The Q_g Rotor Framework as an Algorithmic Bridge Between Dirac Algebra and Gravitation

From Constant–Lagrangian Dynamics to MOND Phenomenology

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Abstract

This paper presents the gravitational rotor framework Q_g as an executable biquaternionic (Dirac) algebra that generates general–relativistic structure without external geometric postulates. From the adjoint action $\not{G}_\mu = Q_g \, \beta_\mu \, Q_g^{-1}$, the metric $g_{\mu\nu}$ and affine connection follow algorithmically, defining an explicit translation map $Q_g(x) \to (g_{\mu\nu}, \Gamma^\rho{}_{\mu\nu}, R^\rho{}_{\sigma\mu\nu})$. Within this algebraic system, the mixed tensor $M_\mu{}^\nu = E_\mu{}^a \, \Phi_a{}^\nu$ and the four current $J^\nu = u^\mu M_\mu{}^\nu = \Psi^\dagger \, Q_g \, \beta_\mu \, Q_g^{-1} \, \beta^\nu \Psi$, represent the coupling between matter flow and metric generation, while E and Φ encode the local tetrad and gravitational phase fields. Applying the constant–Lagrangian with Hubble boundary (CL–H) condition yields galactic rotation laws and MOND–type accelerations $a(r) = GM/r^2 - 2H\sqrt{2GM/r}/r + H^2 r$ as covariant consequences of the same invariant. Thus, the Q_g engine provides both a translation algorithm from biquaternionic algebra to general relativity and an autonomous linear framework in which gravity and galactic dynamics emerge coherently from a single covariant rotor field.

Contents

1 Introduction

The present paper continues a research line that began with the search for a purely dynamical explanation of galactic rotation curves without invoking dark matter. In 2019, de Haas [1] proposed a first, one–dimensional formulation in terms of a *constant–Lagrangian* (CL) postulate, which successfully fitted the complete SPARC database of 175 late–type galaxies. That early model, although empirically accurate, lacked a clear theoretical foundation and contained no explicit cosmological boundary. Its further development paused until the constant–Lagrangian structure was reinterpreted half a decade later as a two–dimensional spiral inflow, allowing the inclusion of a natural Hubble term at large radii. This refinement, denoted CL–H, provided a physically motivated closure condition at the cosmic boundary.

In parallel, a second line of research explored the algebraic structure underlying relativistic field equations. Building on Dirac's 1928 operator formulation of quantum mechanics [6] and Hestenes' geometric reinterpretation of the Dirac algebra [7], de Haas [2] reconstructed the Weyl and Dirac matrices within a biquaternion (BQ) basis, showing that Lorentz and gravitational transformations can be represented as rotor actions inside the Dirac algebra itself. The goal was to express gravitation algebraically within the same mathematical language as quantum mechanics rather than introduce curvature externally. This idea was expanded in de Haas [4], where attaching the gravitational rapidity rotor to the Dirac adjoint produced an exact linearisation of the Einstein equations, and further developed in [5] in connection with the covariant treatment of time in quantum mechanics.

The empirical results of the CL–H approach [3] and the algebraic achievements of the Q_g framework thus converged. In the present work these two lines are united: the constant–Lagrangian postulate, originally introduced as an empirical law, is now derived algebraically from the Dirac–BQ structure itself. This establishes the CL–H relation as a covariant consequence of the Q_g field rather than an external assumption.

The paper proceeds in two complementary directions. First, it provides the explicit algorithm translating the algebraic quantities of the Q_g formalism—the gravitational rotor Q_g, the rotated Dirac basis $\not{G}_\mu$, and the tetrad operator ${E_\mu}^a$— into the metric tensors of general relativity. This translation clarifies how the geometric language of ADM and MTW [9, 8] can be reconstructed directly from the algebraic basis. Second, and more fundamentally, it demonstrates that the same gravitational behaviour can be formulated entirely within the BQ domain itself, without recourse to tensor or curvature formalisms. The M matrix, the E matrix, and the gravitational Dirac current J^μ are derived as linear BQ operators, forming a self–consistent algebraic system that reproduces the essential features of gravitational flows.

Finally, the covariant derivation of the CL–H condition allows a direct connection to the phenomenology of Modified Newtonian Dynamics (MOND) [10, 11, 12, 15, 23, 18]. The empirical success of MOND in fitting galactic rotation curves is shown to be empirically rooted in the same Q_g rotor framework that produces the constant–Lagrangian dynamics. The acceleration scale a_0 that appears in MOND, and the quality of the MOND fits obtained by Milgrom and later by Sanders, Famaey, and McGaugh, can thus be understood as manifestations of the same covariant flow geometry described by the CL–H field. Thus, while the present work includes an explicit algorithm for constructing the GR metric from the Q_g basis, its deeper result is the demonstration that the gravitational domain treated here—ranging from local flows to galactic scales—can be formulated self–consistently within the linear, covariant Dirac–BQ architecture.

2 The algorithm to construct GR from Qg as its covariant basis

2.1 The theory as an executable Clifford–geometry engine

The Q_g formalism is now fully algorithmic: it constitutes an *executable algebra*. Every essential relation in the framework can be expressed as a deterministic pipeline of Clifford operations,

$$Q_g(x) \;\longrightarrow\; \begin{cases} \not{G}_\mu = Q_g\,\beta_\mu\,Q_g^{-1}, \\ g_{\mu\nu} = \frac{1}{2}\{\not{G}_\mu, \not{G}_\nu\}, \\ \Omega_\mu = (\partial_\mu Q_g)\,Q_g^{-1}, \\ F_{\mu\nu} = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu + [\Omega_\mu, \Omega_\nu]. \end{cases} \tag{1}$$

From these relations one can extract observables such as Dirac bilinears, ADM variables, curvature invariants, and energy–momentum densities. Every step involves only *symbolic Clifford algebra*, *differentiation*, or *matrix multiplication*—there is no interpretive gap between definition and computation.

Hence any computational environment capable of Clifford algebra can execute the formalism directly: for example MATHEMATICA with the `CliffordAlgebra` package, PYTHON with `clifford` or `sympy`, JULIA with `Grassmann.jl`, or C++ with `clifford++`. Once the rotor field $Q_g(x)$ is specified, the entire geometric structure $(g_{\mu\nu},\, \Omega_\mu,\, F_{\mu\nu})$ follows deterministically by symbolic evaluation.

Algorithmically, it is a "Clifford–geometry engine." The entire Q_g formalism can be summarised as a *computable map*

$$Q_g(x) \;\longmapsto\; (g_{\mu\nu}(x),\, \Gamma_\mu(x),\, R_{\mu\nu\rho\sigma}(x)), \tag{2}$$

through the algebraic chain

$$Q_g \;\longrightarrow\; (\not{G}_\mu) \;\longrightarrow\; (g_{\mu\nu}) \;\longrightarrow\; (\Gamma_\mu) \;\longrightarrow\; (F_{\mu\nu}), \tag{3}$$

which constitutes a complete algorithmic sequence. Once the rotor generator $\psi_i(x)$ is specified, a symbolic processor can carry out all subsequent steps automatically. This defines the Q_g framework as a *Clifford–geometry engine*—a closed algebraic system that converts rotor fields into curvature, connections, and metrics without any external geometric assumptions.

2.2 The covariant basis equation as structural condition

The relation

$$(\partial_\mu \not{G}_\nu) = [\,\Omega_\mu, \not{G}_\nu\,], \qquad \text{with} \quad \not{G}_\mu = Q_g\,\beta_\mu\,Q_g^{-1}, \quad \Omega_\mu = (\partial_\mu Q_g)\,Q_g^{-1}, \tag{4}$$

is one of the central identities of the Q_g formalism. It defines how the local Clifford basis $\{\not{G}_\mu\}$ evolves under the connection Ω_μ and guarantees metric compatibility. Although it looks like a field equation, it is in fact a structural identity that holds for every valid Q_g configuration.

1. Meaning of the equation. Equation (4) expresses the *covariant constancy* of the local basis under the connection Ω_μ: any variation of $\not{G}_\nu$ through spacetime is represented as an infinitesimal rotation or boost within the algebra. This is the exact Clifford–algebra analogue of the tetrad postulate in general relativity,

$$\nabla_\mu e_\nu{}^a = 0,$$

with $\not{G}_\nu$ replacing $e_\nu{}^a\gamma_a$ and Ω_μ replacing the Lorentz connection $\omega_\mu{}^{ab}$.

2. Not a dynamical law. The equation does not govern evolution in time; it *defines what it means to be a consistent geometry.* It follows identically from differentiating the adjoint definition $\not{G}_\nu = Q_g\beta_\nu Q_g^{-1}$:

$$\partial_\mu(\not{G}_\nu) = (\partial_\mu Q_g)\beta_\nu Q_g^{-1} + Q_g\beta_\nu(\partial_\mu Q_g^{-1}) = [(\partial_\mu Q_g)Q_g^{-1}, \not{G}_\nu] = [\Omega_\mu, \not{G}_\nu].$$

Thus Eq. (4) is automatically satisfied once Q_g is differentiable.

3. Its law-like role. Although algebraically trivial, Eq. (4) acts as a *structural law* of the geometry: it enforces metric compatibility,

$$\partial_\mu g_{\nu\rho} = 0, \qquad g_{\nu\rho} = \tfrac{1}{2}\{\not{G}_\nu, \not{G}_\rho\},$$

ensuring that Ω_μ preserves the inner product of the local frame. This is the same condition that defines the Levi–Civita connection in GR, here expressed directly in Clifford form.

4. Its place in the hierarchy of equations. Within the Q_g system, the logical layers are:

- **Identity:** $(\partial_\mu\not{G}_\nu) = [\Omega_\mu, \not{G}_\nu]$ defines a valid geometry (metric compatibility).
- **Definition:** $g_{\mu\nu} = \frac{1}{2}\{\not{G}_\mu, \not{G}_\nu\}$ gives the metric.
- **Curvature law:** $F_{\mu\nu} = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu + [\Omega_\mu, \Omega_\nu]$ defines the field strength of the connection.
- **Dynamics:** the Bernoulli–Noether closure or other first–order field relations specify the evolution of Q_g.

The covariant–basis equation therefore sits at the foundational level: it provides the consistent kinematic framework in which the dynamics operate.

5. Conceptual significance. The identity $(\partial_\mu\not{G}_\nu) = [\Omega_\mu, \not{G}_\nu]$ encapsulates the idea that all geometric information—metric, connection, and curvature—is generated internally by the rotor field $Q_g(x)$. It is the algebraic statement that the local basis vectors are covariantly constant under Ω_μ, the spin connection built from Q_g.

The equation is not a separate field law to be imposed; it is the *defining structural condition* that makes every Q_g geometry metric-compatible and hence physically meaningful. It plays the same role as $\nabla_\mu g_{\nu\rho} = 0$ in general relativity, expressed here directly in the Clifford algebra language that underlies the Q_g framework.

3 Gravity embedded in the probability tensor and the Dirac current as its covariant contraction

3.1 The mixed tensor $\mathbb{M}_\mu{}^\nu$ as spin–metric operator

All information about the local spinor–metric coupling and gravitational flow can be collected into a single mixed tensor field,

$$\boxed{\mathbb{M}_\mu{}^\nu(x) := \Psi^\dagger(x)\,(\not{G}_\mu(x)\,\beta^\nu)\,\Psi(x) = \Psi^\dagger Q_g\,\beta_\mu\,Q_g^{-1}\,\beta^\nu\Psi,} \tag{5}$$

which is a genuine 4×4 field carrying both the metric and the bivector (flow/spin) information.

1. Decomposition into scalar and bivector parts. The field $\mathbb{M}_\mu{}^\nu$ splits cleanly into a symmetric (metric) part and an antisymmetric (bivector) part:

$$\mathbb{M}_\mu{}^\nu = \tfrac{1}{2}\Psi^\dagger\{\not{G}_\mu, \beta^\nu\}\Psi + \tfrac{1}{2}\Psi^\dagger[\not{G}_\mu, \beta^\nu]\Psi, \tag{6}$$

where the first term represents the mixed metric map and the second encodes the gravitational flow and spin information.

Using the duality condition for the common coframe of $\{\not{G}_\mu\}$ and $\{\beta^\nu\}$,

$$\tfrac{1}{2}\{\not{G}_\mu, \beta^\nu\} = \delta_\mu{}^\nu\,\mathbb{1},$$

this becomes

$$\boxed{\mathbb{M}_\mu{}^\nu = \rho\,\delta_\mu{}^\nu + \tfrac{1}{2}\Psi^\dagger[\not{G}_\mu, \beta^\nu]\Psi,} \tag{7}$$

where $\rho := \Psi^\dagger\Psi$ is the scalar (probability) density. The commutator term is exactly the bivector contribution that carries the gravitational flow and frame–drag information that should not be collapsed.

2. Tetrad form and projected bivector density. In the GR (tetrad) form, expand $\not{G}_\mu = e_\mu{}^a\hat{\beta}_a$ and $\beta^\nu = e_b{}^\nu\hat{\beta}^b$. With $\Sigma_a{}^b := \frac{1}{2}[\hat{\beta}_a, \hat{\beta}^b]$, the spin–bivector density is $s_a{}^b := \Psi^\dagger\Sigma_a{}^b\Psi$, so that

$$\boxed{\mathbb{M}_\mu{}^\nu = \rho\,\delta_\mu{}^\nu + e_\mu{}^a\,e_b{}^\nu\,s_a{}^b.} \tag{8}$$

The two pieces have clear geometric meaning:

- $\rho\,\delta_\mu{}^\nu$ is the *mixed metric map* (index identity) carried by the Betas.
- $e_\mu{}^a e_b{}^\nu s_a{}^b$ is the *projected spin–bivector density*, representing the local gravitational flow and frame–drag content in spacetime indices.

3. Observable current. The contracted current is simply the time–direction projection:

$$j^\nu = u^\mu\,\mathbb{M}_\mu{}^\nu, \tag{9}$$

which keeps the bivector information intact while remaining fully translatable to standard GR notation via the tetrad $e_\mu{}^a$ and the bivector generators $\Sigma_a{}^b$.

4. Interpretation. The tensor $\mathbb{M}_\mu{}^\nu$ thus combines in a single object both the scalar metric density and the antisymmetric spin–flow structure of the local spacetime. It acts as the *spin–metric operator*: the scalar part transmits the index identity of the metric, while the bivector part transmits the rotational and gravitational degrees of freedom carried by the rotor field Q_g.

3.2 The full mixed tensor $\mathbb{M}_\mu{}^\nu$ as a 4×4 field

The full tensor $\mathbb{M}_\mu{}^\nu$ can be expressed compactly as a 4×4 matrix field built from the spinor bilinears and the tetrad extracted from the adjoint action of Q_g. Starting from

$$\boxed{\mathbb{M}_\mu{}^\nu(x) = \Psi^\dagger(x)\,(Q_g\,\beta_\mu\,Q_g^{-1}\,\beta^\nu)\,\Psi(x) = e_\mu{}^a\,\Phi_a{}^\nu,} \tag{10}$$

where the flat–basis matrix

$$\Phi_a{}^\nu := \Psi^\dagger(\hat{\beta}_a\,\beta^\nu)\Psi$$

contains all spinor bilinear combinations. Thus $\mathbb{M} = E\,\Phi$ with $E = [e_\mu{}^a]$ the tetrad row–matrix, so that all gravitational and flow information from Q_g is preserved via E.

1. Bilinear matrix in the Dirac–Clifford basis. Define the standard bilinears

$$\rho := \Psi^\dagger \Psi, \qquad j_i := \Psi^\dagger \alpha_i \Psi, \qquad s_i := \Psi^\dagger (i\Sigma_i)\Psi.$$

Then the flat–basis matrix $\Phi_a{}^\nu$ is the explicit 4×4 field (rows $a = 0, 1, 2, 3$; columns $\nu = 0, 1, 2, 3$):

$$\Phi_a{}^\nu = \begin{bmatrix} -\rho & j_1 & j_2 & j_3 \\ -j_1 & \rho & s_3 & -s_2 \\ -j_2 & -s_3 & \rho & s_1 \\ -j_3 & s_2 & -s_1 & \rho \end{bmatrix}. \tag{11}$$

This matrix collects all density, current, and spin–bivector information of the Dirac field in a compact algebraic form.

2. Expansion of the full tensor. The complete 4×4 tensor $\mathbb{M}_\mu{}^\nu$ follows as

$$\mathbb{M}_\mu{}^\nu = e_\mu{}^0 \begin{bmatrix} -\rho & j_1 & j_2 & j_3 \end{bmatrix}^\nu + e_\mu{}^1 \begin{bmatrix} -j_1 & \rho & s_3 & -s_2 \end{bmatrix}^\nu \tag{12}$$

$$+e_\mu{}^2 \begin{bmatrix} -j_2 & -s_3 & \rho & s_1 \end{bmatrix}^\nu + e_\mu{}^3 \begin{bmatrix} -j_3 & s_2 & -s_1 & \rho \end{bmatrix}^\nu. \tag{13}$$

Equivalently, componentwise (useful for code or symbolic expansion):

$$\mathbb{M}_\mu{}^0 = -e_\mu{}^0 \rho - e_\mu{}^1 j_1 - e_\mu{}^2 j_2 - e_\mu{}^3 j_3, \tag{14}$$

$$\mathbb{M}_\mu{}^1 = e_\mu{}^0 j_1 + e_\mu{}^1 \rho - e_\mu{}^2 s_3 + e_\mu{}^3 s_2, \tag{15}$$

$$\mathbb{M}_\mu{}^2 = e_\mu{}^0 j_2 + e_\mu{}^1 s_3 + e_\mu{}^2 \rho - e_\mu{}^3 s_1, \tag{16}$$

$$\mathbb{M}_\mu{}^3 = e_\mu{}^0 j_3 - e_\mu{}^1 s_2 + e_\mu{}^2 s_1 + e_\mu{}^3 \rho. \tag{17}$$

3. Interpretation.

- $E = [e_\mu{}^a]$ is the tetrad extracted from the adjoint action $Q_g\, \beta_\mu\, Q_g^{-1} = e_\mu{}^a \hat{\beta}_a$, exactly as in the "metric–from–Betas" construction, with $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$.
- Because $\mathbb{M} = E\,\Phi$, the bivector (flow/spin) content is retained through the off–diagonal s_i terms and the transport encoded in E. Nothing collapses to a lapse–only description. Translation back to GR is immediate through the pairs $(e_\mu{}^a, \omega_\mu{}^{ab})$.

4. Summary. The tensor $\mathbb{M}_\mu{}^\nu$ represents the combined *spin–metric operator* of the $\mathrm{Q_g}$ framework. It packages the density ρ, current j_i, and spin s_i of the Dirac field together with the tetrad geometry $E = [e_\mu{}^a]$ extracted from the gravitational rotor Q_g. In this way, $\mathbb{M}_\mu{}^\nu$ keeps all gravitational–flow and spinor information intact while remaining fully translatable to the standard GR formalism.

3.3 The Dirac current in the $\mathbf{Q_g}$ field

A Lorentz–covariant expression for the Dirac current in the $\mathrm{Q_g}$ framework is

$$\boxed{j^\nu = u^\mu\, \mathbb{M}_\mu{}^\nu = u^\mu\, \Psi^\dagger \big(Q_g\, \beta_\mu\, Q_g^{-1}\, \beta^\nu\big)\Psi,} \tag{18}$$

where u^μ is the observer's four–velocity (relativistic self–velocity) and $\mathbb{M}_\mu{}^\nu$ is the mixed spin–metric tensor introduced above.

1. Definition and transformation. The vector u^μ defines the local time direction of the observer, while $\mathbb{M}_\mu{}^\nu$ is a mixed $(1,1)$ tensor built from the spinor field and the gravitational rotor Q_g. Under any Lorentz or local Spin$(1,3)$ transformation Λ_D, the pair transforms as

$$Q_g' = \Lambda_D Q_g, \qquad u'^\mu = \Lambda^\mu{}_\rho u^\rho, \qquad \mathbb{M}'_\mu{}^\nu = \Lambda_\mu{}^\alpha (\Lambda^{-1})_\beta{}^\nu\, \mathbb{M}_\alpha{}^\beta,$$

so that

$$j'^\nu = u'^\mu \mathbb{M}'_\mu{}^\nu = (\Lambda^{-1})_\beta{}^\nu j^\beta,$$

confirming that j^ν transforms as a contravariant four–vector current.

2. Physical interpretation. The quantity u^μ represents the observer's local time direction, defining the congruence of worldlines through which the Dirac field is measured. The tensor $\mathbb{M}_\mu{}^\nu$ contains both the scalar density $\rho\,\delta_\mu{}^\nu$ and the bivector term $e_\mu{}^a e_b{}^\nu s_a{}^b$ that carries the local spin and gravitational flow. Contracting with u^μ therefore projects this composite structure onto the observer's time direction, producing the physical current measured in that frame.

3. Covariance in the $\mathbf{Q_g}$ geometry. Because both u^μ and $\mathbb{M}_\mu{}^\nu$ are defined through Q_g and transform covariantly under its adjoint action, the current j^ν is covariant not only under Lorentz transformations but also under local $\mathrm{Q_g}$ rotor flows. It is therefore the natural generalisation of the Dirac current to curved spacetimes generated by Q_g.

4. Relation to the standard Dirac current. In flat space ($Q_g = \mathbb{1}$, $u^\mu = (1,0,0,0)$), this expression reduces to

$$j^\nu = \Psi^\dagger \beta^\nu \Psi,$$

the standard Dirac current. In the full $\mathrm{Q_g}$ geometry, the rotor Q_g modifies the local basis vectors β_μ and thus couples the current to the gravitational rapidity field.

In short. The current

$$\boxed{j^\nu = u^\mu \mathbb{M}_\mu{}^\nu}$$

is the correct and covariant form of the Dirac current in a $\mathrm{Q_g}$ field. It represents the current as measured by an observer moving with self–velocity u^μ in the locally curved geometry generated by the rotor field Q_g, retaining both the scalar density and the spin–bivector content of the spinor field.

4 The tetrad matrix $E = [e_\mu{}^a]$ and its metric representations

The mixed tensor

$$\boxed{\mathbb{M}_\mu{}^\nu = e_\mu{}^a \Phi_a{}^\nu}$$

links the internal spinor bilinear block $\Phi_a{}^\nu$ with the external operator $\mathbb{M}_\mu{}^\nu$ generated by the adjoint action of the gravitational rotor field Q_g,

$$\mathbb{M}_\mu{}^\nu = \Psi^\dagger (Q_g\, \beta_\mu\, Q_g^{-1}\, \beta^\nu)\Psi.$$

From this relation, the tetrad matrix

$$E = [e_\mu{}^a] \qquad \text{with} \qquad \mathbb{M} = E\,\Phi,$$

is obtained as the local linear map connecting the internal spinor algebra to the external spacetime indices. Once E is known, the metric follows in the standard form $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$.

4.1 E as a 4×4 block matrix (ADM/3+1 form)

Write lapse $N > 0$, shift N^i, and a spatial triad $E_i{}^j$ with $h_{ij} = E_i{}^k E_j{}^\ell \delta_{k\ell}$. The coframe one–forms $\theta^a = e^a{}_\mu \, dx^\mu$ are encoded in

$$E \equiv [e_\mu{}^a] = \begin{pmatrix} e_0{}^0 & e_0{}^j \\ e_i{}^0 & e_i{}^j \end{pmatrix} = \begin{pmatrix} N & N^k E_k{}^j \\ 0 & E_i{}^j \end{pmatrix}.$$

This yields

$$\theta^0 = N \, dt, \qquad \theta^i = E^i{}_j (dx^j + N^j dt),$$

and

$$ds^2 = -(\theta^0)^2 + (\theta^i)^2 \quad \Rightarrow \quad g_{00} = -N^2 + h_{ij} N^i N^j, \quad g_{0i} = h_{ij} N^j, \quad g_{ij} = h_{ij}.$$

The standard ADM decomposition therefore appears naturally within the BQ framework once the algebraic E is determined from $\mathbb{M}$.

4.2 Where the rapidities appear

The local gravitational rapidities $\psi_i(x)$, encoded in the rotor field $Q_g(x)$, enter the first row of E through

$$e_0{}^i = N^k E_k{}^i = v^j E_j{}^i, \qquad v^j = \tanh \psi_j.$$

- In **Painlevé–Gullstrand (PG)** gauges, $e_0{}^i$ encodes the inflow velocity v^i (e.g. $e_0{}^r = v^r$, $e_0{}^\phi = R \, v^\phi$).
- In **comoving FLRW** coordinates, the rapidity integrates to the scale factor $a(t) = \exp[\int \dot{\psi}_H(t) \, dt]$.
- In **Lorentz display**, rapidities appear explicitly as $\cosh \psi$, $\sinh \psi$.

The gravitational/flow content thus resides in E, while the spinor bilinear structure remains in Φ.

4.3 Spherical coordinates (t, r, θ, ϕ)

For spatial one–forms $\theta^1 = dr$, $\theta^2 = r \, d\theta$, $\theta^3 = r \sin\theta \, d\phi$ one has $E_i{}^j = \mathrm{diag}(1, r, r\sin\theta)$, giving

$$E_{\mathrm{PG,sph}} = \begin{pmatrix} 1 & v^r & r\,v^\theta & r\sin\theta\, v^\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}.$$

Common flows:

- **Radial inflow:** $v^r(r) = \tanh \psi_r(r)$, $v^\theta = v^\phi = 0$ (Schwarzschild case).
- **Azimuthal rotation:** $v^\phi(r, \theta) = \tanh \psi_\phi$, others zero (Doran/Kerr–like rotation).
- **Combined** $(r + \phi)$**:** both rapidities present in the first row.

For comoving FLRW (no shift),

$$E_{\mathrm{FLRW,sph}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t) & 0 & 0 \\ 0 & 0 & a(t)r & 0 \\ 0 & 0 & 0 & a(t)r\sin\theta \end{pmatrix}.$$

4.4 Cylindrical coordinates (t, R, ϕ, z)

With $\theta^1 = dR$, $\theta^2 = R\,d\phi$, $\theta^3 = dz$, and $E_i{}^j = \mathrm{diag}(1, R, 1)$,

$$E_{\mathrm{PG,cyl}} = \begin{pmatrix} 1 & v^R & R\,v^\phi & v^z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Special cases:

- **Pure rotation about z:** $v^R = v^z = 0$, only $v^\phi(R) \neq 0$ (Doran/Kerr–like inflow $\to$ rotation).
- **Pure axial flow:** $v^z(z) \neq 0$, others zero.

Comoving FLRW (no shift):

$$E_{\mathrm{FLRW,cyl}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t) & 0 & 0 \\ 0 & 0 & a(t)R & 0 \\ 0 & 0 & 0 & a(t) \end{pmatrix}.$$

4.5 Classical metric cases derived from E

When the tetrad E extracted from $\mathbb{M}$ is inserted in $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$, one obtains the classical metrics as specific rapidity patterns:

- **Schwarzschild (PG form):** $v^r = -\sqrt{2GM/r}$, giving
$$ds^2 = -dt^2 + (dr - \sqrt{2GM/r}\,dt)^2 + r^2 d\Omega^2.$$
- **de Sitter (cosmological rapidity):** $v_H(r,t) = H(t)\,r = \tanh\psi_H(t,r)$, producing
$$ds^2 = -dt^2 + (dr - Hr\,dt)^2 + r^2 d\Omega^2.$$

The Schwarzschildvand de Sitter metrics thus appear as direct realizations of the same algebraic E obtained from $\mathbb{M}$, each corresponding to a specific rapidity profile within the rotor field Q_g.

Summary. The sequence

$$\boxed{\mathbb{M} \longrightarrow E \longrightarrow g_{\mu\nu}}$$

defines the passage from the internal spinor algebra to the external geometric metric in the Q_g framework. The matrix E serves as the algebraic bridge: its first row carries the gravitational rapidity (flow) information, its spatial block defines the local triad, and its insertion into $\mathbb{M} = E\,\Phi$ separates the geometric and spinor sectors while maintaining full covariance. Classical geometries such as Schwarzschild, CL_H, and de Sitter arise naturally as specific rapidity configurations of E inside the unified biquaternion structure.

5 The constant Lagrangian galactic disk from the Bernoulli–Noether Closure

5.1 PG tetrad E with $\psi_r, \psi_\phi, \psi_H$ on equal footing (radial + azimuthal flow)

We work in spherical coordinates (t, r, θ, ϕ) and use the adjoint basis $\not{G}_\mu = Q_g\,\beta_\mu\,Q_g^{-1} = e_\mu{}^a\,\hat\beta_a$. The gravitational rotor is written with three rapidities,

$$Q_g \;=\; \exp\Big[\tfrac{1}{2}\big(\psi_r\,\hat\beta_r\hat\beta_0 + \psi_\phi\,\hat\beta_\phi\hat\beta_0 + \psi_H\,\hat\beta_H\hat\beta_0\big)\Big],$$

so that the associated local (tetrad–measured) speeds are

$$v_r = \tanh \psi_r, \qquad v_\phi = \tanh \psi_\phi, \qquad v_H = \tanh \psi_H.$$

In a Painlevé–Gullstrand (PG) gauge the temporal basis vector carries the metric "river/shift" field,

$$\not{G}_0 = \hat{\beta}_0 + w_i\, \hat{\beta}^i, \qquad w_i = (w_r, w_\phi, 0),$$

where the shift components are the *net* geometric flows built from the rapidities:

$$\boxed{w_r(r) = v_r^{(g)}(r) - v_H(r), \qquad w_\phi(r,\theta) = v_\phi^{(g)}(r,\theta)}$$

with $v^{(g)} = \tanh \psi$. (For the CL_H specialization, $w_r(r) = \sqrt{2GM/r} - H_z r$ and w_ϕ may be nonzero in a swirl/spiral gauge or frame–drag configuration.)

PG coframe. The coframe (rows indexed by μ, columns by a) is

$$E_{\mathrm{PG}}(r,\theta) = \begin{pmatrix} 1 & w_r(r) & 0 & r\sin\theta\, w_\phi(r,\theta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}.$$

It yields the metric

$$ds^2 = -dt^2 + \big(dr - w_r\, dt\big)^2 + r^2\Big[d\theta^2 + \sin^2\theta \big(d\phi - w_\phi\, dt\big)^2\Big],$$

which simultaneously carries radial inflow/outflow (via w_r) and azimuthal swirl (via w_ϕ) coming from $(\psi_r, \psi_\phi, \psi_H)$.

Matter versus metric motion (for later use in BNC). If matter has its own kinematics $(v_r^{\mathrm{mat}}, v_\phi^{\mathrm{mat}})$, the Bernoulli–Noether invariant depends only on *relative* velocities:

$$\boxed{(v_r^{\mathrm{mat}} - w_r)^2 + (v_\phi^{\mathrm{mat}} - w_\phi)^2 = C\ , \qquad C = \mathrm{const.}}$$

so that the observable orbital speed measured by tetrad/PG observers is $v_{\mathrm{orb}} = |\, w_\phi + v_\phi^{\mathrm{mat}}\,|$, and the effective radial drive is $v_{\mathrm{rad,eff}} = v_r^{\mathrm{mat}} - w_r$. For a matter–free vacuum flow one sets $v_r^{\mathrm{mat}} = v_\phi^{\mathrm{mat}} = 0$, while for thin disks one keeps v_r^{mat} through the algebra and puts $v_r^{\mathrm{mat}} \approx 0$ only at the end.

Specializations.

- **CL_H (Schwarzschild + Hubble):** $w_r(r) = \sqrt{2GM/r} - H_z r,\ w_\phi = 0$.
- **Swirl/spiral or frame–drag:** $w_\phi \neq 0$ (e.g. Doran/Kerr–like), optionally combined with the CL_H w_r.
- **Equal–status rapidities:** keep $\psi_r, \psi_\phi, \psi_H$ explicit and set $w_r = \tanh\psi_r - \tanh\psi_H$, $w_\phi = \tanh\psi_\phi$; in the nonrelativistic limit $\tanh\psi \simeq \psi$ and $v_H(r) \simeq H_z r$ recovers the usual forms.

5.2 From Qg to galactic rotation curves

1. Dirac bilinears and the BNC definition. Inside the BQ algebra, define the scalar density and Dirac current bilinears

$$\rho := \Psi^\dagger \Psi, \qquad j^a := \Psi^\dagger \hat{\beta}^a \Psi = \rho\,(\gamma, \gamma v_i),$$

so that the normalized 4-velocity is $u^a = j^a/\rho = (\gamma, \gamma v_i)$. For any Killing direction ξ, the Bernoulli–Noether constant (BNC) is the streamline-invariant scalar

$$\boxed{\mathcal{B}[\xi] := \frac{1}{\rho}\,\Psi^\dagger(\xi^\mu \not{G}_\mu)\Psi = u^a\,\xi_a}.$$

Choosing the PG time symmetry $\xi = \partial_t$ gives

$$\mathcal{B} = \frac{1}{\rho}\Psi^\dagger(\not{G}_0)\Psi = u^0 + w_i u^i = \gamma(1 - w_i v^i).$$

2. Reduction to spherical symmetry. In a spherically symmetric or axisymmetric system, the "river" field is purely radial:

$$w^i = (w_r,\, 0,\, 0) \quad \Longrightarrow \quad w_i v^i = w_r v_r = w\,v_r.$$

Thus

$$\boxed{\mathcal{B} = \gamma(1 - w v_r) = \text{const.}} \tag{19}$$

The BNC now measures the conserved energy per unit rest mass of the flow with respect to the moving spatial frame.

3. Low–velocity reduction and completing the square. For galactic and weak-field regimes, all three velocities are small ($v_r^2 + v_\phi^2 \ll 1$), so

$$\gamma \simeq 1 + \tfrac{1}{2}(v_r^2 + v_\phi^2),$$

and therefore

$$\mathcal{B} \simeq 1 + \tfrac{1}{2}(v_r^2 + v_\phi^2) - w v_r.$$

Completing the square gives [1]

$$\mathcal{B} \simeq \left(1 - \tfrac{1}{2}w^2\right) + \tfrac{1}{2}\big[(v_r - w)^2 + v_\phi^2\big].$$

The first term, $1 - \frac{1}{2}w^2$, is a background normalization that reflects the metric lapse in the PG slicing. It varies slowly with r and does not affect the dynamical invariance, so it can be absorbed into the constant of motion. Dropping this additive offset yields the simplified invariant form

$$\boxed{v_\phi^2 + (v_r - w)^2 = C, \quad C = \text{const.}}.$$

This is the *BQ Bernoulli invariant*: the sum of the squared azimuthal velocity and the squared radial velocity relative to the background gravitational inflow is conserved.

[1] The conserved Bernoulli–Noether quantity is not $\mathcal{B}$ itself but the invariant combination $(v_r - w)^2 + v_\phi^2 - w^2 =$ const; hence, even for $v_r = 0$, the $w(r)$ terms remain active and do not cancel.

4. Boundary and disk results. At the smooth bulge–disk boundary $r = R$, impose the virial condition

$$v_\phi^2(R) = v_{\rm orb}^2(R) = \tfrac{1}{2}\Big(\sqrt{\tfrac{2GM}{R}} - H_z R\Big)^2.$$

Hence

$$C = v_\phi^2(R) + \Big(\sqrt{\tfrac{2GM}{R}} - H_z R\Big)^2 = \tfrac{3}{2}\Big(\sqrt{\tfrac{2GM}{R}} - H_z R\Big)^2.$$

For the thin, nearly steady disk ($v_r \simeq 0$, so matter that is present is effectively in free fall with zero velocity relative to the velocity field),

$$v_{\rm rad,eff}(r) = \sqrt{\tfrac{2GM}{r}} - H_z r,$$

and the observable orbital law becomes

$$\boxed{v_\phi^2(r) = \tfrac{3}{2}\Big(\sqrt{\tfrac{2GM}{R}} - H_z R\Big)^2 - \Big(\sqrt{\tfrac{2GM}{r}} - H_z r\Big)^2.}$$

The effective radial velocity was

$$\boxed{v_{\rm rad,eff}^2(r) = \Big(\sqrt{\tfrac{2GM}{r}} - H_z r\Big)^2},$$

and at the turn–around radius we have

$$v_{\rm rad,eff}(r_c) = 0 \;\Rightarrow\; r_c = \Big(\tfrac{2GM}{H_z^2}\Big)^{1/3}, \qquad v_\phi^2(r_c) = \tfrac{3}{2}\Big(\sqrt{\tfrac{2GM}{R}} - H_z R\Big)^2.$$

Summary. All steps remain internal to the BQ algebra: the rotor Q_g encodes the rapidities ψ_i that generate the flow; the adjoint basis $\not{G}_\mu$ introduces the shift $w(r)$; the BNC provides the conserved scalar $\mathcal{B} = \gamma(1 - w v_r)$; and its low–velocity limit produces the Bernoulli invariant $v_\phi^2 + (v_r - w)^2 = \text{const.}$ used in the disk fits.

5.3 Boundary condition: effective velocities *inside* a smooth bulge

In this part we derive the boundary conditons for an effective Newtonian-Hubble potential. Assume a regular (smooth) bulge with spherical symmetry and an interior Newtonian potential[2]

$$\Phi_{\rm in}(r) = -\frac{GM}{2R}\left(3 - \frac{r^2}{R^2}\right), \qquad 0 \le r \le R.$$

Then

$$-\partial_r \Phi_{\rm in} = \frac{GM}{R^3}\, r, \quad v_c^2(r) = r\,(-\partial_r \Phi_{\rm in}) = \frac{GM}{R^3}\, r^2.$$

At the boundary $r = R$, $v_c^2(R) = GM/R$ and the *exterior* free–fall term is $\sqrt{2GM/R}$. To include the horizon background H_z we work with the subtracted speeds $\sqrt{2GM/R} - H_z R$ (at $r = R$) and $\sqrt{2GM/r} - H_z r$ (general r).

Orbital term (inside). Because $v_c^2(r) \propto r^2$, write it with the boundary value as

$$\boxed{v_{\rm orb}^2(r) = \frac{1}{2}\left(\sqrt{\frac{2GM}{R}} - H_z R\right)^2 \frac{r^2}{R^2}.}$$

For $H_z = 0$ this reduces to $v_{\rm orb}^2 = GM\, r^2/R^3$ as expected.

[2]Uniform density gives this exactly; any smooth profile with a harmonic interior has the same leading form.

Radial effective term (inside). The interior free–fall speed referenced to infinity is

$$v_{\mathrm{ff,in}}^2(r) = 2|\Phi_{\mathrm{in}}(r)| = \frac{GM}{R}\left(3 - \frac{r^2}{R^2}\right).$$

Subtracting the Hubble background gives

$$\boxed{v_{\mathrm{rad,eff}}^2(r, H_z) = \left(\sqrt{\frac{GM}{R}\left(3 - \frac{r^2}{R^2}\right)} - H_z r\right)^2.}$$

At $r = R$ this becomes $(\sqrt{2GM/R} - H_z R)^2$, matching the exterior form.

Combined (projected) effective speed. The helical (Bernoulli) projection inside adds the tangential part with the same geometric factor that yields 3/2 outside; inside this gives

$$\boxed{v_L^2(r) = \frac{1}{2}\left(\sqrt{\frac{2GM}{R}} - H_z R\right)^2 \frac{r^2}{R^2} + \left(\sqrt{\frac{GM}{R}\left(3 - \frac{r^2}{R^2}\right)} - H_z r\right)^2.}$$

Evaluated at $r = R$ this reproduces $v_L^2(R) = \frac{3}{2}\left(\sqrt{2GM/R} - H_z R\right)^2$, ensuring smooth matching to the exterior plateau.

Checks.

- Continuity at $r = R$: all three expressions reduce to the corresponding exterior forms with $r \mapsto R$.
- Newtonian limit $H_z \to 0$: $v_{\mathrm{orb}}^2 = GM\, r^2/R^3$, $v_{\mathrm{rad,eff}}^2 = GM/R\,(3 - r^2/R^2)$, and $v_L^2 = GM\, r^2/R^3 + GM/R\,(3 - r^2/R^2)$.

Interpretation. A smooth bulge has a harmonic interior potential; circular speed grows linearly with r while the interior free–fall speed follows $v_{\mathrm{ff,in}}^2 \propto (3 - r^2/R^2)$. Including the background flow by the subtractions $(\cdot - H_z r)$ and projecting the helical (inflow+azimuthal) motion yields the formulas above, which match continuously to the exterior CL_H relations at $r = R$.

6 MOND–type acceleration laws derived from the CL_H orbital velocity

6.1 Background: origin of MOND and its covariant problem

Modified Newtonian Dynamics (MOND) was introduced by Milgrom [10, 11, 12] to explain the flat rotation curves of spiral galaxies without invoking dark matter. In its original, nonrelativistic form the dynamical law

$$a\,\mu\!\left(\frac{a}{a_0}\right) = a_N = \frac{GM}{r^2}$$

introduced a new constant acceleration scale $a_0 \sim 10^{-10}\,\mathrm{m\,s^{-2}}$ and an interpolation function $\mu(x)$ connecting the Newtonian regime ($\mu \to 1$) to the "deep–MOND" limit ($\mu \approx x$). Although phenomenologically successful [15, 18], this construction was not relativistic and could not be embedded consistently in a cosmological context.

Relativistic and covariant extensions were later developed, most notably the Tensor–Vector–Scalar (TeVeS) theory [14] and its descendants [20, 21]. These frameworks aimed to reproduce the

MOND limit while providing a metric description compatible with cosmology and gravitational lensing. A remaining challenge has been to obtain the MOND regimes *and* their transitions directly from a geometric flow or covariant Lagrangian without introducing auxiliary fields or interpolation functions by hand.

The CL_H formalism derived in this paper offers such a route: it provides a single kinematic law—based on a constant–Lagrangian condition with a cosmological (Hubble) boundary—that reduces to MOND–type behaviour in the appropriate acceleration range while remaining fully covariant and explicitly cosmological through its dependence on the Hubble parameter H.

6.2 Derivation of MOND–type acceleration laws

Starting point ($\mathbf{CL}_H$ kinematics). In the constant–Lagrangian with Hubble boundary (CL_H) configuration the Painlevé–Gullstrand (PG) river picture gives the effective radial flow

$$v_r^{\mathrm{eff}}(r) = \sqrt{\frac{2GM}{r}} - H\,r,$$

and the disk azimuthal speed (for the matter–free baseline) follows from the CL closure as

$$v_\phi^2(r) = \tfrac{3}{2}\,v_r^{\mathrm{eff}\,2}(R) - v_r^{\mathrm{eff}\,2}(r),$$

where the bulge rim $r = R$ fixes the integration constant through the virial–continuity condition.

From velocity to acceleration. The centripetal acceleration observed in the disk is

$$a(r) = \frac{v_\phi^2(r)}{r}.$$

Expanding $v_r^{\mathrm{eff}\,2}(r)$ gives

$$v_r^{\mathrm{eff}\,2}(r) = \frac{2GM}{r} - 2H\sqrt{2GM\,r} + H^2 r^2,$$

so that

$$a(r) = \underbrace{\frac{GM}{r^2}}_{\text{Newtonian}} - \underbrace{2H\sqrt{\frac{2GM}{r}}\frac{1}{r}}_{\text{transition (MOND-like)}} + \underbrace{H^2 r}_{\text{cosmic tail}} \quad (+\text{ constant from } R). \tag{20}$$

Up to the rim constant, this already has the canonical MOND structure: a Newtonian term, a geometric–mean transition term, and a large–radius cosmological tail reminiscent of relativistic MOND extensions [19, 13].

MOND notation. Introduce MOND's characteristic acceleration a_0 by $H = a_0/c$ (as first suggested in [24, 25]). Then Eq. (20) reads

$$a(r) = \frac{GM}{r^2} - \frac{2a_0}{c}\sqrt{\frac{2GM}{r}}\frac{1}{r} + \frac{a_0^2}{c^2}\,r. \tag{21}$$

This single analytic expression spans the Newtonian, MOND–transition, and dark–energy (DE) tail regimes without any empirical interpolation function. It represents the acceleration–level projection of the first–order CL_H velocity invariant.

Exact MOND–form identities (μ/ν language). Defining $a_N(r) = GM/r^2$, one obtains directly from the CL_H law:

$$a(r) = \frac{1}{r}\Big[\tfrac{3}{2}A(R) - A(r)\Big], \qquad A(x) = \frac{2GM}{x} - 2H\sqrt{2GM}\,x + H^2x^2, \tag{22}$$

$$a(r) = \nu_{\mathrm{CLH}}(r)\,a_N(r), \qquad \nu_{\mathrm{CLH}}(r) = \frac{r}{GM}\Big[\tfrac{3}{2}A(R) - A(r)\Big], \tag{23}$$

$$\mu_{\mathrm{CLH}}(r) = \frac{a_N}{a} = \big[\nu_{\mathrm{CLH}}(r)\big]^{-1}. \tag{24}$$

These relations are exact and parameter–free; they play the same role as MOND's empirical interpolation functions [22, 17] but arise from the geometric CL_H flow itself.

Asymptotic regimes.

- **Inner (Newtonian):** for $r \ll R$ and $Hr \ll \sqrt{2GM/r}$, $a \simeq a_N$, reproducing classical gravity.
- **Transition / deep–MOND:** the second term dominates; balancing the first two terms gives $a \sim \sqrt{a_0\,a_N}$, yielding the baryonic Tully–Fisher relation $v_{\text{flat}}^4 \propto G\,a_0\,M$ [16, 18].
- **Outer / DE–tail:** for $r \gtrsim r_c = (2GM/H^2)^{1/3}$, $a \simeq H^2 r = (a_0^2/c^2)r$, consistent with the de–Sitter background and relativistic MOND limits [21].

External–field and environment. An ambient gravitational field shifts the effective boundary rate to $H_{\text{eff}} = H + g_{\text{ext}}/c$, so all instances of H in the formulas above are replaced by H_{eff}. This gives a direct, quantitative expression for MOND's external–field effect (EFE) [12, 23] without introducing an extra μ–function.

Interpretation. Equation (21) shows that the three empirical MOND regimes arise naturally as the three analytic components of a single Bernoulli–Noether (CL_H) flow: Newtonian curvature (GM/r^2), the inflow–expansion coupling ($\propto H\sqrt{GM/r}/r$), and the de–Sitter tail (H^2r). In the Q_g/BNC framework, the MOND acceleration constant is not an additional parameter but the cosmological boundary $a_0 = cH$. The observed MOND phenomenology thus appears as the *acceleration–level projection* of the covariant CL_H invariant.

6.3 Interpretations of the baryonic mass term M

1. Classical (Newtonian) interpretation. In the standard Newtonian picture and in the original formulation of Modified Newtonian Dynamics (MOND) [10, 11, 15], the parameter M appearing in the acceleration law $a_N = GM/r^2$ is simply the *baryonic mass* of the galaxy: stars, gas, and any other luminous matter contained within the radius r. This is the mass directly inferred from photometry and gas content. Under this interpretation $M = M_{\text{bar}}(r)$ is an input function fixed by observations, and the apparent "missing mass" arises solely from the modified dynamics or geometry.

2. Effective (metric–flow) interpretation. In the CL_H framework the same symbol M also appears in the geometric flow term

$$w_r(r) = \sqrt{\frac{2GM}{r}} - H\,r,$$

which defines the local gravitational inflow through the tetrad component ${E_0}^r = w_r$. Here M plays the role of an *effective source parameter* that sets the strength of the local space–time flow. Because the CL_H system is self–consistent and covariant, this M represents not just the static baryonic content but the dynamically coupled mass that determines the magnitude of the Dirac–Q_g rotor field. In this sense, M is the mass parameter of the metric, rather than a mere baryonic input.

3. Dynamical or enclosed–mass interpretation. At the level of the observable rotation curves, the combination GM/r^2 entering the acceleration law (21) can be rewritten as

$$\frac{GM_{\text{eff}}(r)}{r^2} = \frac{GM}{r^2} - \frac{2H\sqrt{2GM\,r}}{r} + \frac{H^2r^2}{r^2}\,r,$$

so that the total dynamical or "enclosed" mass seen by the rotation curve is

$$M_{\text{eff}}(r) = M + \frac{r^{3/2}}{\sqrt{2G}}\Big(2H\sqrt{M} - \frac{H^2r^{3/2}}{\sqrt{2G}}\Big).$$

This $M_{\text{eff}}(r)$ grows with radius, mimicking the mass discrepancy normally attributed to dark matter in the outer disk. The apparent "dark" component thus arises as an *emergent metric mass* caused by the coupling of the local inflow to the cosmological term H, rather than by an additional particle component.

4. Relation to MOND and scaling laws. In the MOND interpretation the empirical relation $v_{\text{flat}}^4 \simeq G\,a_0\,M_{\text{bar}}$ McGaugh et. al (2005, 2011) [16, 18] uses M_{bar} as the true baryonic mass, whereas in the CL_H expression the same scaling follows naturally from the geometric cross term $-2H\sqrt{2GM/r}/r$ once $a_0 = cH$ is identified. Therefore the CL_H framework distinguishes explicitly between:

- the *luminous baryonic mass* M_{bar},
- the *metric source parameter* M controlling the local inflow,
- and the *effective dynamical mass* $M_{\text{eff}}(r)$ emerging from the coupling between the baryonic term and the Hubble field.

In practice, M_{bar} and M coincide in the inner (Newtonian) region, while $M_{\text{eff}}(r) > M$ in the outer disk, reproducing the same mass–discrepancy behaviour that MOND attributes to modified inertia.

5. Interpretation within the $\mathbf{Q}_g$ field. From the Dirac–Q_g perspective, M is encoded algebraically in the scalar amplitude of the gravitational spinor field and couples directly to the curvature through the rotor rapidity ψ_H. The apparent mass discrepancy is thus a geometric effect: the gravitational rotor links the local baryonic source to the global cosmological flow. In this sense, the parameter M in the CL_H law acts simultaneously as a baryonic constant, a curvature parameter, and a source of the metric flow.

6.4 Covariance and cosmological embedding of MOND within the $\mathbf{CL}_H$ and $\mathbf{Q}_g$ framework

1. Revisiting MOND's original aims. Since its introduction by Milgrom in 1983 [10, 11, 12], the MOND programme has pursued two principal theoretical goals: (a) to obtain a fully *covariant* formulation compatible with relativistic field theory and gravitational lensing, and (b) to achieve a consistent *cosmological embedding* in which the characteristic acceleration scale a_0 arises naturally from the large–scale structure of space–time. Despite notable progress through TeVeS [14] and related tensor–vector–scalar approaches [21], these theories still relied on additional fields or modified actions and did not link the MOND constant a_0 directly to the cosmic expansion rate H.

2. Covariance through the Q_g adjoint structure. In the present work, covariance is achieved at the algebraic level rather than through an extended field content. The gravitational rotor Q_g acts in the adjoint of the Dirac–BQ algebra, producing the local tetrad and metric through $\not{G}_\mu = Q_g \beta_\mu Q_g^{-1}$. All dynamical relations, including the Bernoulli–Noether invariant and the CL_H constant–Lagrangian condition, are therefore written directly in covariant form. No nonrelativistic limit or "modified inertia" assumption is required: the apparent MOND behaviour emerges from the geometry of the covariant flow itself.

3. Natural cosmological embedding. The cosmological boundary enters through the Hubble parameter H, which appears intrinsically in the PG shift field $w_r(r) = \sqrt{2GM/r} - H\,r$ and therefore in the acceleration law (21). Identifying $a_0 = cH$ connects the local MOND transition directly to the cosmic expansion, providing the embedding that TeVeS and other relativistic extensions sought but had to introduce phenomenologically [24, 25]. The MOND scale thus ceases to be a new constant of nature and becomes the boundary value of the covariant flow.

4. Beyond the Newtonian paradigm. While MOND historically attempted to remain within a "modified Newtonian" framework—adjusting the force or inertia laws of point masses—the CL_H system belongs to a different paradigm. Here the dynamics of galaxies arise not from altered forces acting on test particles, but from the structure of a self–consistent *matter–metric flow*. The same rotor Q_g that defines the Dirac geometry also generates the effective metric responsible for galactic kinematics. In this view, MOND–like behaviour is not a correction to Newtonian gravity but an emergent property of the covariant space–time flow.

5. Implications. This shift of viewpoint resolves the two persistent issues in the MOND programme:

(i) The theory is covariant by construction, since the underlying algebra is Lorentz– and gauge–covariant at each step.

(ii) The cosmological scale $a_0 = cH$ appears automatically, making the connection between galaxy dynamics and cosmic expansion explicit.

At the same time, the framework abandons the expectation that modified Newtonian mechanics must underlie the observed phenomenology. Instead, the observed MOND regimes arise as different projections of a single covariant rotor field Q_g whose dynamics include both local curvature and global expansion.

6. Conclusion. In summary, the present formalism supplies MOND with the two theoretical elements it lacked: explicit covariance and a built–in cosmological boundary. The price is a departure from the Newtonian paradigm; the reward is a unified algebraic and geometric language in which the MOND acceleration scale, the flat rotation curves, and the de–Sitter asymptote all emerge from one covariant flow. This suggests that the MOND phenomenology may not be a modification of Newtonian dynamics at all, but rather a low–velocity manifestation of the covariant Q_g field and its constant–Lagrangian (CL_H) dynamics. Because, to the extent that the Q_g construction proves correct and experimentally verifiable, it demonstrates that phenomenologically the MOND approach was right all along—the observed regularities indeed reflect an underlying gravitational self-consistency of the cosmos.

7 Conclusion

The present study began by establishing the translation procedure from the biquaternionic Q_g formalism to the familiar general–relativistic metric representation. Through the adjoint operation $\not{G}_\mu = Q_g\,\beta_\mu\,Q_g^{-1}$ and its decomposition $\not{G}_\mu = e_\mu{}^a\hat{\beta}_a$, the standard geometric quantities $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ and the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ can be recovered directly from the algebraic basis. This provides an explicit algorithm connecting the algebraic rotor description to the metric structure usually employed in general relativity.

Through the course of the paper it has been shown that the same physical results—including the definition of the metric, the construction of the tetrad operators, the Bernoulli–Noether invariant, and the constant–Lagrangian (CL_H) dynamics leading to MOND–type acceleration laws—can be derived entirely within the linear Dirac–BQ algebra itself. In this algebraic domain, gravity is expressed as a rotor–induced transformation of the Dirac basis, and the metric arises as the symmetric scalar part of the biquaternionic products of these rotated elements. The resulting framework therefore performs gravitational analysis without explicit recourse to the tensor calculus of GR, remaining covariant and internally consistent at the algebraic level.

At the same time, general relativity has provided a remarkably successful phenomenological framework for over a century, across domains ranging from astrophysics to cosmology. Nothing in the present approach contradicts its empirical results within the range where they have been verified. Rather, the Q_g formalism offers an alternative and possibly deeper representation of the same phenomena—one that may clarify why GR works so well in its tested regimes and how it might extend or connect to new domains such as galactic dynamics and cosmology.

The analysis presented here thus shows that the gravitational and MOND–related regimes considered can indeed be constructed and interpreted within the Q_g algebraic domain itself, while recognising that general relativity remains the established geometric language for interpreting observations in curved space–time. The two approaches are not in conflict but complementary: the Q_g formulation exposes the covariant algebraic roots of gravitational flow, and GR provides its historically and empirically grounded geometric expression.

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Implementation recipe for the Q_g algorithm (CL_H branch)

This summary outlines the minimal executable sequence for reproducing the constant–Lagrangian (CL_H) and Bernoulli–Noether (BNC) results.

Step 1: Construct the gravitational rotor Q_g.
Use the local rapidities $(\psi_r, \psi_\phi, \psi_H)$ for radial inflow, azimuthal swirl, and horizon expansion:

$$Q_g = \exp\Big[\tfrac{1}{2}(\psi_r\,\hat{\beta}_r\hat{\beta}_0 + \psi_\phi\,\hat{\beta}_\phi\hat{\beta}_0 + \psi_H\,\hat{\beta}_H\hat{\beta}_0)\Big].$$

Step 2: Extract the tetrad $E = [e_\mu{}^a]$ and the metric.
From the adjoint $\not{G}_\mu = Q_g\,\beta_\mu\,Q_g^{-1} = e_\mu{}^a\hat{\beta}_a$ read off the PG-like shift (river) components:

$$w_r(r) = \sqrt{\tfrac{2GM}{r}} - H_z r, \qquad w_\phi(r,\theta)\ \text{(optional spiral/frame–drag)}.$$

Step 3: Fix the BNC constant K at the bulge rim $r = R$.
Apply the virial and continuity conditions:

$$v_\phi^{\text{mat}}(R) = v_{\text{orb}}(R) = \tfrac{1}{2}^{1/2}\left(\sqrt{\tfrac{2GM}{R}} - H_z R\right), \qquad v_r^{\text{mat}}(R) \simeq 0,$$

leading to

$$K = \tfrac{3}{2}\left(\sqrt{\tfrac{2GM}{R}} - H_z R\right)^2.$$

Step 4: Propagate along the disk (Bernoulli–Noether invariant).
The conserved relative combination is

$$(v_r^{\text{mat}} - w_r)^2 + (v_\phi^{\text{mat}} - w_\phi)^2 - w^2 = K.$$

For a thin steady disk set $v_r^{\text{mat}} \approx 0$ *after* the constant is fixed:

$$v_\phi^{\text{mat}}(r) = \sqrt{K - w_r^2(r)}, \qquad v_{\text{rad,eff}}(r) = -w_r(r).$$

Step 5: Predict observable rotation curves.
The observable orbital velocity (PG/tetrad frame) is

$$v_{\text{orb}}(r) = \big|\,w_\phi(r) + v_\phi^{\text{mat}}(r)\,\big|.$$

With $w_\phi \approx 0$ this reduces to

$$v_{\text{orb}}^2(r) = \tfrac{3}{2}\left(\sqrt{\tfrac{2GM}{R}} - H_z R\right)^2 - \left(\sqrt{\tfrac{2GM}{r}} - H_z r\right)^2.$$

Step 6: Outer edge and plateau.
The turn–around radius follows from $v_{\text{rad,eff}}(r_c) = 0$:

$$r_c = \left(\tfrac{2GM}{H_z^2}\right)^{1/3}, \qquad v_{\text{orb}}^2(r_c) = K.$$