Extended Collatz Function as a Markov Chain

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Abstract
We model the iteration of the extended Collatz function using an infinite-state Markov chain. This probabilistic model provides a framework for analyzing the dynamics of Collatz sequences. We demonstrate that the state corresponding to the number 1 is the unique absorbing state and argue that, under this model, the system inevitably converges toward this equilibrium.

1. Introduction

The Collatz conjecture is a famous unsolved problem in mathematics.
For any positive integer a_0, the sequence is generated by the recursive function.

$$a_{n+1} = f(a_n) = \begin{cases} 1 & if\ a_n = 1 \\ \frac{a_n}{2} & if\ a_n\ is\ even \\ \frac{3a_n+1}{2} & if\ a_n\ is\ odd\ and\ a_n > 1 \end{cases}$$

The conjecture asserts that for every positive integer starting value a_0, the sequence eventually converges to 1.

2. The Markov Chain Model

We define a Markov chain vector X_n , $n \geq 0$ with countably infinite states $\mathbb{N} = \{1, 2, 3, \ldots\}$, where each i-th component represents the probability that the system is in state i at time n. The transition matrix P is defined such that P(i,j) represents the probability of transitioning from state j to state i. For the Collatz function f, this transition is deterministic:

P(1,1) = 1 (1 is an absorbing state)
For even j > 1, P(j/2, j) = 1
For odd j > 1, P((3j+1)/2, j) = 1

All other entries of P are zero. The evolution of the system is governed by:

$$X_{n+1} = P\,X_n$$

3. Convergence Analysis

The long-term behavior of the Markov chain is determined by its fixed points, which satisfy:

$$X = PX$$

where $X = \lim_{n\to\infty} X_n$. Let $A = P - I$, where I is the identity matrix, then

$$AX = 0.$$

The infinite-dimensional matrix A is defined as follows:

$A(1,1) = 0$

$A(j,j) = -1$ for $j > 1$

$A(j/2,j) = 1$ if j is even

$A((3j+1)/2,j) = 1$ if j is odd and $j > 1$

All other entries are zero. We seek all nonnegative sequences $X = (x_1, x_2, x_3, \ldots)^T$ satisfying $AX = 0$, $x_i \geq 0$, and $\sum_{i=1}^{\infty} x_i = 1$.

Equation for r = 1:

From $A(1,1) = 0$ and $A(1,2) = 1$ (since n = 2 is even and n/2 = 1).

$$0.x_1 + 1.x_2 = 0 \quad \Rightarrow x_2 = 0$$

Equation for general r > 1:

For r > 1, the nonzero entries in row r are:

$A(r,r) = -1$,

$A(r,2r) = 1$,

and possibly $A(r,(2r-1)/3) = 1$ if $(2r-1)/3$ is an integer > 1 and odd.

Hence:

$x_r = x_{2r} + \delta_r\, x_{(2r-1)/3}$, where $\delta_r = 1$ if $(2r-1)/3$ is integer > 1, otherwise 0.

4. Absence of k-cycle solutions for K > 1

Suppose X has finite support $S \subset N$ of size k with $x_i = 1/k$ for $i \in S$ and all others zero. For any $r \in S$ from $x_r = x_{2r} + \delta_r x_{(2r-1)/3}$, where $\delta_r = 1$ if $(2r-1)/3$ is integer > 1, otherwise 0, at least one of 2r or (2r-1)/3 must also belong to S. Continuing this process indefinitely contradicts the finiteness of S. Hence there is no finite non empty support S containing any k > 1. Therefore, there is no finite nonempty support S with |S| = k > 1.

5. Conclusion

The unique nonnegative, normalized solution to AX = 0 is:
X = (1, 0, 0, 0, …). Hence, we have constructed a simple Markov chain model whose dynamics mirror the iteration of the Collatz function. This model reframes the deterministic Collatz conjecture as a probabilistic convergence problem and shows that state 1 is the unique, globally attracting absorbing state of the chain.

References

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[2] J . C. Lagarias. The 3x+1 Problem: An overview. https://arxiv.org/abs/2111.02635, 2021.