text{ref}})$ and the derived uncertainty in k_e.

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