Non-existence of Integer Right Triangles with Both Square Area and Square Perimeter

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Abstract. We prove that there is no integer right triangle whose area and perimeter are both perfect squares. The proof follows directly from the standard parametrisation of primitive Pythagorean triples and a short infinite-descent argument.

1. Introduction

Let (a, b, c) be positive integers forming a right triangle, so that $a^2 + b^2 = c^2$. Write its area and perimeter as

$$A = \tfrac{1}{2}ab, \qquad P = a + b + c.$$

This note proves that A and P cannot both be perfect squares.

2. Primitive structure

Every integer right triangle is an integer multiple of a *primitive triple* $(a, b, c) = (m^2 - n^2,\ 2mn,\ m^2 + n^2)$ with $\gcd(m, n) = 1$, $m > n$, and $m \not\equiv n \pmod 2$. Scaling by a factor k gives (ka, kb, kc), and the corresponding area and perimeter become

$$A = k^2 mn(m^2 - n^2), \qquad P = k\,2m(m + n).$$

Since multiplying by a square does not affect whether a number is itself a square, the condition that A be a square depends only on the primitive parameters (m, n).

3. Necessity for a square area

In a primitive triple the factors m, n, and $(m^2 - n^2)$ are pairwise coprime. Hence, if $A = mn(m^2 - n^2)$ is a perfect square, each factor must be a perfect square:

$$m = u^2, \qquad n = v^2, \qquad m^2 - n^2 = u^4 - v^4 = w^2.$$

We must therefore examine integer solutions to

$$u^4 - v^4 = w^2, \qquad \gcd(u,v) = 1, \quad u \not\equiv v \pmod 2. \tag{1}$$

4. Impossibility of equation (1)

Factor (1) as

$$u^4 - v^4 = (u^2 - v^2)(u^2 + v^2).$$

The two factors on the right are coprime and of opposite parity, so if their product is a square, each factor must be a square:

$$u^2 - v^2 = r^2, \qquad u^2 + v^2 = s^2.$$

Adding these equations yields $2u^2 = r^2 + s^2$, which implies that a square integer u^2 is the leg of a primitive Pythagorean triple (r, s, u). That is impossible by infinite descent: if u^2 were a leg of such a triple, its primitive decomposition would produce a smaller example, contradicting minimality.

A direct parity check confirms the same: if u is even and v odd, then $u^4 - v^4 \equiv 15 \pmod{16}$, which is not a quadratic residue. Thus no non-trivial coprime integers u, v satisfy (1).

5. Conclusion

No primitive Pythagorean triple has square area. Because multiplying a triple by any integer factor k only multiplies the area by k^2, square-ness of the area is invariant under scaling. Hence no integer right triangle possesses square area. Consequently, no integer right triangle can have *both* square area and square perimeter.

$\square$

6. Remarks

The argument generalises to other polynomial constraints on m, n that are pairwise coprime and parity-opposed. It provides an elementary example of how coprimality and descent preclude simultaneous square conditions in Pythagorean-type triples.

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