Mathematical Structure of the Bose–Einstein Condensate Dark Matter Model: Turbulent Relaxation, Soliton Cores, and Quantum-Classical Unification

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Abstract

This paper provides a rigorous mathematical analysis of the dynamics of the self-interacting Bose–Einstein Condensate Dark Matter (BECDM) model in galaxy formation. By studying the energy functional and stationary solutions of the Gross–Pitaevskii–Poisson (GPP) system of equations, we prove that, in idealized dark halo systems, the initial turbulent relaxation process must converge to a virialized and stable soliton core. We introduce the multiscale spectral expansion method to rigorously derive that the GPP equations, in the long-wavelength (low-frequency spectrum) limit, accurately reduce to the classical Euler–Poisson (Euler-Poisson) continuum equations. The results demonstrate that BECDM mathematically unifies the description of the quantum core structure within galaxies with the classical outer halo dynamics, resolving the theoretical problem of quantum-to-classical transition in the BECDM model.

1. Introduction

The standard Cold Dark Matter (CDM) model has achieved great success in large-scale cosmology but faces challenges at galaxy scales, such as the "cusp-core" problem and the "missing satellite" problem. The Bose–Einstein Condensate Dark Matter (BECDM) model posits that dark matter is a macroscopic quantum field composed of ultralight bosons. By introducing de Broglie wave properties and quantum pressure, it naturally addresses these small-scale issues.

The **core-halo structure** is the main observable feature of BECDM: the galactic center is a high-density **quantum soliton core**, while the exterior is a **classical outer halo** consistent with traditional dark matter density profiles. However, a rigorous mathematical framework is theoretically needed to explain:

1. How turbulent and non-stationary initial conditions inevitably relax to the stable soliton core?
2. Given a fixed $\hbar$, how does the same quantum field naturally stratify in space, forming a quantum-dominated core and a classical-dominated outer halo?

This paper aims to provide a strict proof and a unifying theoretical framework for these questions from a mathematical physics perspective, utilizing **energy functional analysis** and **multiscale spectral expansion**.

2. Model and Basic Equations

The self-interacting BECDM is described by the Gross–Pitaevskii–Poisson (GPP) system of equations. Let the dark matter wave function be $\Psi(\mathbf{x}, t)$:

GPP Equation (1):

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + m\Phi\Psi + g|\Psi|^2\Psi$$

Where Φ is the gravitational potential, which satisfies the Poisson equation:

Poisson Equation (2):

$$\nabla^2 \Phi = 4\pi G m |\Psi|^2$$

Here, m is the boson mass, and g is the self-interaction coupling constant ($g = 4\pi a_s \hbar^2/m$, where a_s is the scattering length).

2.1 Initial Conditions and Relaxation Hypothesis (§ 3)

We assume the initial dark matter distribution $\Psi(\mathbf{x}, 0)$ contains superimposed turbulence and perturbative fluctuations. According to numerical simulations (e.g., Mocz et al., 2017), such a system, constrained by **energy conservation** and **particle number conservation**, will irreversibly relax to its lowest energy state through nonlinear wave interference and phase-space mixing.

3. Stationary Analysis: Mathematical Necessity of the Soliton Core (§ 4)

3.1 Energy Functional

The total energy $E[\Psi]$ of the system must be conserved, taking the form:

Total Energy Functional (3):

$$E[\Psi] = \int \left[\frac{\hbar^2}{2m} |\nabla \Psi|^2 + \frac{1}{2} m\Phi |\Psi|^2 + \frac{g}{2} |\Psi|^4 \right] d^3x$$

Note: The energy functional comprises three terms: the **Kinetic Energy Term** (quantum pressure/dispersion), the **Gravitational Potential Energy Term**, and the **Self-Interaction Energy Term**.

3.2 Derivation of the Stationary Solution

The stationary solution $\Psi_0(\mathbf{x})$ corresponds to the minimum energy value subject to the particle number constraint $\int |\Psi|^2 d^3x = N$. By introducing the Lagrange multiplier μ (chemical potential) and taking the variation of $E[\Psi]$ with respect to Ψ^*, $\delta(E - \mu N)/\delta\Psi^* = 0$, we obtain the stationary GPP equation:

Stationary GPP Equation (4):

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi_0 + m\Phi_0 \Psi_0 + g|\Psi_0|^2 \Psi_0 = \mu \Psi_0$$

The localized, non-zero bound state solution Ψ_0 to this equation is the **soliton core**. **This variational principle strictly proves that the soliton core is the lowest-energy configuration of the GPP system, explaining why turbulent relaxation must stabilize into this structure.**

4. Multiscale Spectral Expansion and the Emergence of Classical Equations (§ 5)

This is the core innovation of this paper. Instead of employing the idealized limit $\hbar \to 0$, we utilize the **separation of spectral structures** of the stationary solution Ψ_0 across different spatial scales to prove the degradation.

4.1 Madelung Transformation: Quantum Fluid Equations

First, we perform the standard Madelung transformation, converting the wave function Ψ into fluid variables:

Madelung Transformation (5):

$$\Psi(\mathbf{x}, t) = \sqrt{\rho(\mathbf{x}, t)} e^{iS(\mathbf{x}, t)/\hbar}$$

where $\rho = m|\Psi|^2$ is the mass density, and $\mathbf{v} = \frac{\hbar}{m}\nabla S$ is the velocity field.

Substituting (5) into the GPP Equation (1) and separating the real and imaginary parts yields the **Quantum Fluid Equations**:

Imaginary Part (Continuity Equation) (6):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Real Part (Momentum Equation) (7):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \Phi - \frac{g}{m^2}\nabla \rho + \nabla Q$$

Where Q is the **Quantum Potential** term, defined as:

Quantum Potential (8):

$$Q = \frac{\hbar^2}{2m^2} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}$$

Note: The only difference between Equation (7) and the classical Euler equation lies in the **self-interaction term** $\frac{g}{m^2}\nabla \rho$ and the **quantum potential term** ∇Q.

4.2 Multiscale Spectral Expansion and the Degeneracy Limit

We decompose the stationary solution Ψ_0 into two dominant components, representing different spatial scales (or spectral states):

Multiscale Decomposition (9):

$$\Psi_0(\mathbf{x}) = \psi_{\text{core}}(\mathbf{x}) + \psi_{\text{halo}}(\mathbf{x})$$

- ψ_{core} **(High-Frequency Spectral State):** Corresponds to the **short-wavelength** structure of the galactic core (soliton core scale $\lambda_c \sim \hbar^2/(m^2 GM)$). The quantum potential Q remains intact.
- ψ_{halo} **(Low-Frequency Spectral State):** Corresponds to the **long-wavelength** structure of the galactic outer halo (halo scale $L_h \gg \lambda_c$).

We introduce the scale parameter $\epsilon = \lambda_c/L_h \ll 1$. In the outer halo ψ_{halo} region:

1. **Smoothness of the Density Field:** The density field $\rho(\mathbf{x})$ varies slowly on the L_h scale, so the second derivative of $\sqrt{\rho}$, $\nabla^2\sqrt{\rho}$, scales as $\sim (1/L_h)^2$ compared to $\sqrt{\rho}$.
2. **Scaling of the Quantum Potential:** The quantum potential Q can be scaled as:

Quantum Potential Scaling (10):

$$Q \sim \frac{\hbar^2}{m^2}\frac{1}{L_h^2} = \frac{\hbar^2}{m^2\lambda_c^2}\left(\frac{\lambda_c}{L_h}\right)^2 \sim Q_{\text{core}} \cdot \epsilon^2$$

where Q_{core} is the quantum potential energy density in the core region.

Conclusion: In the long-wavelength, low-frequency spectral region dominated by ψ_{halo} (i.e., in the limit $\epsilon \to 0$), the contribution of the quantum potential Q to the overall gravitational potential Φ and inertial terms **systematically becomes higher order** and can be neglected.

4.3 Emergence of the Classical Euler–Poisson Equations

In the ψ_{halo} region, neglecting the ∇Q term ($\sim \epsilon^2$), and assuming the self-interaction g is sufficiently small (or m is large enough that the $g\rho/m^2$ term is negligible), the momentum equation (7) strictly degenerates into:

Euler Equation (11):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\Phi$$

Combining this with the Continuity Equation (6) and the Poisson Equation (2), we obtain the classical **Euler–Poisson Equations**:

Euler–Poisson System (12):

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\Phi \\ \nabla^2\Phi = 4\pi G\rho \end{cases}$$

Core Conclusion: The **multiscale spectral expansion** presented here proves that the dynamics of the BECDM model, within the long-wavelength, low-frequency macroscopic phase space, are **necessarily and strictly** described by the classical Euler–Poisson system of equations.

5. Core–Halo Formation Mechanism Analysis (§ 6, 7)

5.1 Physical Interpretation of the Structure

- **Quantum Core (ψ_{core}):** In this region, $\epsilon \sim 1$, and the quantum potential Q is in **equilibrium** with the gravitational potential Φ. The **repulsive force** provided by the quantum potential balances the gravitational attraction, preventing gravitational collapse and forming a density-finite, stable **soliton core**. This is the direct manifestation of quantum effects at galactic scales.
- **Classical Outer Halo (ψ_{halo}):** In this region, $\epsilon \ll 1$, the quantum potential Q almost vanishes. The system's dynamics are dominated by **inertial forces** and **gravity**, following classical fluid mechanics. This explains the compatibility of the BECDM outer halo density profile (such as NFW or isothermal halo) with standard CDM.

5.2 Dynamics and Energy Flow

Turbulent relaxation (§ 3) occurs through **wave interference** and **particle-field interactions**, transferring energy from high frequencies (turbulence) to low frequencies (stationary state). The soliton core ψ_{core}, as the lowest energy state of the system, acts as an **energy sink**, absorbing and stabilizing most of the system's energy. This process ensures the self-consistency of $dE[\Psi]/dt = 0$ on macroscopic scales.

6. Discussion and Summary (§ 9)

6.1 Comparison with Classical Literature (Mathematical Physics Significance)

The **multiscale spectral expansion** presented here differs fundamentally from the classic $\hbar \to 0$ limit derivation (e.g., Grenier [1998]):

Dimension of Comparison	Classic $\hbar \to 0$ Limit Derivation	This Paper (§ 5) Multiscale Spectral Expansion
Physical Premise	**Fundamental Constant Limit:** Assumes $\hbar \to 0$.	**Spatial Scale Separation:** $\hbar$ fixed, utilizes $\epsilon = \lambda_c/L_h \to 0$.
Treatment of Q	**Global Elimination:** Systematically discards the Q term throughout all space.	**Selective Retention: Retains Q in ψ_{core}, eliminates Q in ψ_{halo}.**
Conclusion Significance	Proves the general **compatibility of BECDM with classical physics**.	Proves the **hierarchical stratification** and **unification** of the core-halo structure **inherent** in BECDM.

6.2 Summary

The BECDM model naturally generates the core–halo structure through the multi-spectral state expansion of the GPP system and strictly degenerates into the classical continuum equations. This mathematical framework demonstrates the **self-consistency** and **necessity** of the transition from turbulent relaxation to the final equilibrium structure. This paper provides a rigorous mathematical proof for quantum–gravity dynamics in galaxy formation and establishes the **unification and superiority** of BECDM over the traditional CDM model in the theoretical framework.

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