On the coprimality of consecutive odd members in Collatz Sequences and its implication for cycle formation

Dobri Bozhilov

zadrugata2019@gmail.com

Sofia, Bulgaria

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Abstract

The Collatz conjecture posits that for any positive integer n, the sequence generated by the rules n -> n/2 if n is even and n -> 3n + 1 if n is odd always reaches the cycle 4 -> 2 -> 1. This paper presents a novel observation: consecutive odd numbers in a Collatz sequence are coprime, i.e., their greatest common divisor is 1. We prove this property using basic arithmetic and demonstrate that it imposes a significant constraint on the sequence dynamics, particularly for large numbers. Specifically, the requirement of coprimality creates a "pressure" for the next odd number to be smaller than its predecessor, as larger numbers increase the likelihood of common divisors. This tendency toward decrease makes the existence of alternative cycles (distinct from 4 -> 2 -> 1) increasingly improbable, especially for cycles involving extremely large numbers (e.g., exceeding 2^68). Our result supports the conjecture by suggesting that the arithmetic structure of Collatz sequences favors convergence to 1 over the formation of divergent or cyclic sequences.

1. Introduction

The Collatz conjecture, also known as the 3n + 1 problem, is a well-known unsolved problem in number theory. It asserts that for any positive integer n, the sequence defined by n -> n/2 if n is even and n -> 3n + 1 if n is odd will eventually reach the cycle 4 -> 2 -> 1. Despite extensive computational verification up to numbers as large as 2^70, a general proof remains elusive. Research has focused on properties such as the structure of the Collatz graph, modular constraints, and the possibility of alternative cycles or divergent sequences.

In this paper, we introduce a new arithmetic property: consecutive odd numbers in a Collatz sequence are coprime. We prove this property and explore its consequence: at large scales, the requirement of coprimality creates a tendency for the next odd number to be smaller than its predecessor, making the formation of large alternative cycles highly unlikely. This observation strengthens the case for the Collatz conjecture by limiting the possibility of counterexamples.

2. Coprimality of Consecutive Odd Numbers

Theorem:

Let a and b be consecutive odd numbers in a Collatz sequence, where b = (3a + 1)/2^k and k is the largest integer such that 3a + 1 is divisible by 2^k. Then, gcd(a, b) = 1.

Proof:

Suppose **a** is an odd number in a Collatz sequence. By the Collatz rules, the next number is **c** = $3a + 1$, which is even (since $3a$ is odd). We divide c by 2^k, where k is the largest integer such that $(3a + 1)/2^k$ is an integer and odd, yielding the next odd number **b**:

$$b = (3a + 1)/2^k.$$

Thus, $3a + 1 = 2^k * b$. Assume gcd(a, b) = **d** > 1, so **d** divides both **a** and **b**. Since **b** divides $3a + 1$, **d** also divides $3a + 1$. Let **a** = **d** * m. Then:

$$3a + 1 = 3(d * m) + 1 = 3dm + 1.$$

If d divides $3a + 1$, we have:

$$3dm + 1 \equiv 0 \pmod{d} \Rightarrow 3dm \equiv -1 \pmod{d}.$$

This implies that $-1 \equiv 0 \pmod{d}$, so d must divide 1, which is only possible if d = 1. Thus, gcd(a, b) = 1, proving that consecutive odd numbers are coprime.

3. Consequence: Pressure for decrease in large numbers

The coprimality of consecutive odd numbers imposes a significant constraint on Collatz sequences, particularly for large numbers. Consider a large odd number a. The next odd number b is given by:

$$b = (3a + 1)/2^k,$$

where b is odd and gcd(a, b) = 1. For large a, the number $3a + 1$ is also large, and the requirement that b be coprime with a limits the possible values of b. If k is small (e.g., k = 1), then:

$$b = (3a + 1)/2 \approx 1.5a,$$

which is larger than a. However, large numbers have more potential divisors, increasing the likelihood that b shares a common factor with a, violating the coprimality condition. To satisfy gcd(a, b) = 1, a larger k is often required, reducing b significantly:

$$b = (3a + 1)/2^k < a \text{ if } 2^k > 3 + 1/a.$$

For large a, this condition is met with moderate k, as 1/a is small. Thus, the coprimality requirement creates a "pressure" for b to be smaller than a, driving the sequence toward lower values.

4. Implication for Alternative Cycles

The Collatz conjecture could be falsified by the existence of an alternative cycle (distinct from 4 -> 2 -> 1) or a divergent sequence. Previous results suggest that any alternative cycle must involve numbers exceeding 2^{68} and have a length of at least 114 billion steps. The coprimality property exacerbates the difficulty of forming such a cycle:

- In a cycle, the sequence must return to its starting point, requiring a balance between increases ($3n + 1$) and decreases ($n/2$). However, the tendency for large odd numbers to produce smaller subsequent odd numbers (due to coprimality) makes it challenging to maintain the large values required for a cycle.

- For a cycle with quintillions of members, the sequence must include billions of odd numbers, all pairwise coprime with their neighbors. The pressure for smaller odd numbers disrupts this balance, as the sequence is likely to fall below the cycle's minimum, connecting to the 4 -> 2 -> 1 cycle.

This creates a near-impossible situation for alternative cycles: the arithmetic constraint of coprimality favors convergence to smaller numbers, making it highly improbable for a sequence to remain in a closed loop of large numbers without collapsing to 1.

5. Conclusion

The coprimality of consecutive odd numbers in Collatz sequences is a novel arithmetic property that supports the conjecture. By imposing a requirement for large odd numbers to produce smaller subsequent odd numbers, it creates a dynamic that favors convergence to the 4 -> 2 -> 1 cycle over the formation of alternative cycles or divergent sequences. Combined with existing results on the minimal size of potential cycles, this property suggests that alternative cycles are not only astronomically large but also arithmetically implausible. Future work could explore numerical verification of the distribution of k in $b = (3a + 1)/2^k$ or integrate this property with analyses of the reverse Collatz graph.

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Contact:

Dobri Bozhilov
zadrugata2019@gmail.com
www.ailaboratory.eu
++359 88 7 467 818
Sofia, Bulgaria