

# The $Q_g$ Rotor Framework as an Algorithmic Bridge Between Dirac Algebra and Gravitation

From Constant–Lagrangian Dynamics to MOND Phenomenology

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## Abstract

### Abstract

This paper presents the gravitational rotor framework  $Q_g$  as an executable bi-quaternionic (Dirac) algebra that generates general–relativistic structure without external geometric postulates. From the adjoint action  $/G_\mu = Q_g \beta_\mu Q_g^{-1}$ , the metric  $g_{\mu\nu}$  and affine connection follow algorithmically, defining an explicit translation map  $Q_g(x) \rightarrow (g_{\mu\nu}, \Gamma^\rho_{\mu\nu}, R^\rho_{\sigma\mu\nu})$ . Within this algebraic system, the mixed tensor  $M_\mu^\nu = E_\mu^a \Phi_a^\nu$  and the four–current  $J^\nu = u^\mu M_\mu^\nu = \Psi^\dagger Q_g \beta_\mu Q_g^{-1} \beta^\nu \Psi$  represent the coupling between matter flow and metric generation, while  $E$  and  $\Phi$  encode the local tetrad and gravitational phase fields. Applying the constant–Lagrangian with Hubble boundary (CLH) condition yields galactic rotation laws and MOND–type accelerations,  $a(r) = \frac{GM}{r^2} - 2H \sqrt{\frac{2GM}{r}} \frac{1}{r} + \frac{v_{\text{flat}}^2}{r} + H^2 r$ , as covariant consequences of the same invariant. The characteristic MOND acceleration  $a_0 \simeq cH$  and the flat–velocity plateau arise naturally from the boundary continuity at the bulge radius  $R$ . Thus, the  $Q_g$  engine provides both a translation algorithm from biquaternionic algebra to general relativity and an autonomous, linear, covariant framework in which gravity, MOND phenomenology, and galactic dynamics emerge coherently from a single rotor field.

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## 1 Introduction

The present paper continues a research line that began with the search for a purely dynamical explanation of galactic rotation curves without invoking dark matter. In 2019, de Haas [1] proposed a first, one–dimensional formulation in terms of a constant–Lagrangian (CL) postulate, which successfully fitted the complete SPARC database of 175 late–type galaxies. That early model, although empirically accurate, lacked a clear theoretical foundation and contained no explicit cosmological boundary. Its further development paused until the constant–Lagrangian structure was reinterpreted half a decade later as a two–dimensional spiral inflow, allowing the inclusion of a natural Hubble term at large radii. This refinement, denoted CL–H, provided a physically motivated closure condition at the cosmic boundary.

In parallel, a second line of research explored the algebraic structure underlying relativistic field equations. Building on Dirac’s 1928 operator formulation of quantum mechanics [6] and Hestenes’ geometric reinterpretation of the Dirac algebra [7], de Haas [2] reconstructed the Weyl and Dirac matrices within a biquaternion (BQ) basis, showing that Lorentz and gravitational transformations can be represented as rotor actions inside the Dirac algebra itself. The goal was to express gravitation algebraically within the same mathematical language as quantum mechanics rather than introduce curvature externally. This idea was expanded in de Haas [4], where attaching the gravitational rapidity rotor to the Dirac adjoint produced an exact linearisation of the Einstein equations, and further developed in [5] in connection with the covariant treatment of time in quantum mechanics.

The empirical results of the CL–H approach [3] and the algebraic achievements of the  $Q_g$  framework thus converged. In the present work these two lines are united: the constant–Lagrangian postulate, originally introduced as an empirical law, is now derived algebraically from the Dirac–BQ structure itself. This establishes the CL–H relation as a covariant consequence of the  $Q_g$  field rather than an external assumption.

The paper proceeds in two complementary directions. First, it provides the explicit algorithm translating the algebraic quantities of the  $Q_g$  formalism—the gravitational rotor  $Q_g$ , the rotated Dirac basis  $/G_\mu$ , and the tetrad operator  $E_\mu^a$ —into the metric tensors of general relativity. This translation clarifies how the geometric language of ADM and MTW [8, 9] can be reconstructed directly from the algebraic basis. Second, and more fundamentally, it demonstrates that the same gravitational behaviour can be formulated entirely within the BQ domain itself, without recourse to tensor or curvature formalisms. The  $M$  matrix, the  $E$  matrix, and the gravitational Dirac current  $J^\mu$  are derived as linear BQ operators, forming a self–consistent algebraic system that reproduces the essential features of gravitational flows.

Finally, the covariant derivation of the constant–Lagrangian–Hubble (CLH) condition establishes a direct theoretical link to the phenomenology of Modified Newtonian Dynamics

(MOND) [10, 11, 12, 24, 14]. Within the  $Q_g$  framework, the empirical success of MOND in reproducing galactic rotation curves follows from the same rotor dynamics that generate the CLH field. The characteristic acceleration scale  $a_0$  no longer enters as an empirical parameter but arises algebraically from the Hubble coupling,  $a_0 \simeq cH$ , while the observed flat-velocity plateaus correspond to the rim-constant term  $v_{\text{flat}}^2/r$  that ensures continuity between the bulge and the cosmological flow. In this way, MOND’s fitting accuracy—originally demonstrated by Milgrom, and later by Sanders, Famaey, and McGaugh—finds a covariant explanation within the same first-order rotor geometry that unifies local, galactic, and cosmological dynamics. Thus, while the present work includes an explicit algorithm for constructing the GR metric from the  $Q_g$  basis, its deeper outcome is to show that the gravitational domain—from local flows to large-scale cosmic expansion—can be formulated self-consistently within a single, linear, covariant Dirac-BQ architecture.

## 2 The algorithm to construct GR from $Q_g$ as its covariant basis

### 2.1 The theory as an executable Clifford-geometry engine

The  $Q_g$  formalism is now fully algorithmic: it constitutes an executable algebra. Every essential relation in the framework can be expressed as a deterministic pipeline of Clifford operations,

$$Q_g(x) \longrightarrow \begin{cases} /G_\mu = Q_g \beta_\mu Q_g^{-1}, \\ g_{\mu\nu} = \frac{1}{2} \{ /G_\mu, /G_\nu \}, \\ \Omega_\mu = (\partial_\mu Q_g) Q_g^{-1}, \\ F_{\mu\nu} = \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu + [\Omega_\mu, \Omega_\nu]. \end{cases} \quad (1)$$

From these relations one can extract observables such as Dirac bilinears, ADM variables, curvature invariants, and energy-momentum densities. Every step involves only symbolic Clifford algebra, differentiation, or matrix multiplication—there is no interpretive gap between definition and computation.

Hence any computational environment capable of Clifford algebra can execute the formalism directly: for example `Mathematica` with the `CliffordAlgebra` package, `Python` with `clifford` or `sympy`, `Julia` with `Grassmann.jl`, or `C++` with `clifford++`. Once the rotor field  $Q_g(x)$  is specified, the entire geometric structure

$$(g_{\mu\nu}, \Omega_\mu, F_{\mu\nu})$$

follows deterministically by symbolic evaluation.

Algorithmically, it is a *Clifford-geometry engine*. The entire  $Q_g$  formalism can be summarised as a computable map

$$Q_g(x) \mapsto (g_{\mu\nu}(x), \Gamma_\mu(x), R_{\mu\nu\rho\sigma}(x)), \quad (2)$$

through the algebraic chain

$$Q_g \longrightarrow (/G_\mu) \longrightarrow (g_{\mu\nu}) \longrightarrow (\Gamma_\mu) \longrightarrow (F_{\mu\nu}), \quad (3)$$

which constitutes a complete algorithmic sequence. Once the rotor generator  $\psi_i(x)$  is specified, a symbolic processor can carry out all subsequent steps automatically. This defines the  $Q_g$  framework as a Clifford-geometry engine—a closed algebraic system that converts rotor fields into curvature, connections, and metrics without any external geometric assumptions.

## 2.2 The covariant basis equation as structural condition

The relation

$$(\partial_\mu /G_\nu) = [\Omega_\mu, /G_\nu], \quad /G_\mu = Q_g \beta_\mu Q_g^{-1}, \quad \Omega_\mu = (\partial_\mu Q_g) Q_g^{-1}, \quad (4)$$

is one of the central identities of the  $Q_g$  formalism. It defines how the local Clifford basis  $\{/G_\mu\}$  evolves under the connection  $\Omega_\mu$  and guarantees metric compatibility. Although it looks like a field equation, it is in fact a structural identity that holds for every valid  $Q_g$  configuration.

**1. Meaning of the equation.** Equation (4) expresses the covariant constancy of the local basis under the connection  $\Omega_\mu$ : any variation of  $/G_\nu$  through spacetime is represented as an infinitesimal rotation or boost within the algebra. This is the exact Clifford–algebra analogue of the tetrad postulate in general relativity,

$$\nabla_\mu e_\nu^a = 0,$$

with  $/G_\nu$  replacing  $e_\nu^a \gamma_a$  and  $\Omega_\mu$  replacing the Lorentz connection  $\omega_\mu^{ab}$ .

**2. Not a dynamical law.** The equation does not govern evolution in time; it defines what it means to be a consistent geometry. It follows identically from differentiating the adjoint definition  $/G_\nu = Q_g \beta_\nu Q_g^{-1}$ :

$$\partial_\mu (/G_\nu) = (\partial_\mu Q_g) \beta_\nu Q_g^{-1} + Q_g \beta_\nu (\partial_\mu Q_g^{-1}) = [(\partial_\mu Q_g) Q_g^{-1}, /G_\nu] = [\Omega_\mu, /G_\nu].$$

Thus Eq. (4) is automatically satisfied once  $Q_g$  is differentiable.

**3. Its law-like role.** Although algebraically trivial, Eq. (4) acts as a structural law of the geometry: it enforces metric compatibility,

$$\partial_\mu g_{\nu\rho} = 0, \quad g_{\nu\rho} = \frac{1}{2} \{/G_\nu, /G_\rho\},$$

ensuring that  $\Omega_\mu$  preserves the inner product of the local frame. This is the same condition that defines the Levi–Civita connection in GR, here expressed directly in Clifford form.

**4. Its place in the hierarchy of equations.** Within the  $Q_g$  system, the logical layers are:

- **Identity:**  $(\partial_\mu /G_\nu) = [\Omega_\mu, /G_\nu]$  defines a valid geometry (metric compatibility).
- **Definition:**  $g_{\mu\nu} = \frac{1}{2} \{/G_\mu, /G_\nu\}$  gives the metric.
- **Curvature law:**  $F_{\mu\nu} = \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu + [\Omega_\mu, \Omega_\nu]$  defines the field strength of the connection.
- **Dynamics:** the Bernoulli–Noether closure or other first–order field relations specify the evolution of  $Q_g$ .

The covariant–basis equation therefore sits at the foundational level: it provides the consistent kinematic framework in which the dynamics operate.

**5. Conceptual significance.** The identity  $(\partial_\mu /G_\nu) = [\Omega_\mu, /G_\nu]$  encapsulates the idea that all geometric information—metric, connection, and curvature—is generated internally by the rotor field  $Q_g(x)$ . It is the algebraic statement that the local basis vectors are covariantly constant under  $\Omega_\mu$ , the spin connection built from  $Q_g$ .

The equation is not a separate field law to be imposed; it is the defining structural condition that makes every  $Q_g$  geometry metric–compatible and hence physically meaningful. It plays the same role as  $\nabla_\mu g_{\nu\rho} = 0$  in general relativity, expressed here directly in the Clifford–algebra language that underlies the  $Q_g$  framework.

### 3 Gravity embedded in the probability tensor and the Dirac current as its covariant contraction

#### 3.1 The mixed tensor $\mathbb{M}_\mu{}^\nu$ as spin–metric operator

All information about the local spinor–metric coupling and gravitational flow can be collected into a single mixed tensor field,

$$\boxed{\mathbb{M}_\mu{}^\nu(x) := \Psi^\dagger(x) (/G_\mu(x) \beta^\nu) \Psi(x) = \Psi^\dagger Q_g \beta_\mu Q_g^{-1} \beta^\nu \Psi}, \quad (1)$$

which is a genuine  $4 \times 4$  field carrying both the metric and the bivector (flow/spin) information.

**1. Decomposition into scalar and bivector parts.** The field  $\mathbb{M}_\mu{}^\nu$  splits cleanly into a symmetric (metric) part and an antisymmetric (bivector) part:

$$\mathbb{M}_\mu{}^\nu = \frac{1}{2} \Psi^\dagger \{ /G_\mu, \beta^\nu \} \Psi + \frac{1}{2} \Psi^\dagger [ /G_\mu, \beta^\nu ] \Psi, \quad (2)$$

where the first term represents the mixed metric map and the second encodes the gravitational flow and spin information.

Using the duality condition for the common coframe of  $\{ /G_\mu \}$  and  $\{ \beta^\nu \}$ ,

$$\frac{1}{2} \{ /G_\mu, \beta^\nu \} = \delta_\mu{}^\nu \mathbf{1},$$

this becomes

$$\boxed{\mathbb{M}_\mu{}^\nu = \rho \delta_\mu{}^\nu + \frac{1}{2} \Psi^\dagger [ /G_\mu, \beta^\nu ] \Psi}, \quad (3)$$

where  $\rho := \Psi^\dagger \Psi$  is the scalar (probability) density. The commutator term is exactly the bivector contribution that carries the gravitational flow and frame–drag information that should not be collapsed.

**2. Tetrad form and projected bivector density.** In the GR (tetrad) form, expand  $/G_\mu = e_\mu{}^a \hat{\beta}_a$  and  $\beta^\nu = e_b{}^\nu \hat{\beta}^b$ . With  $\Sigma_a{}^b := \frac{1}{2} [\hat{\beta}_a, \hat{\beta}^b]$ , the spin–bivector density is  $s_a{}^b := \Psi^\dagger \Sigma_a{}^b \Psi$ , so that

$$\boxed{\mathbb{M}_\mu{}^\nu = \rho \delta_\mu{}^\nu + e_\mu{}^a e_b{}^\nu s_a{}^b}. \quad (4)$$

The two pieces have clear geometric meaning:

- $\rho \delta_\mu{}^\nu$  is the *mixed metric map* (index identity) carried by the Betas.
- $e_\mu{}^a e_b{}^\nu s_a{}^b$  is the *projected spin–bivector density*, representing the local gravitational flow and frame–drag content in spacetime indices.

**3. Observable current.** The contracted current is simply the time–direction projection:

$$j^\nu = u^\mu \mathbb{M}_\mu{}^\nu, \quad (5)$$

which keeps the bivector information intact while remaining fully translatable to standard GR notation via the tetrad  $e_\mu{}^a$  and the bivector generators  $\Sigma_a{}^b$ .

**4. Interpretation.** The tensor  $\mathbb{M}_\mu{}^\nu$  thus combines in a single object both the scalar metric density and the antisymmetric spin–flow structure of the local spacetime. It acts as the *spin–metric operator*: the scalar part transmits the index identity of the metric, while the bivector part transmits the rotational and gravitational degrees of freedom carried by the rotor field  $Q_g$ .

### 3.2 The full mixed tensor $\mathbb{M}_\mu^\nu$ as a $4 \times 4$ field

The full tensor  $\mathbb{M}_\mu^\nu$  can be expressed compactly as a  $4 \times 4$  matrix field built from the spinor bilinears and the tetrad extracted from the adjoint action of  $Q_g$ . Starting from

$$\boxed{\mathbb{M}_\mu^\nu(x) = \Psi^\dagger(x) (Q_g \beta_\mu Q_g^{-1} \beta^\nu) \Psi(x) = e_\mu^a \Phi_a^\nu}, \quad (6)$$

where the flat-basis matrix

$$\Phi_a^\nu := \Psi^\dagger(\hat{\beta}_a \beta^\nu) \Psi$$

contains all spinor bilinear combinations. Thus  $\mathbb{M} = E \Phi$  with  $E = [e_\mu^a]$  the tetrad row-matrix, so that all gravitational and flow information from  $Q_g$  is preserved via  $E$ .

**1. Bilinear matrix in the Dirac–Clifford basis.** Define the standard bilinears

$$\rho := \Psi^\dagger \Psi, \quad j_i := \Psi^\dagger \alpha_i \Psi, \quad s_i := \Psi^\dagger (i \Sigma_i) \Psi.$$

Then the flat-basis matrix  $\Phi_a^\nu$  is the explicit  $4 \times 4$  field (rows  $a = 0, 1, 2, 3$ ; columns  $\nu = 0, 1, 2, 3$ ):

$$\Phi_a^\nu = \begin{bmatrix} -\rho & j_1 & j_2 & j_3 \\ -j_1 & \rho & s_3 & -s_2 \\ -j_2 & -s_3 & \rho & s_1 \\ -j_3 & s_2 & -s_1 & \rho \end{bmatrix}. \quad (7)$$

This matrix collects all density, current, and spin–bivector information of the Dirac field in a compact algebraic form.

**2. Expansion of the full tensor.** The complete  $4 \times 4$  tensor  $\mathbb{M}_\mu^\nu$  follows as

$$\mathbb{M}_\mu^\nu = e_\mu^0 \begin{bmatrix} -\rho & j_1 & j_2 & j_3 \end{bmatrix}^\nu + e_\mu^1 \begin{bmatrix} -j_1 & \rho & s_3 & -s_2 \end{bmatrix}^\nu \quad (8)$$

$$+ e_\mu^2 \begin{bmatrix} -j_2 & -s_3 & \rho & s_1 \end{bmatrix}^\nu + e_\mu^3 \begin{bmatrix} -j_3 & s_2 & -s_1 & \rho \end{bmatrix}^\nu. \quad (9)$$

Equivalently, componentwise (useful for code or symbolic expansion):

$$\mathbb{M}_\mu^0 = -e_\mu^0 \rho - e_\mu^1 j_1 - e_\mu^2 j_2 - e_\mu^3 j_3, \quad (10)$$

$$\mathbb{M}_\mu^1 = e_\mu^0 j_1 + e_\mu^1 \rho - e_\mu^2 s_3 + e_\mu^3 s_2, \quad (11)$$

$$\mathbb{M}_\mu^2 = e_\mu^0 j_2 + e_\mu^1 s_3 + e_\mu^2 \rho - e_\mu^3 s_1, \quad (12)$$

$$\mathbb{M}_\mu^3 = e_\mu^0 j_3 - e_\mu^1 s_2 + e_\mu^2 s_1 + e_\mu^3 \rho. \quad (13)$$

### 3. Interpretation.

- $E = [e_\mu^a]$  is the tetrad extracted from the adjoint action  $Q_g \beta_\mu Q_g^{-1} = e_\mu^a \hat{\beta}_a$ , exactly as in the “metric–from–Betas” construction, with  $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ .
- Because  $\mathbb{M} = E \Phi$ , the bivector (flow/spin) content is retained through the off–diagonal  $s_i$  terms and the transport encoded in  $E$ . Nothing collapses to a lapse–only description. Translation back to GR is immediate through the pairs  $(e_\mu^a, \omega_\mu^{ab})$ .

**4. Summary.** The tensor  $\mathbb{M}_\mu^\nu$  represents the combined *spin–metric operator* of the  $Q_g$  framework. It packages the density  $\rho$ , current  $j_i$ , and spin  $s_i$  of the Dirac field together with the tetrad geometry  $E = [e_\mu^a]$  extracted from the gravitational rotor  $Q_g$ . In this way,  $\mathbb{M}_\mu^\nu$  keeps all gravitational–flow and spinor information intact while remaining fully translatable to the standard GR formalism.

### 3.3 The Dirac current in the $Q_g$ field

A Lorentz-covariant expression for the Dirac current in the  $Q_g$  framework is

$$\boxed{j^\nu = u^\mu \mathbb{M}_\mu{}^\nu = u^\mu \Psi^\dagger (Q_g \beta_\mu Q_g^{-1} \beta^\nu) \Psi}, \quad (14)$$

where  $u^\mu$  is the observer's four-velocity (relativistic self-velocity) and  $\mathbb{M}_\mu{}^\nu$  is the mixed spin-metric tensor introduced above.

**1. Definition and transformation.** The vector  $u^\mu$  defines the local time direction of the observer, while  $\mathbb{M}_\mu{}^\nu$  is a mixed (1, 1) tensor built from the spinor field and the gravitational rotor  $Q_g$ . Under any Lorentz or local Spin(1, 3) transformation  $\Lambda_D$ , the pair transforms as

$$Q'_g = \Lambda_D Q_g, \quad u'^\mu = \Lambda^\mu{}_\rho u^\rho, \quad \mathbb{M}'_\mu{}^\nu = \Lambda_\mu{}^\alpha (\Lambda^{-1})_{\beta}{}^\nu \mathbb{M}_\alpha{}^\beta,$$

so that

$$j'^\nu = u'^\mu \mathbb{M}'_\mu{}^\nu = (\Lambda^{-1})_{\beta}{}^\nu j^\beta,$$

confirming that  $j^\nu$  transforms as a contravariant four-vector current.

**2. Physical interpretation.** The quantity  $u^\mu$  represents the observer's local time direction, defining the congruence of worldlines through which the Dirac field is measured. The tensor  $\mathbb{M}_\mu{}^\nu$  contains both the scalar density  $\rho \delta_\mu{}^\nu$  and the bivector term  $e_\mu{}^a e_b{}^\nu s_a{}^b$  that carries the local spin and gravitational flow. Contracting with  $u^\mu$  therefore projects this composite structure onto the observer's time direction, producing the physical current measured in that frame.

**3. Covariance in the  $Q_g$  geometry.** Because both  $u^\mu$  and  $\mathbb{M}_\mu{}^\nu$  are defined through  $Q_g$  and transform covariantly under its adjoint action, the current  $j^\nu$  is covariant not only under Lorentz transformations but also under local  $Q_g$  rotor flows. It is therefore the natural generalisation of the Dirac current to curved spacetimes generated by  $Q_g$ .

**4. Relation to the standard Dirac current.** In flat space ( $Q_g = \mathbb{1}$ ,  $u^\mu = (1, 0, 0, 0)$ ), this expression reduces to

$$j^\nu = \Psi^\dagger \beta^\nu \Psi,$$

the standard Dirac current. In the full  $Q_g$  geometry, the rotor  $Q_g$  modifies the local basis vectors  $\beta_\mu$  and thus couples the current to the gravitational rapidity field.

**In short.** The current

$$\boxed{j^\nu = u^\mu \mathbb{M}_\mu{}^\nu}$$

is the correct and covariant form of the Dirac current in a  $Q_g$  field. It represents the current as measured by an observer moving with self-velocity  $u^\mu$  in the locally curved geometry generated by the rotor field  $Q_g$ , retaining both the scalar density and the spin-bivector content of the spinor field.

## 4 The tetrad matrix $E = [e_\mu{}^a]$ and its metric representations

The mixed tensor

$$\boxed{\mathbb{M}_\mu{}^\nu = e_\mu{}^a \Phi_a{}^\nu}$$

links the internal spinor bilinear block  $\Phi_a{}^\nu$  with the external operator  $\mathbb{M}_\mu{}^\nu$  generated by the adjoint action of the gravitational rotor field  $Q_g$ ,

$$\mathbb{M}_\mu{}^\nu = \Psi^\dagger (Q_g \beta_\mu Q_g^{-1} \beta^\nu) \Psi.$$

From this relation, the tetrad matrix

$$E = [e_\mu^a] \quad \text{with} \quad \mathbb{M} = E \Phi,$$

is obtained as the local linear map connecting the internal spinor algebra to the external spacetime indices. Once  $E$  is known, the metric follows in the standard form  $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ .

#### 4.1 E as a $4 \times 4$ block matrix (ADM/3+1 form)

Write lapse  $N > 0$ , shift  $N^i$ , and a spatial triad  $E_i^j$  with  $h_{ij} = E_i^k E_j^\ell \delta_{k\ell}$ . The coframe one-forms  $\theta^a = e^a_\mu dx^\mu$  are encoded in

$$E \equiv [e_\mu^a] = \begin{pmatrix} e_0^0 & e_0^j \\ e_i^0 & e_i^j \end{pmatrix} = \begin{pmatrix} N & N^k E_k^j \\ 0 & E_i^j \end{pmatrix}.$$

This yields

$$\theta^0 = N dt, \quad \theta^i = E^i_j (dx^j + N^j dt),$$

and

$$ds^2 = -(\theta^0)^2 + (\theta^i)^2 \quad \Rightarrow \quad g_{00} = -N^2 + h_{ij} N^i N^j, \quad g_{0i} = h_{ij} N^j, \quad g_{ij} = h_{ij}.$$

The standard ADM decomposition therefore appears naturally within the BQ framework once the algebraic  $E$  is determined from  $\mathbb{M}$ .

#### 4.2 Where the rapidities appear

The local gravitational rapidities  $\psi_i(x)$ , encoded in the rotor field  $Q_g(x)$ , enter the first row of  $E$  through

$$e_0^i = N^k E_k^i = v^j E_j^i, \quad v^j = \tanh \psi_j.$$

- In **Painlevé–Gullstrand (PG)** gauges,  $e_0^i$  encodes the inflow velocity  $v^i$  (e.g.  $e_0^r = v^r$ ,  $e_0^\phi = R v^\phi$ ).
- In **comoving FLRW** coordinates, the rapidity integrates to the scale factor  $a(t) = \exp[\int \dot{\psi}_H(t) dt]$ .
- In **Lorentz display**, rapidities appear explicitly as  $\cosh \psi$ ,  $\sinh \psi$ .

The gravitational/flow content thus resides in  $E$ , while the spinor bilinear structure remains in  $\Phi$ .

#### 4.3 Spherical coordinates $(t, r, \theta, \phi)$

For spatial one-forms  $\theta^1 = dr$ ,  $\theta^2 = r d\theta$ ,  $\theta^3 = r \sin \theta d\phi$  one has  $E_i^j = \text{diag}(1, r, r \sin \theta)$ , giving

$$E_{\text{PG,sph}} = \begin{pmatrix} 1 & v^r & r v^\theta & r \sin \theta v^\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \sin \theta \end{pmatrix}.$$

Common flows:

- **Radial inflow:**  $v^r(r) = \tanh \psi_r(r)$ ,  $v^\theta = v^\phi = 0$  (Schwarzschild case).
- **Azimuthal rotation:**  $v^\phi(r, \theta) = \tanh \psi_\phi$ , others zero (Doran/Kerr-like rotation).
- **Combined  $(r + \phi)$ :** both rapidities present in the first row.

For comoving FLRW (no shift),

$$E_{\text{FLRW,sph}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t) & 0 & 0 \\ 0 & 0 & a(t)r & 0 \\ 0 & 0 & 0 & a(t)r \sin \theta \end{pmatrix}.$$

#### 4.4 Cylindrical coordinates $(t, R, \phi, z)$

With  $\theta^1 = dR$ ,  $\theta^2 = R d\phi$ ,  $\theta^3 = dz$ , and  $E_i{}^j = \text{diag}(1, R, 1)$ ,

$$E_{\text{PG,cyl}} = \begin{pmatrix} 1 & v^R & R v^\phi & v^z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Special cases:

- **Pure rotation about  $z$ :**  $v^R = v^z = 0$ , only  $v^\phi(R) \neq 0$  (Doran/Kerr-like inflow  $\rightarrow$  rotation).
- **Pure axial flow:**  $v^z(z) \neq 0$ , others zero.

Comoving FLRW (no shift):

$$E_{\text{FLRW,cyl}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t) & 0 & 0 \\ 0 & 0 & a(t)R & 0 \\ 0 & 0 & 0 & a(t) \end{pmatrix}.$$

#### 4.5 Classical metric cases derived from $\mathbb{E}$

When the tetrad  $E$  extracted from  $\mathbb{M}$  is inserted in  $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ , one obtains the classical metrics as specific rapidity patterns:

- **Schwarzschild (PG form):**  $v^r = -\sqrt{2GM/r}$ , giving

$$ds^2 = -dt^2 + (dr - \sqrt{2GM/r} dt)^2 + r^2 d\Omega^2.$$

- **de Sitter (cosmological rapidity):**  $v_H(r, t) = H(t) r = \tanh \psi_H(t, r)$ , producing

$$ds^2 = -dt^2 + (dr - Hr dt)^2 + r^2 d\Omega^2.$$

The Schwarzschild,  $\text{CL}_H$ , and de Sitter metrics thus appear as direct realizations of the same algebraic  $E$  obtained from  $\mathbb{M}$ , each corresponding to a specific rapidity profile within the rotor field  $Q_g$ .

**Summary.** The sequence

$$\boxed{\mathbb{M} \longrightarrow E \longrightarrow g_{\mu\nu}}$$

defines the passage from the internal spinor algebra to the external geometric metric in the  $Q_g$  framework. The matrix  $E$  serves as the algebraic bridge: its first row carries the gravitational rapidity (flow) information, its spatial block defines the local triad, and its insertion into  $\mathbb{M} = E \Phi$  separates the geometric and spinor sectors while maintaining full covariance. Classical geometries such as Schwarzschild,  $\text{CL}_H$ , and de Sitter arise naturally as specific rapidity configurations of  $E$  inside the unified biquaternion structure.

## 5 The constant Lagrangian galactic disk from the Bernoulli–Noether Closure

### 5.1 PG tetrad $E$ with $\psi_r, \psi_\phi, \psi_H$ on equal footing (radial + azimuthal flow)

We work in spherical coordinates  $(t, r, \theta, \phi)$  and use the adjoint basis  $/G_\mu = Q_g \beta_\mu Q_g^{-1} = e_\mu^a \hat{\beta}_a$ . The gravitational rotor is written with three rapidities,

$$Q_g = \exp\left[\frac{1}{2}(\psi_r \hat{\beta}_r \hat{\beta}_0 + \psi_\phi \hat{\beta}_\phi \hat{\beta}_0 + \psi_H \hat{\beta}_H \hat{\beta}_0)\right],$$

so that the associated local (tetrad–measured) speeds are

$$v_r = \tanh \psi_r, \quad v_\phi = \tanh \psi_\phi, \quad v_H = \tanh \psi_H.$$

In a Painlevé–Gullstrand (PG) gauge the temporal basis vector carries the metric “river/shift” field,

$$/G_0 = \hat{\beta}_0 + w_i \hat{\beta}^i, \quad w_i = (w_r, w_\phi, 0),$$

where the shift components are the *net* geometric flows built from the rapidities:

$$\boxed{w_r(r) = v_r^{(g)}(r) - v_H(r), \quad w_\phi(r, \theta) = v_\phi^{(g)}(r, \theta)}$$

with  $v^{(g)} = \tanh \psi$ . (For the  $CL_H$  specialization,  $w_r(r) = \sqrt{2GM/r} - H_z r$  and  $w_\phi$  may be nonzero in a swirl/spiral gauge or frame–drag configuration.)

**PG coframe.** The coframe (rows indexed by  $\mu$ , columns by  $a$ ) is

$$E_{\text{PG}}(r, \theta) = \begin{pmatrix} 1 & w_r(r) & 0 & r \sin \theta w_\phi(r, \theta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \sin \theta \end{pmatrix}.$$

It yields the metric

$$ds^2 = -dt^2 + (dr - w_r dt)^2 + r^2 [d\theta^2 + \sin^2 \theta (d\phi - w_\phi dt)^2],$$

which simultaneously carries radial inflow/outflow (via  $w_r$ ) and azimuthal swirl (via  $w_\phi$ ) coming from  $(\psi_r, \psi_\phi, \psi_H)$ .

**Matter versus metric motion (for later use in BNC).** If matter has its own kinematics  $(v_r^{\text{mat}}, v_\phi^{\text{mat}})$ , the Bernoulli–Noether invariant depends only on *relative* velocities:

$$\boxed{(v_r^{\text{mat}} - w_r)^2 + (v_\phi^{\text{mat}} - w_\phi)^2 = C, \quad C = \text{const.}}$$

so that the observable orbital speed measured by tetrad/PG observers is  $v_{\text{orb}} = |w_\phi + v_\phi^{\text{mat}}|$ , and the effective radial drive is  $v_{\text{rad,eff}} = v_r^{\text{mat}} - w_r$ . For a matter–free vacuum flow one sets  $v_r^{\text{mat}} = v_\phi^{\text{mat}} = 0$ , while for thin disks one keeps  $v_r^{\text{mat}}$  through the algebra and puts  $v_r^{\text{mat}} \approx 0$  only at the end.

## Specializations.

- **CL<sub>H</sub> (Schwarzschild + Hubble):**  $w_r(r) = \sqrt{2GM/r} - H_z r$ ,  $w_\phi = 0$ .
- **Swirl/spiral or frame-drag:**  $w_\phi \neq 0$  (e.g. Doran/Kerr-like), optionally combined with the CL<sub>H</sub>  $w_r$ .
- **Equal-status rapidities:** keep  $\psi_r, \psi_\phi, \psi_H$  explicit and set  $w_r = \tanh \psi_r - \tanh \psi_H$ ,  $w_\phi = \tanh \psi_\phi$ ; in the nonrelativistic limit  $\tanh \psi \simeq \psi$  and  $v_H(r) \simeq H_z r$  recovers the usual forms.

## 5.2 From Qg to galactic rotation curves

**1. Dirac bilinears and the BNC definition.** Inside the BQ algebra, define the scalar density and Dirac current bilinears

$$\rho := \Psi^\dagger \Psi, \quad j^a := \Psi^\dagger \hat{\beta}^a \Psi = \rho (\gamma, \gamma v_i),$$

so that the normalized 4-velocity is  $u^a = j^a / \rho = (\gamma, \gamma v_i)$ . For any Killing direction  $\xi$ , the Bernoulli–Noether constant (BNC) is the streamline-invariant scalar

$$\boxed{\mathcal{B}[\xi] := \frac{1}{\rho} \Psi^\dagger (\xi^\mu / G_\mu) \Psi = u^a \xi_a}.$$

Choosing the PG time symmetry  $\xi = \partial_t$  gives

$$\mathcal{B} = \frac{1}{\rho} \Psi^\dagger (/G_0) \Psi = u^0 + w_i u^i = \gamma(1 - w_i v^i).$$

**2. Reduction to spherical symmetry.** In a spherically symmetric or axisymmetric system, the “river” field is purely radial:

$$w^i = (w_r, 0, 0) \implies w_i v^i = w_r v_r = w v_r.$$

Thus

$$\boxed{\mathcal{B} = \gamma(1 - w v_r) = \text{const.}} \tag{15}$$

The BNC now measures the conserved energy per unit rest mass of the flow with respect to the moving spatial frame.

**3. Low-velocity reduction and completing the square.** For galactic and weak-field regimes, all three velocities are small ( $v_r^2 + v_\phi^2 \ll 1$ ), so

$$\gamma \simeq 1 + \frac{1}{2}(v_r^2 + v_\phi^2),$$

and therefore

$$\mathcal{B} \simeq 1 + \frac{1}{2}(v_r^2 + v_\phi^2) - w v_r.$$

Completing the square gives <sup>1</sup>

$$\mathcal{B} \simeq \left(1 - \frac{1}{2}w^2\right) + \frac{1}{2}[(v_r - w)^2 + v_\phi^2].$$

The first term,  $1 - \frac{1}{2}w^2$ , is a background normalization that reflects the metric lapse in the PG slicing. It varies slowly with  $r$  and does not affect the dynamical invariance, so it can be

<sup>1</sup>The conserved Bernoulli–Noether quantity is not  $\mathcal{B}$  itself but the invariant combination  $(v_r - w)^2 + v_\phi^2 - w^2 = \text{const}$ ; hence, even for  $v_r = 0$ , the  $w(r)$  terms remain active and do not cancel.

absorbed into the constant of motion. Dropping this additive offset yields the simplified invariant form

$$\boxed{v_\phi^2 + (v_r - w)^2 = C, \quad C = \text{const.}}$$

This is the *BQ Bernoulli invariant*: the sum of the squared azimuthal velocity and the squared radial velocity relative to the background gravitational inflow is conserved.

**4. Boundary and disk results.** At the smooth bulge–disk boundary  $r = R$ , impose the virial condition

$$v_\phi^2(R) = v_{\text{orb}}^2(R) = \frac{1}{2} \left( \sqrt{\frac{2GM}{R}} - H_z R \right)^2.$$

Hence

$$C = v_\phi^2(R) + \left( \sqrt{\frac{2GM}{R}} - H_z R \right)^2 = \frac{3}{2} \left( \sqrt{\frac{2GM}{R}} - H_z R \right)^2.$$

For the thin, nearly steady disk ( $v_r \simeq 0$ , so matter that is present is effectively in free fall with zero velocity relative to the velocity field),

$$v_{\text{rad,eff}}(r) = \sqrt{\frac{2GM}{r}} - H_z r,$$

and the observable orbital law becomes

$$\boxed{v_\phi^2(r) = \frac{3}{2} \left( \sqrt{\frac{2GM}{R}} - H_z R \right)^2 - \left( \sqrt{\frac{2GM}{r}} - H_z r \right)^2.}$$

The effective radial velocity was

$$\boxed{v_{\text{rad,eff}}^2(r) = \left( \sqrt{\frac{2GM}{r}} - H_z r \right)^2},$$

and at the turn-around radius we have

$$v_{\text{rad,eff}}(r_c) = 0 \Rightarrow r_c = \left( \frac{2GM}{H_z^2} \right)^{1/3}, \quad v_\phi^2(r_c) = \frac{3}{2} \left( \sqrt{\frac{2GM}{R}} - H_z R \right)^2.$$

**Summary.** All steps remain internal to the BQ algebra: the rotor  $Q_g$  encodes the rapidities  $\psi_i$  that generate the flow; the adjoint basis  $/G_\mu$  introduces the shift  $w(r)$ ; the BNC provides the conserved scalar  $\mathcal{B} = \gamma(1 - wv_r)$ ; and its low-velocity limit produces the Bernoulli invariant  $v_\phi^2 + (v_r - w)^2 = \text{const.}$  used in the disk fits.

### 5.3 Boundary condition: effective velocities *inside* a smooth bulge

In this part we derive the boundary conditions for an effective Newtonian-Hubble potential. Assume a regular (smooth) bulge with spherical symmetry and an interior Newtonian potential<sup>2</sup>

$$\Phi_{\text{in}}(r) = -\frac{GM}{2R} \left( 3 - \frac{r^2}{R^2} \right), \quad 0 \leq r \leq R.$$

Then

$$-\partial_r \Phi_{\text{in}} = \frac{GM}{R^3} r, \quad v_c^2(r) = r (-\partial_r \Phi_{\text{in}}) = \frac{GM}{R^3} r^2.$$

At the boundary  $r = R$ ,  $v_c^2(R) = GM/R$  and the *exterior* free-fall term is  $\sqrt{2GM/R}$ . To include the horizon background  $H_z$  we work with the subtracted speeds  $\sqrt{2GM/R} - H_z R$  (at  $r = R$ ) and  $\sqrt{2GM/r} - H_z r$  (general  $r$ ).

<sup>2</sup>Uniform density gives this exactly; any smooth profile with a harmonic interior has the same leading form.

**Orbital term (inside).** Because  $v_c^2(r) \propto r^2$ , write it with the boundary value as

$$v_{\text{orb}}^2(r) = \frac{1}{2} \left( \sqrt{\frac{2GM}{R}} - H_z R \right)^2 \frac{r^2}{R^2}.$$

For  $H_z = 0$  this reduces to  $v_{\text{orb}}^2 = GM r^2/R^3$  as expected.

**Radial effective term (inside).** The interior free-fall speed referenced to infinity is

$$v_{\text{ff,in}}^2(r) = 2|\Phi_{\text{in}}(r)| = \frac{GM}{R} \left( 3 - \frac{r^2}{R^2} \right).$$

Subtracting the Hubble background gives

$$v_{\text{rad,eff}}^2(r, H_z) = \left( \sqrt{\frac{GM}{R} \left( 3 - \frac{r^2}{R^2} \right)} - H_z r \right)^2.$$

At  $r = R$  this becomes  $(\sqrt{2GM/R} - H_z R)^2$ , matching the exterior form.

**Combined (projected) effective speed.** The helical (Bernoulli) projection inside adds the tangential part with the same geometric factor that yields 3/2 outside; inside this gives

$$v_L^2(r) = \frac{1}{2} \left( \sqrt{\frac{2GM}{R}} - H_z R \right)^2 \frac{r^2}{R^2} + \left( \sqrt{\frac{GM}{R} \left( 3 - \frac{r^2}{R^2} \right)} - H_z r \right)^2.$$

Evaluated at  $r = R$  this reproduces  $v_L^2(R) = \frac{3}{2} (\sqrt{2GM/R} - H_z R)^2$ , ensuring smooth matching to the exterior plateau.

**Checks.**

- Continuity at  $r = R$ : all three expressions reduce to the corresponding exterior forms with  $r \mapsto R$ .
- Newtonian limit  $H_z \rightarrow 0$ :  $v_{\text{orb}}^2 = GM r^2/R^3$ ,  $v_{\text{rad,eff}}^2 = GM/R(3 - r^2/R^2)$ , and  $v_L^2 = GM r^2/R^3 + GM/R(3 - r^2/R^2)$ .

**Interpretation.** A smooth bulge has a harmonic interior potential; circular speed grows linearly with  $r$  while the interior free-fall speed follows  $v_{\text{ff,in}}^2 \propto (3 - r^2/R^2)$ . Including the background flow by the subtractions  $(\cdot - H_z r)$  and projecting the helical (inflow+azimuthal) motion yields the formulas above, which match continuously to the exterior  $\text{CL}_H$  relations at  $r = R$ .

## 6 MOND-type acceleration laws derived from the $\text{CL}_H$ orbital velocity

### 6.1 Background: origin of MOND and its covariant problem

Modified Newtonian Dynamics (MOND) was introduced by Milgrom [10, 11, 12] to explain the flat rotation curves of spiral galaxies without invoking dark matter. In its original, nonrelativistic form the dynamical law

$$a \mu \left( \frac{a}{a_0} \right) = a_N = \frac{GM}{r^2}$$

introduced a new constant acceleration scale  $a_0 \sim 10^{-10} \text{ m s}^{-2}$  and an interpolation function  $\mu(x)$  connecting the Newtonian regime ( $\mu \rightarrow 1$ ) to the “deep-MOND” limit ( $\mu \approx x$ ). Although phenomenologically successful [17, 14], this construction was not relativistic and could not be embedded consistently in a cosmological context.

Relativistic and covariant extensions were later developed, most notably the Tensor–Vector–Scalar (TeVeS) theory [16] and its descendants [21, 22]. These frameworks aimed to reproduce the MOND limit while providing a metric description compatible with cosmology and gravitational lensing. A remaining challenge has been to obtain the MOND regimes *and* their transitions directly from a geometric flow or covariant Lagrangian without introducing auxiliary fields or interpolation functions by hand.

The  $\text{CL}_H$  formalism derived in this paper offers such a route: it provides a single kinematic law—based on a constant–Lagrangian condition with a cosmological (Hubble) boundary—that reduces to MOND–type behaviour in the appropriate acceleration range while remaining fully covariant and explicitly cosmological through its dependence on the Hubble parameter  $H$ .

## 6.2 Derivation of MOND–type acceleration laws

**Starting point (CLH kinematics).** In the constant–Lagrangian with Hubble boundary (CLH) configuration, the Painlevé–Gullstrand (PG) river model gives the effective radial flow

$$v_r^{\text{eff}}(r) = \sqrt{\frac{2GM}{r}} - Hr,$$

and the disk azimuthal velocity (for the matter–free baseline) follows from the CL closure as

$$v_\phi^2(r) = \frac{3}{2} v_r^{\text{eff}2}(R) - v_r^{\text{eff}2}(r) = v_{\text{flat}}^2 - v_r^{\text{eff}2}(r),$$

where the bulge rim  $r = R$  fixes the integration constant through the virial–continuity condition. The quantity  $v_{\text{flat}}^2$  is the empirical plateau velocity at the disk edge.

**From velocity to acceleration.** The centripetal acceleration observed in the disk is

$$a(r) = \frac{v_\phi^2(r)}{r}.$$

Expanding  $v_r^{\text{eff}2}(r)$  gives

$$v_r^{\text{eff}2}(r) = \frac{2GM}{r} - 2H\sqrt{2GM}r + H^2r^2,$$

so that

$$a(r) = \underbrace{\frac{GM}{r^2}}_{\text{Newtonian}} - \underbrace{2H\sqrt{\frac{2GM}{r}} \frac{1}{r}}_{\text{transition (MOND-like)}} + \underbrace{H^2r}_{\text{cosmic tail}} + \underbrace{\frac{v_{\text{flat}}^2}{r}}_{\text{rim constant}}. \quad (16)$$

**Interpretation of the terms.** Equation (16) contains all four components required for a complete galactic acceleration law:

- The *Newtonian term*  $\frac{GM}{r^2}$  dominates inside the bulge and recovers standard gravity.
- The *geometric–mean transition term*  $-2H\sqrt{2GM/r} 1/r$  reproduces the MOND interpolation behaviour  $a \propto \sqrt{a_N a_0}$ , with  $H$  acting as the natural acceleration scale.
- The *cosmic tail*  $H^2r$  describes the large–radius coupling to the global expansion field and approaches a de–Sitter background.
- The *rim constant*  $v_{\text{flat}}^2/r$  retains the boundary condition from  $r = R$  and fixes the asymptotic flat velocity of the disk. It provides an explicit algebraic realization of the MOND plateau.

**Relation to MOND.** Up to the rim constant, Eq. (16) already exhibits the canonical MOND structure: a Newtonian term, a geometric-mean transition, and a large-radius cosmological tail. Including the rim term makes the correspondence exact: in MOND, the flat-velocity plateau emerges implicitly through the interpolation function, whereas in the CLH law it appears explicitly through the boundary condition  $v_{\text{flat}}^2 = \frac{3}{2}(\sqrt{2GM/R} - HR)^2$ . The CLH expression thus provides a direct algebraic derivation of the MOND form without introducing a phenomenological constant  $a_0$ :

$$a_0 \equiv cH.$$

**Unified interpretation.** The complete acceleration law

$$a(r) = \frac{GM}{r^2} - 2H\sqrt{\frac{2GM}{r}}\frac{1}{r} + \frac{v_{\text{flat}}^2}{r} + H^2r$$

naturally spans four dynamical regimes:

Regime	Dominant term	Scaling	Physical meaning
Bulge	$\frac{GM}{r^2}$	$r^{-2}$	Newtonian core
Disk (transition)	$-2H\sqrt{2GM/r}1/r$	$r^{-3/2}$	MOND-like interpolation
Outer rim	$v_{\text{flat}}^2/r$	$r^{-1}$	Flat velocity plateau
Cosmic	$H^2r$	$r^{+1}$	Cosmological tail

**Summary.** The CLH acceleration law unifies the Newtonian, MOND, and cosmological regimes in a single analytic expression derived from the  $Q_g$  rotor framework. The flat-velocity plateau arises from the rim constant, the MOND scaling from the Hubble-coupled term, and the cosmic expansion from  $H^2r$ . No empirical interpolation function or new scale parameter is required: the observed MOND phenomenology emerges as the low-acceleration limit of the first-order CLH invariant.

### 6.3 Interpretations of the baryonic mass term $M$

The baryonic mass  $M$  in the CLH expression no longer appears as a purely Newtonian parameter. It participates in a composite dynamical structure where the local baryonic field, the Hubble coupling, and the rim constant jointly define the effective gravitational potential. From the acceleration law (16),

$$a(r) = \frac{GM}{r^2} - 2H\sqrt{\frac{2GM}{r}}\frac{1}{r} + H^2r + \frac{v_{\text{flat}}^2}{r},$$

we can formally associate an *effective mass function*

$$M_{\text{eff}}(r) = M - \frac{2Hr^{3/2}}{G}\sqrt{2GM} + \frac{H^2r^3}{G} + \frac{rv_{\text{flat}}^2}{G}.$$

Each term in  $M_{\text{eff}}$  describes a distinct dynamical contribution to the observed rotation field:

1. **Baryonic core.** The Newtonian component  $M$  represents the luminous matter, dominating the bulge region.
2. **Transition coupling.** The mixed term proportional to  $H\sqrt{GM}r^{3/2}$  expresses the dynamical feedback between local mass and the background expansion.
3. **Cosmic component.** The  $H^2r^3$  term acts as an algebraic representation of the cosmological background density that connects the galactic flow to the Hubble field.

4. **Rim constant.** The additional boundary term  $rv_{\text{flat}}^2/G$  retains the memory of the bulge–disk transition at  $r = R$ . It fixes the flat velocity plateau and must therefore be considered an intrinsic part of the galactic effective mass distribution.

This interpretation differs from MOND, in which the baryonic mass alone determines the rotation amplitude through a fixed acceleration scale  $a_0$ . In the CLH and  $Q_g$  framework, the baryonic and cosmological terms are algebraically linked, and the rim constant provides the closure condition that connects the Newtonian and asymptotic regimes. The result is a self-consistent definition of an effective mass that already encodes the observed mass discrepancy without introducing dark matter as a separate component.

**Summary.** The baryonic mass  $M$  in the  $Q_g$  framework thus represents the Newtonian anchor of a fourfold flow equilibrium. The total effective mass  $M_{\text{eff}}(r)$  combines the luminous content with the dynamical and cosmological corrections that arise naturally from the rotor coupling, ensuring continuity between bulge, disk, and cosmic domains.

#### 6.4 Covariance and cosmological embedding of MOND within the CLH and $Q_g$ framework

The CLH acceleration law reproduces MOND phenomenology while maintaining explicit covariance within the  $Q_g$  rotor formalism. Its terms derive from first-order flow equations rather than from a modified second-order metric dynamics. The resulting equations are covariant under local Lorentz transformations of the spinor basis, ensuring that both matter flow and gravitational field originate from a single algebraic adjoint current.

1. **Covariant structure.** In the  $Q_g$  language, the cosmic expansion is described by the rapidity field  $\psi_H$  through  $\dot{\psi}_H = -4\pi G\rho_\Psi + \Lambda$ . The MOND acceleration scale  $a_0$  corresponds directly to the Hubble rapidity,  $a_0 = cH$ , and therefore transforms as a scalar under coordinate changes. No external modification of inertia or gravity is required: the MOND law follows from the covariant evolution of the flow field itself.

2. **Natural cosmological embedding.** Because  $H$  and  $\rho_\Psi$  enter the same first-order relation, the galactic flow and the cosmological background are two aspects of a single field. The rim constant  $v_{\text{flat}}^2/r$  acts as the algebraic boundary through which the local disk couples to the global Hubble expansion, completing the continuity between galactic and cosmic regimes. At large radii the  $H^2r$  term of Eq. (16) becomes dominant, producing the correct asymptotic transition to de-Sitter behaviour.

3. **Relation to MOND covariance.** While relativistic extensions of MOND such as TeVeS introduce separate tensor and scalar fields to restore covariance, the  $Q_g$  formalism achieves the same result within the Dirac algebra itself. Covariance is maintained automatically because the gravitational rotor transforms as a spinor operator rather than as a metric tensor.

**Summary.** The CLH acceleration law therefore provides both the empirical success of MOND and a natural cosmological embedding within a fully covariant algebraic theory. The identification  $a_0 = cH$  arises directly from the first-order flow law, and the rim constant supplies the boundary condition that links local and global dynamics without additional fields or parameters.

## 7 Conclusion

The present study began by establishing the translation procedure from the biquaternionic  $Q_g$  formalism to the familiar general-relativistic metric representation. Through the adjoint operation  $/G_\mu = Q_g \beta_\mu Q_g^{-1}$  and its decomposition  $/G_\mu = e_\mu^a \hat{\beta}_a$ , the standard geometric quantities  $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$  and the line element  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$  can be recovered directly from the algebraic basis. This establishes an explicit algorithm connecting the algebraic rotor description to the metric structure employed in general relativity.

Through the course of the paper it has been shown that the same physical results—including the definition of the metric, the construction of the tetrad operators, the Bernoulli–Noether invariant, and the constant–Lagrangian (CLH) dynamics leading to MOND–type acceleration laws—can be derived entirely within the linear Dirac–BQ algebra itself. In this algebraic domain, gravity is represented as a rotor–induced transformation of the Dirac basis, and the metric arises as the symmetric scalar projection of biquaternionic products of these rotated elements. The resulting framework performs gravitational analysis without explicit recourse to tensor calculus, remaining covariant and self-consistent at the algebraic level.

The enhanced CLH dynamics derived here demonstrate that the full range of galactic and cosmological phenomena—from Newtonian cores through MOND–like transition zones to Hubble–coupled tails—can be obtained directly from the rotor flow law. The inclusion of the rim constant  $v_{\text{flat}}^2/r$  ensures dynamical continuity between bulge and cosmic regimes, reproducing the flat–velocity plateau and providing a natural algebraic origin for MOND behaviour. In this sense, the  $Q_g$  framework completes the gravitational dynamics at the algebraic level: GR reappears as its geometric projection, with the information about the underlying flow encoded but not explicit.

General relativity remains an exceptionally successful phenomenological framework across astrophysics and cosmology, and nothing in the present work contradicts its verified results. The  $Q_g$  formalism instead offers an alternative representation—one that reveals why GR’s geometric formulation functions so effectively in its tested domains, and how it can be extended to regimes where information about the flow of space becomes physically relevant.

The analysis thus shows that the gravitational and MOND–related regimes considered can indeed be constructed and interpreted within the  $Q_g$  algebraic domain itself, while recognising that general relativity continues to provide the established geometric language for observations in curved space–time. The two approaches are complementary: the  $Q_g$  formulation exposes the covariant algebraic roots of gravitational flow, and GR provides its historically and empirically grounded geometric expression.

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## Implementation recipe for the $Q_g$ algorithm (CL<sub>H</sub> branch)

This summary outlines the minimal executable sequence for reproducing the constant-Lagrangian (CL<sub>H</sub>) and Bernoulli–Noether (BNC) results.

### Step 1: Construct the gravitational rotor $Q_g$ .

Use the local rapidities  $(\psi_r, \psi_\phi, \psi_H)$  for radial inflow, azimuthal swirl, and horizon expansion:

$$Q_g = \exp\left[\frac{1}{2}(\psi_r \hat{\beta}_r \hat{\beta}_0 + \psi_\phi \hat{\beta}_\phi \hat{\beta}_0 + \psi_H \hat{\beta}_H \hat{\beta}_0)\right].$$

### Step 2: Extract the tetrad $E = [e_\mu^a]$ and the metric.

From the adjoint  $/G_\mu = Q_g \beta_\mu Q_g^{-1} = e_\mu^a \hat{\beta}_a$  read off the PG-like shift (river) components:

$$w_r(r) = \sqrt{\frac{2GM}{r}} - H_z r, \quad w_\phi(r, \theta) \text{ (optional spiral/frame-drag).}$$

### Step 3: Fix the BNC constant $K$ at the bulge rim $r = R$ .

Apply the virial and continuity conditions:

$$v_\phi^{\text{mat}}(R) = v_{\text{orb}}(R) = \frac{1}{2}^{1/2} \left( \sqrt{\frac{2GM}{R}} - H_z R \right), \quad v_r^{\text{mat}}(R) \simeq 0,$$

leading to

$$K = \frac{3}{2} \left( \sqrt{\frac{2GM}{R}} - H_z R \right)^2.$$

### Step 4: Propagate along the disk (Bernoulli–Noether invariant).

The conserved relative combination is

$$(v_r^{\text{mat}} - w_r)^2 + (v_\phi^{\text{mat}} - w_\phi)^2 - w^2 = K.$$

For a thin steady disk set  $v_r^{\text{mat}} \approx 0$  after the constant is fixed:

$$v_\phi^{\text{mat}}(r) = \sqrt{K - w_r^2(r)}, \quad v_{\text{rad,eff}}(r) = -w_r(r).$$

### Step 5: Predict observable rotation curves.

The observable orbital velocity (PG/tetrad frame) is

$$v_{\text{orb}}(r) = |w_\phi(r) + v_\phi^{\text{mat}}(r)|.$$

With  $w_\phi \approx 0$  this reduces to

$$v_{\text{orb}}^2(r) = \frac{3}{2} \left( \sqrt{\frac{2GM}{R}} - H_z R \right)^2 - \left( \sqrt{\frac{2GM}{r}} - H_z r \right)^2.$$

### Step 6: Outer edge and plateau.

The turn-around radius follows from  $v_{\text{rad,eff}}(r_c) = 0$ :

$$r_c = \left( \frac{2GM}{H_z^2} \right)^{1/3}, \quad v_{\text{orb}}^2(r_c) = K.$$