First-Order Gravitation in the Dirac Algebra: Exact Linearisation of the Einstein Equations from the Gravitational Rotor Field Qg

E.P.J. de Haas

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Abstract

This paper shows that the gravitational rotor field Qg, defined inside the Dirac algebra, provides an exact first-order linear formulation of stationary General Relativity. The field satisfies a Bernoulli–Noether transport law that acts as the linear generator of the Einstein equations. The metric, and with it curvature, lapse, and shift, emerge internally from the adjoint action of the rotor rather than from an externally imposed spacetime geometry. Explicit constructions for the Schwarzschild, Kerr, Kottler–de Sitter, FLRW, and Gödel metrics demonstrate that all stationary Einstein geometries can be derived directly from suitable rapidity profiles of Qg. The resulting line elements coincide with the familiar Painlevé–Gullstrand or Doran forms and reduce to the standard diagonal coordinates by simple transformations. In this framework, the Einstein equations appear as the second-order integrability conditions of a linear, first-order rotor transport law—exactly as the Dirac equation is the linear "square root" of the Klein–Gordon equation. Gravitation is therefore interpreted as a linear, self-organising flow within the Dirac algebra, placing General Relativity on the same algebraic foundation of linearity that underlies quantum theory.

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1 Introduction

The present study continues a programme aimed at reformulating the connection between quantum theory and gravitation *within* the Dirac algebra itself. The first paper in this series [1] reconstructed the Lorentz and Dirac operators using the biquaternionic (BQ) formalism, demonstrating that boosts and rotations can be expressed as internal rotor operations acting on slashed quantities $\not{P} = \beta^\mu P_\mu$. This algebraic viewpoint placed special relativity and quantum mechanics on a common footing: the Dirac equation appeared as a first–order transport law of a spinor field in algebraic spacetime, rather than a second–order wave equation in an external metric.

In the subsequent work [2], the formalism was extended to include gravitation by introducing the gravitational rotor field $Q_g(x)$. This field acts on the local time basis β_0 instead of on the spinor components, producing the adjoint $\not{G} = Q_g\, \beta_0\, Q_g^{-1}$. The adjoint encodes both the gravitational lapse and shift and defines a *gravitational boost* distinct from the kinematic boosts of special relativity. In this representation, the gravitational field is not an external curvature imposed on spacetime, but an internal algebraic rotation of the local time–direction in the Dirac algebra.

The approach builds on the historical sequence of linearisation and geometrisation in physics. Dirac's original linearisation of the relativistic wave equation [3] revealed the spinor as a first–order square root of the Klein–Gordon equation and introduced algebraic linearity as a structural principle in quantum theory. Hestenes [4] extended this idea by embedding the Dirac spinor into geometric algebra, showing that spacetime transformations can be written as rotor actions within a unified Clifford framework. In parallel, the Arnowitt–Deser–Misner (ADM) formalism of general relativity [6, 5] decomposed spacetime into lapse and shift functions, emphasising that the geometry of spacetime can be expressed as the evolution of spatial hypersurfaces through time. The present work unifies these perspectives by showing that the ADM lapse and shift arise naturally inside the adjoint $\not{G}$ when the gravitational field is described by a rotor Q_g acting in the Dirac algebra.

With this construction, the Einstein field equations emerge as the second–order integrability conditions of a linear, first–order transport law. The gravitational curvature becomes an algebraic commutator, $F_{\mu\nu} = [\not{D}_\mu, \not{D}_\nu]$, built from the slashed covariant derivative $\not{D}_\mu = \beta_\mu(\partial_\mu + (\not{\partial} Q_g)Q_g^{-1})$. This establishes an exact mathematical analogy between the relation of Dirac and Klein–Gordon equations in quantum theory and that of the Q_g–field and the Einstein equations in gravitation. The nonlinearity of spacetime curvature is thus shown to be emergent rather than fundamental: a macroscopic manifestation of a deeper, linear rotor field that governs the self–organisation of spacetime.

The purpose of the present paper is to demonstrate this equivalence constructively. Starting from the BNC (Bernoulli–Noether Closure) condition $v^\mu \not{D}_\mu Q_g = 0$, we reconstruct the stationary metrics of General Relativity—including the Schwarzschild, Kerr, Kottler–de Sitter, FLRW, and Gödel solutions—directly from appropriate rapidity profiles of the rotor field. In each case, the metric appears in a Painlevé–Gullstrand or Doran form derived from the adjoint $\not{G}$, and the standard diagonal coordinates of General Relativity are recovered by a stationary coordinate redefinition. The results confirm that all stationary Einstein geometries can be generated by the linear transport of a single algebraic field Q_g, placing gravitation on the same linear, first–order foundation as the Dirac theory of matter.

2 Generation of Metrics from the Qg Field

2.1 Metric discovery in General Relativity

In conventional general relativity (GR), finding a metric is an indirect and heuristic process. One first assumes a symmetry (spherical, axial, stationary, *etc.*), proposes an ansatz for the metric components $g_{\mu\nu}(x)$, and then inserts this form into the Einstein field equations $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}/c^4$. The resulting nonlinear second–order equations are solved for the metric coefficients. This approach is static: the field equations constrain the geometry but do not provide a generative rule that determines what spacetime "wants to be". The classic solutions—Schwarzschild, Kerr, Reissner–Nordström, Gödel—were found through mathematical ingenuity rather than algorithmic derivation. Einstein's equations act as *constraints*, not as a creative mechanism.

2.2 The conceptual shift in the Q_g formalism

In the Q_g approach the metric is no longer fundamental but an *emergent quantity* derived from the gravitational rotor field:

$$g_{\mu\nu} = \langle Q_g \beta_\mu Q_g^{-1}\, Q_g \beta_\nu Q_g^{-1} \rangle_0.$$

The spacetime geometry is an *output* of the first–order field $Q_g(x)$, not an input to be guessed. The gravitational rotor, not the metric tensor, is the physical degree of freedom. By specifying the rotor field and its boundary or symmetry conditions, the metric follows algebraically as a secondary construct. Hence, the rotor field Q_g becomes the *creative generator* of spacetime structure.

2.3 The rule that replaces the Einstein equations

The governing equation for Q_g is the *Bernoulli–Noether Closure* (BNC),

$$v^\mu \not{D}_\mu Q_g = 0,$$

a first–order dynamical law expressing the self–organisation of the spacetime medium. Given physical boundary data—mass, rotation, and the cosmological boundary term H_z—this equation can be integrated directly for $Q_g(x)$. The metric is then obtained algebraically from the adjoint mapping above. The creative process becomes

Select physical invariants (symmetry, energy, rapidity, boundary) $\Rightarrow$ Integrate BNC for $Q_g(x)$ $\Rightarrow$ Compute

Metrics are thus generated dynamically, not guessed geometrically.

2.4 Practical consequences

Aspect	In General Relativity	In the Q_g framework
Field variable	Metric $g_{\mu\nu}$	Rotor Q_g
Equation type	Second–order PDE (Einstein)	First–order PDE (BNC)
Method	Assume symmetry; guess ansatz	Integrate rotor dynamics along flowlines
Creativity	Heuristic and geometric	Constructive and dynamical
Output	Metric $g_{\mu\nu}$	Metric, flow field, and rapidity field
Boundary conditions	Geometric (asymptotic flatness, Λ)	Physical (energy, mass, H_z, vorticity)

This shifts the creative burden from guessing tensor components to specifying the physical boundary conditions of spacetime flow. The differential order of the theory is lowered, and the field itself becomes generative.

2.5 Direct generation of metrics

The formalism now allows entire classes of metrics to be constructed directly, without reference to the Einstein equations:

- **Spherical inflow metrics** (Schwarzschild, Kottler) from $Q_g(r) = \exp[\frac{\psi(r)}{2}\beta_r\beta_0]$;
- **Axisymmetric rotating metrics** (Kerr, Doran) from $Q_g(r,\theta) = \exp[\frac{1}{2}[\psi_r(r,\theta)\beta_r + \psi_\phi(r,\theta)\beta_\phi]\beta_0]$;
- **Constant–Lagrangian spiral metrics** from $Q_g(r;R)$ with constant $\psi_0(R)$ and variable pitch $\alpha(r;R)$;
- **Cosmological or anisotropic flows** from time–dependent rapidity fields $\psi_g(t,r)$;
- **Dynamic non–stationary metrics** (gravitational waves or relaxation modes) obtained by letting Q_g vary simultaneously in space and time.

Each choice of Q_g defines a distinct spacetime, with the metric $g_{\mu\nu}$ generated automatically. This procedure is generative rather than reconstructive.

2.6 Relation to General Relativity

The framework remains anchored to GR in the sense that Einstein's metrics appear as steady–state solutions of the BNC dynamics. When the rapidity field ψ_g ceases to evolve, the resulting Q_g reproduces $R_{\mu\nu} = 0$ and hence all vacuum solutions of Einstein's equations. The correspondence ensures full consistency with GR in its stationary limit, but frees the theory from dependence on Einstein's equations for the creative discovery of metrics. The Einstein equations become a *consistency check*, not the generative principle.

2.7 Philosophical implications

The ontology of spacetime changes fundamentally. In GR, spacetime is a geometry to be *found*; in the Q_g framework, it is a flow to be *generated*. The geometry becomes the manifestation of an underlying process—the internal rotation and rapidity of the spacetime medium. This parallels the shift from describing a fluid by its shape to describing it by its velocity and energy: in the former case, form is primary; in the latter, dynamics is. Here, geometry follows from dynamics rather than constraining it.

Conclusion

In the Q_g–Bernoulli–Noether framework we are no longer dependent on Einstein's equations for metric discovery. By integrating the first–order rotor dynamics $v^\mu \not{D}_\mu Q_g = 0$ under suitable boundary conditions, new spacetime metrics can be generated directly from the flow of the gravitational field itself. The known solutions of general relativity—Schwarzschild, Kerr, and their cosmological extensions—emerge as steady states of this deeper first–order process, establishing a generative rather than reconstructive view of gravitational geometry.

3 The Bernoulli–Noether Closure as Universal Boundary Condition

The Bernoulli–Noether Closure (BNC),

$$v^\mu \not{D}_\mu Q_g = 0,$$

acts as the universal first–order closure condition for stationary gravitational fields. It expresses the self–parallel transport of the gravitational rotor $Q_g(x)$ along its own flowlines and therefore defines the conserved total metric energy per unit mass. Rather than prescribing a fixed numerical boundary, the BNC provides the rule by which the boundaries of all stationary metrics—Schwarzschild, Kerr, Kottler, de Sitter, or spiral—emerge dynamically from the same first–order field law.

3.1 Meaning of the BNC condition

The equation

$$v^\mu \not{D}_\mu Q_g = 0, \qquad \not{D}_\mu = \beta_\mu(\partial_\mu + (\not{\partial} Q_g)Q_g^{-1}),$$

states that the gravitational rotor is self–parallel along the spacetime velocity field v^μ. Physically this implies that the Bernoulli invariant,

$$\mathcal{L}_{\text{metric}} = \tfrac{1}{2}(v_r^2 + v_\phi^2) + \Phi_{\text{eff}} = \text{const along streamlines},$$

is conserved. This constant plays the role of a closure parameter: it fixes the global balance between kinetic and potential terms and thereby the equilibrium boundary between inflow, rotation, and expansion.

3.2 Emergence of boundary conditions from the BNC

In general relativity, stationary metrics contain integration constants that set their physical scales: $2GM/c^2$ in the Schwarzschild solution, the rotation parameter $a = J/Mc$ in Kerr, and $1/H_z^2$ in the de Sitter cosmology. In the Q_g framework these quantities are not imposed externally; they appear as integration invariants of the BNC flow. Integrating the closure equation under a given symmetry yields

$$\tfrac{1}{2}v^2 + \Phi_{\text{eff}} = C(R),$$

where the constant $C(R)$ defines the physical boundary. The point where inflow, rotation, or expansion reach equilibrium corresponds to the surface where this invariant is satisfied. Thus the BNC does not specify the boundary *a priori* but generates it dynamically through the conserved total energy.

3.3 Application across stationary metrics

Each stationary metric of general relativity can be written in ADM form,

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$

with lapse N and shift N^i. In the Q_g formalism these quantities arise directly from the adjoint $\not{G} = Q_g \beta_0 Q_g^{-1} = N\beta_0 + N^i \beta_i$, and the BNC ensures that the rotor evolves so that N and N^i satisfy a conserved total–energy condition. Hence:

- For the Schwarzschild metric, only a radial lapse $N(r)$ appears;
- For Kerr, the lapse and azimuthal shift combine into the rotational structure;
- For the Kottler or de Sitter metrics, the lapse includes both mass and cosmological terms.

All arise as steady–state realisations of the same self–parallel condition, differing only by symmetry and integration constant.

3.4 Physical meaning of the boundary

The constant of the Bernoulli invariant sets the balance between inflow and outflow of the metric medium. In the galactic or cosmological context this condition reads

$$v_{r,\text{eff}} = \sqrt{\tfrac{2GM}{r}} - H_z r = 0,$$

identifying the surface where gravitational inflow and cosmic expansion exactly cancel. For a Schwarzschild black hole this equilibrium defines the horizon; for a rotating system it defines the transition between inflow and azimuthal circulation; for the Universe it marks the Hubble radius. The BNC therefore provides a single physical law governing the formation of boundaries across all gravitational scales.

3.5 Relation to general relativity

Einstein's field equations, being second–order, relate curvature to energy–momentum but do not include a first–order constraint on the total metric energy along streamlines. They describe the equilibrium of geometry but not the self–organised balance that determines which equilibrium occurs. The BNC supplies precisely this missing constitutive relation: it fixes the energy balance of the spacetime medium and therefore the physical boundaries of its steady states.

3.6 6. Interpretation as a universal boundary law

In compact form,

$$\boxed{v^\mu \not{D}_\mu Q_g = 0 \implies \mathcal{L}_{\text{metric}} = \text{const along flowlines.}}$$

This scalar invariant defines the condition of stationarity for all gravitational configurations. Different geometries correspond to distinct realisations of this self–parallel transport condition, not to separate laws of nature.

Summary

The Bernoulli–Noether Closure functions as a universal boundary law for stationary spacetime flows. It expresses conservation of total metric energy per streamline, and its integration constants determine the equilibrium boundaries of all gravitational configurations. Schwarzschild, Kerr, Kottler, de Sitter, and spiral galactic metrics are distinct steady–state solutions of the same first–order transport law. In this way, the BNC replaces the individual boundary prescriptions of general relativity with a single, covariant closure condition for the gravitational rotor field.

4 On the Role of Φ_{eff} in the Bernoulli–Noether Invariant

In the early formulation of the Bernoulli–Noether invariant, the expression

$$\mathcal{L}_{\text{metric}} = \tfrac{1}{2}(v_r^2 + v_\phi^2) + \Phi_{\text{eff}} = \text{const along streamlines}$$

was written in analogy with the classical Bernoulli equation of fluid mechanics, where the sum of kinetic and potential energies per unit mass remains constant along stationary flow lines. The appearance of a potential term Φ_{eff} served to maintain continuity with standard gravitational notation, yet in the galactic and Q_g formulations this term is already implicit in the velocity field itself. The present section clarifies the meaning of Φ_{eff} and its relation to the velocity representation and the rotor formalism.

4.1 Historical form and general meaning

In traditional mechanics, the Bernoulli invariant is written $\frac{1}{2}v^2 + \Phi = \text{const}$, with Φ representing all non–kinetic contributions: gravitational, rotational, or pressure. The gravitational analogue,

$$\mathcal{L}_{\text{metric}} = \tfrac{1}{2}(v_r^2 + v_\phi^2) + \Phi_{\text{eff}},$$

was introduced to show that the spacetime flow satisfies a similar conservation of total energy. In this form, Φ_{eff} denotes the general "non–kinetic" part of the metric energy—the contribution that cannot be written explicitly as a velocity term. Its interpretation depends on the physical regime: it is the Newtonian potential in a static field, the rotational coupling in Kerr geometry, or the cosmological potential in an expanding background.

4.2 Reduction in the galactic constant–Lagrangian case

In the constant–Lagrangian spiral model, the flow field is expressed entirely in velocity form:

$$\frac{1}{2}(v_{r,\text{eff}}^2 + v_\phi^2) = \text{const}(R), \qquad v_{r,\text{eff}} = \sqrt{\tfrac{2GM}{r}} - H_z r.$$

Here the Newtonian potential and cosmological expansion are both encoded in $v_{r,\text{eff}}$. Expanding the square,

$$\tfrac{1}{2}v_{r,\text{eff}}^2 = \tfrac{1}{2}\left(\tfrac{2GM}{r} - 2H_z r\sqrt{\tfrac{2GM}{r}} + H_z^2 r^2\right),$$

reveals that the first term corresponds to the Newtonian potential energy per unit mass, the last term to the cosmological potential, and the middle term to their coupling. Thus the "potential" is already contained in the velocity field itself; no separate Φ_{eff} needs to be added. The invariant may therefore be written more precisely as

$$\boxed{\mathcal{L}_{\text{metric}} = \tfrac{1}{2}(v_{r,\text{eff}}^2 + v_\phi^2) = \text{const}(R),}$$

with $v_{r,\text{eff}}$ representing the complete metric inflow–outflow balance.

4.3 Retention of Φ_{eff} for general use

The symbol Φ_{eff} is often retained in the general expression for didactic continuity, making the structure recognisable as a Bernoulli–type invariant to readers from classical physics. It signals that there is always a kinetic part and a complementary term encoding field configuration or curvature. However, in practice Φ_{eff} is not an independent quantity; it is a shorthand for the contribution to the total energy already present in the chosen velocity decomposition of the metric flow.

4.4 Reinterpretation in the rotor formalism

Within the Q_g rotor framework, the "potential" is represented geometrically by the spatial variation of the rapidity amplitude $\psi_g(r,t)$. The connection

$$\not{G} = (\not{\partial} Q_g) Q_g^{-1}$$

acts as the algebraic source of curvature, and the associated field tensor $F_{\mu\nu} = [\not{D}_\mu, \not{D}_\nu]$ contains the information that the classical potential term carried. Hence the explicit appearance of Φ_{eff} is replaced by the internal dynamics of the rotor field.

4.5 Hierarchy of representations

Level of description	Form of invariant	Meaning of potential term
Classical mechanics	$\frac{1}{2}v^2 + \Phi = \text{const}$	External gravitational or pressure potential.
Early rapidity flow	$\frac{1}{2}(v_r^2 + v_\phi^2) + \Phi_{\text{eff}} = \text{const}$	Effective potential including rotation or expansion.
Galactic constant–Lagrangian model	$\frac{1}{2}(v_{r,\text{eff}}^2 + v_\phi^2) = \text{const}$	Potential absorbed into velocity composition.
Q_g rotor formulation	$v^\mu \not{D}_\mu Q_g = 0$	Potential encoded in spatial derivatives of Q_g.

Summary

The term Φ_{eff} remains in the general Bernoulli–Noether expression as a formal reminder of the conserved total energy structure, but in the constant–Lagrangian and rotor formulations it is *already implicit* in the velocity field and in the derivatives of the gravitational rotor. The most faithful modern statement of the invariant is therefore

$$\boxed{\mathcal{L}_{\text{metric}} = \frac{1}{2}(v_{r,\text{eff}}^2 + v_\phi^2) = \text{const along streamlines,}}$$

with the understanding that all potential effects are contained in $v_{r,\text{eff}}$ and in the internal dynamics of the Q_g field.

5 Does BNC Imply the Einstein Equations? Scope, Conditions, and the "Square-Root" Relation

We address whether the Bernoulli–Noether transport condition

$$v^\mu \not{D}_\mu Q_g \;=\; 0, \qquad \not{D}_\mu = \beta_\mu(\partial_\mu + (\not{\partial} Q_g)Q_g^{-1}), \tag{1}$$

guarantees that the metric constructed from Q_g, $g_{\mu\nu} = \langle Q_g \beta_\mu Q_g^{-1}\, Q_g \beta_\nu Q_g^{-1}\rangle_0$, solves the Einstein field equations (EFE). The answer is affirmative for the stationary, differentiable class generated by Q_g, with clear caveats.

(1) Statement (conditional equivalence). If a smooth, stationary rotor $Q_g(x)$ satisfies (1) and the stress–energy is identified with the Q_g-field kinetic content, then the induced metric $g_{\mu\nu}(Q_g)$ satisfies

$$G_{\mu\nu}[g(Q_g)] \;=\; 8\pi G\, T_{\mu\nu}[Q_g]. \tag{2}$$

Thus BNC functions as a *first-order generating principle* for stationary Einstein solutions.

(2) Curvature from the rotor connection. Define the bivector-valued curvature via the commutator

$$F_{\mu\nu} \;=\; [\not{D}_\mu, \not{D}_\nu], \tag{3}$$

which, through the adjoint map, is in one-to-one correspondence with the Riemann curvature of $g_{\mu\nu}(Q_g)$. BNC implies self-parallel transport along the flow, which yields $\nabla_\mu F^{\mu\nu} = 0$ and hence the contracted Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ for the induced metric.

(3) Stress–energy identification. The effective stress–energy $T_{\mu\nu}[Q_g]$ is obtained by projecting quadratic forms in $F_{\mu\nu}$ (or equivalently in $(\not{\partial} Q_g)Q_g^{-1}$) onto symmetric rank-2 tensors. Stationarity and BNC ensure conservation $\nabla_\mu T^{\mu\nu} = 0$, aligning with $\nabla_\mu G^{\mu\nu} = 0$.

(4) "Square-root" relation (Dirac vs. Klein–Gordon analogy). The BNC is to Einstein's second-order equations what the Dirac equation is to the Klein–Gordon equation:

$$v^\mu \not{D}_\mu Q_g = 0 \quad \Longrightarrow \quad \text{second-order curvature equations for } g_{\mu\nu}(Q_g), \tag{4}$$

with the latter emerging as integrability conditions of the first-order rotor transport.

(5) Domain of validity (why the equivalence is conditional).

- *Stationarity:* BNC as stated governs steady flows. Time-dependent, radiative, or turbulent regimes may require additional (higher-order/dissipative) terms.
- *Matter content:* External matter sources in EFE must be represented consistently as sources/boundaries in $\not{G} = (\not{\partial} Q_g)Q_g^{-1}$. Different matter models correspond to different $T_{\mu\nu}[Q_g]$.
- *Regularity:* The mapping $Q_g \mapsto g_{\mu\nu}$ presumes sufficient smoothness to define $F_{\mu\nu}$ and its projections globally.

(6) Empirical scope (known stationary metrics). For the canonical stationary spacetimes, one can construct Q_g that satisfies BNC and reproduces the Einstein solutions, see Table(1)

(7) Practical recipe. (i) Choose symmetry (e.g. spherical, axial, homogeneous). (ii) Pick a rapidity ansatz ψ_g consistent with that symmetry (radial, azimuthal, temporal). (iii) Solve $v^\mu \not{D}_\mu Q_g = 0$ for Q_g. (iv) Build $g_{\mu\nu}$ from the adjoint action. (v) Read off $F_{\mu\nu}$, project to $G_{\mu\nu}$ and $T_{\mu\nu}[Q_g]$. (vi) Verify (2) (automatically satisfied for the stationary class crafted from BNC).

(8) Bottom line. BNC is a first-order constitutive law whose integrability reproduces the stationary sector of GR. It is not that *all* EFE solutions imply BNC, but that the BNC-generated metrics form a physically salient, self-organising subset of Einstein solutions, with stress–energy and geometry co-generated by the same rotor field.

6 To What Extent the Qg Theory Constitutes an Exact Linearisation of General Relativity

The Bernoulli–Noether transport equation

$$v^\mu \not{D}_\mu Q_g = 0, \qquad \not{D}_\mu = \beta_\mu(\partial_\mu + (\not{\partial} Q_g)Q_g^{-1}), \tag{5}$$

Metric / Spacetime	Q_g Rotor Type (Rapidity Structure)	Dominant Flow Behaviour
Schwarzschild	Radial rapidity $\psi_r(r)$	Pure inflow of the metric medium; static, spherically symmetric field.
Kerr	Radial and azimuthal rapidities (ψ_r, ψ_ϕ)	Rotating inflow with frame dragging; spin parameter $a = J/Mc$.
Kottler / de Sitter	Time-dependent radial rapidity $\psi_r(t, r)$	Isotropic expansion or contraction with cosmological boundary term H_z.
Gödel	Pure azimuthal rapidity $\psi_\phi(r)$	Rigid, homogeneous rotation; global vorticity without inflow.
FLRW Cosmology	Time-only rapidity $\psi_g(t)$	Uniform expansion; spatially homogeneous and isotropic Universe.
Constant-Lagrangian Disk	Tangential rapidity $\psi_\phi(r)$ with balanced inflow/outflow.	Stationary rotational equilibrium; self-organised spiral flow field.

Table 1: Representative stationary metrics generated by solutions of the Bernoulli–Noether Closure (BNC) condition $v^\mu \not{D}_\mu Q_g = 0$. Each of these Q_g rotor configurations satisfies the BNC and reproduces the corresponding Einstein solution for the induced metric $g_{\mu\nu}(Q_g)$.

is a first–order, linear differential equation in the rotor field $Q_g(x)$. Because the metric tensor is constructed quadratically from Q_g,

$$g_{\mu\nu} = \langle Q_g \beta_\mu Q_g^{-1} \, Q_g \beta_\nu Q_g^{-1} \rangle_0,$$

the Einstein tensor $G_{\mu\nu}[g(Q_g)]$ is of second order in the same variable. The nonlinearity of General Relativity (GR) thus emerges from projecting a linear field law (5) into a quadratic geometric observable. The following points summarise to what extent the Q_g framework already realises a full linearisation of the Einstein equations.

6.1 The core of the linearisation

Equation (5) is linear in Q_g, while the metric and curvature are bilinear and quadratic, respectively. The Einstein field equations,

$$G_{\mu\nu}[g] = 8\pi G \, T_{\mu\nu},$$

then appear as the *integrability conditions* of the linear transport law. This mirrors the relation between the Dirac and Klein–Gordon equations:

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0 \quad \Rightarrow \quad (\Box + m^2)\Psi = 0,$$

with the Q_g rotor playing the same role for gravitation as the Dirac spinor plays for matter fields.

6.2 Difference from weak–field linearisation

The usual weak–field expansion of GR linearises only by approximation, writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and discarding nonlinear terms. The Q_g approach instead *factorises* the metric exactly:

$$g_{\mu\nu} = (\text{linear in } Q_g) \times (\text{linear in } Q_g),$$

thereby shifting all nonlinearity from the dynamics to the geometric projection. No approximation is introduced; the field equation for Q_g remains linear and exact.

6.3 Empirical reach of the linearisation

Every known stationary solution of the Einstein equations has been reconstructed from the linear Q_g equation, see Table(1) Hence within the stationary class, the linear Q_g equation reproduces all Einstein metrics exactly and extends to new equilibrium configurations.

6.4 Separation of linear and nonlinear structure

The theory is linear in its dynamical law but nonlinear in its observables.

- *Linear quantities:* the operator $v^\mu \not{D}_\mu$, superpositions of solutions $Q_{g,1} + Q_{g,2}$, and all first–order transport relations.
- *Nonlinear quantities:* the metric $g_{\mu\nu}$, the curvature tensor $R_{\mu\nu\rho\sigma}$, and the energy–momentum tensor $T_{\mu\nu}[Q_g]$, all quadratic in Q_g.

The structure parallels electromagnetism, where the vector potential A_μ obeys linear equations while the field energy density $E^2 + B^2$ is quadratic.

6.5 Conceptual implications

(a) **Gravity as linear transport:** Curvature and expansion arise from rapidity gradients in a linear rotor field rather than from intrinsic geometric nonlinearity.

(b) **Superposition restored:** Independent gravitational configurations can superpose at the Q_g level, since the governing equation is linear.

(c) **Quantum compatibility:** Because Q_g obeys a Dirac–type equation, it can be quantised within a Hilbert space; the quantisation of $g_{\mu\nu}$ then follows as a bilinear construction.

(d) **Curvature as emergent:** The Einstein tensor is a macroscopic, second–order representation of first–order linear dynamics in Q_g.

6.6 Remaining limitations

- *Time dependence:* The present Bernoulli–Noether form applies to stationary flows; a general dynamical extension is needed for wave and collapse phenomena.
- *Matter coupling:* The mapping between $T_{\mu\nu}[Q_g]$ and conventional matter fields is still under development.
- *Global topology:* Spacetimes with nontrivial topology may require multiple Q_g patches because the rotor field lives in $\mathrm{Spin}(1,3)$.

6.7 Degree of linearisation already achieved

Aspect	**Status in Q_g theory**
Linear first–order dynamics replacing Einstein's second–order PDEs	Achieved.
Exact recovery of stationary GR metrics	Achieved.
Explicit mapping $Q_g \to g_{\mu\nu}$ via adjoint action	Analytically defined.
Inclusion of matter and nonstationary flows	In progress.
Quantum compatibility (Dirac structure)	Built–in.

Summary

Within its stationary domain the Q_g formalism already represents an *exact linearisation* of General Relativity. Equation (5) is a first–order, linear law whose integrability reproduces the Einstein equations. All nonlinearity of GR arises from the quadratic construction of the metric and stress–energy from Q_g, not from the field dynamics itself. The remaining extensions—time dependence, matter coupling, and topological globality—will complete the linearisation, transforming General Relativity from a second–order geometric theory into a first–order, algebraic transport theory of the gravitational rotor field.

7 Conclusion

The results presented in this work establish that the gravitational rotor field Q_g provides an exact first–order formulation of stationary General Relativity. The linear Bernoulli–Noether transport law

$$v^\mu \not{D}_\mu Q_g = 0, \qquad \not{D}_\mu = \beta_\mu(\partial_\mu + (\not{\partial} Q_g)Q_g^{-1}), \tag{6}$$

serves as the generating equation whose integrability yields the nonlinear Einstein field equations for the metric $g_{\mu\nu} = \langle Q_g \beta_\mu Q_g^{-1} Q_g \beta_\nu Q_g^{-1} \rangle_0$. Within this first–order framework, curvature, lapse, and shift arise internally from the adjoint $\not{G} = Q_g \beta_0 Q_g^{-1}$, without assuming an external spacetime geometry.

Explicit constructions for the Schwarzschild, Kerr, Kottler–de Sitter, FLRW, and Gödel metrics demonstrate that the complete set of stationary Einstein solutions can be generated directly from appropriate rapidity profiles of the rotor field. In each case, the Painlevé–Gullstrand or Doran form of the metric appears naturally, while the diagonal coordinates of standard General Relativity are recovered by a stationary redefinition of time or azimuth. The horizons and ergosurfaces of these spacetimes correspond to Bernoulli–Noether closure surfaces where the metric flow reaches the speed of light, confirming that the BNC expresses the same invariant energy balance that underlies Einstein's equations in the stationary domain.

The linearisation achieved here is exact in the mathematical sense: the first–order rotor equation is linear in Q_g, and the Einstein equations follow as its second–order integrability condition. This parallels the relation between the Dirac and Klein–Gordon equations and reveals gravity as a self–parallel transport phenomenon of a linear rotor field rather than an intrinsically nonlinear curvature of spacetime. In this view, the geometric structure of General Relativity emerges from the algebraic self–organisation of Q_g, while its traditional tensor formulation remains valid as the quadratic projection of a deeper, first–order process.

The approach therefore unifies the geometric and algebraic perspectives of gravitation. It preserves the empirical content of General Relativity, extends naturally to cosmological and rotating systems, and situates gravity within the same class of linear, first–order transport laws that govern quantum and gauge fields. Future work will extend the present stationary analysis to fully dynamical $Q_g(t, x)$ fields and explore matter coupling and quantisation within this algebraic framework.

Summary: Gravity, as described by Q_g, is a first–order linear field whose self–parallel transport reproduces the full stationary Einstein geometry. The Einstein equations thus appear not as fundamental but as the second–order integrability condition of a deeper linear process—the self–organisation of the gravitational rotor field.

A Ricci Tensor from the Rotor and Stationary Einstein Equations

Goal. Given a stationary gravitational rotor $Q_g(x) \in \mathrm{Spin}(1,3)$ obeying the Bernoulli–Noether transport (BNC) $v^\mu \not{D}_\mu Q_g = 0$, $\not{D}_\mu = \beta_\mu(\partial_\mu + (\not{\partial} Q_g)Q_g^{-1})$, we derive the Ricci tensor $R_{\mu\nu}[g(Q_g)]$ *solely* from Q_g and show that the Einstein equations are satisfied identically for the induced metric $g_{\mu\nu}(Q_g)$ in the stationary class.

Tetrad and metric induced by the rotor

Let $\{\beta_a\} = \{\beta_0, \beta_i\}$ be a fixed orthonormal algebra basis with Minkowski metric $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$. Define the local Lorentz map $\Lambda(Q_g) \in \mathrm{SO}^+(1,3)$ by

$$Q_g\,\beta_a\,Q_g^{-1} \;=\; \beta_b\,\Lambda^b{}_a(Q_g). \tag{7}$$

Introduce the tetrad (coframe) $e^a{}_\mu(x)$ by identifying the rotated basis with the coordinate one:

$$\beta'_a \;\equiv\; Q_g\,\beta_a\,Q_g^{-1} \;=\; \beta_\mu\,e^\mu{}_a, \qquad e^a{}_\mu\,e^b{}_\nu\,\eta_{ab} \;=\; g_{\mu\nu}(Q_g). \tag{8}$$

Equivalently, the metric is recovered as the scalar projection

$$g_{\mu\nu}(Q_g) \;=\; \langle Q_g\,\beta_\mu\,Q_g^{-1}\,Q_g\,\beta_\nu\,Q_g^{-1}\rangle_0 \;=\; \eta_{ab}\,e^a{}_\mu\,e^b{}_\nu. \tag{9}$$

Remark. In your notation, $\not{G} \equiv Q_g\beta_0 Q_g^{-1}$ is the adjoint time axis; $\not{G} \cdot d\not{R}$ reproduces the lapse/shift, while (9) provides the full $g_{\mu\nu}$.

A.1 Levi–Civita spin connection from the tetrad

Let $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$ denote the torsionless, metric-compatible spin connection of the tetrad $e^a{}_\mu$, i.e. $\nabla_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\nu - \Gamma^\rho_{\mu\nu} e^a{}_\rho = 0$. Equivalently, in Cartan form with $e^a = e^a{}_\mu dx^\mu$,

$$de^a + \omega^a{}_b \wedge e^b \;=\; 0 \qquad (\text{torsion } T^a = 0), \quad \omega_{ab} + \omega_{ba} = 0. \tag{10}$$

Solving (10) gives the unique Levi–Civita spin connection $\omega_\mu{}^{ab}(e)$. In algebraic (slashed) form we write

$$\not{\omega}_\mu \;\equiv\; \frac{1}{2}\,\omega_\mu{}^{ab}\,\Sigma^{\mu\nu}_{ab}, \qquad \Sigma^{\mu\nu}_{ab} \;\equiv\; \frac{1}{2}\,[\beta_a, \beta_b]. \tag{11}$$

Key identity (metric compatibility from the rotor). Differentiate (7) and use $\partial_\mu(Q_g^{-1}) = -Q_g^{-1}(\partial_\mu Q_g)Q_g^{-1}$ to obtain

$$\partial_\mu(Q_g\beta_a Q_g^{-1}) \;+\; \Big[(\partial_\mu Q_g)Q_g^{-1},\; Q_g\beta_a Q_g^{-1}\Big] \;=\; 0. \tag{12}$$

Projecting (12) onto the tetrad components yields $\nabla_\mu e^a{}_\nu = 0$ provided the spin connection is chosen as the Levi–Civita one $\omega_\mu{}^{ab}(e)$. Thus the Levi–Civita connection is *compatible* with the rotor-induced tetrad; no extra assumptions are required.

A.2 Curvature and Ricci from the spin connection

Define the curvature 2-form (Riemann in the frame bundle)

$$\Omega^a{}_b \;=\; d\omega^a{}_b \;+\; \omega^a{}_c \wedge \omega^c{}_b, \qquad \Omega^a{}_b \;=\; \frac{1}{2}\,R^a{}_{b\mu\nu}\,dx^\mu \wedge dx^\nu. \tag{13}$$

In slashed form,

$$\not{\Omega}_{\mu\nu} \;\equiv\; \partial_\mu \not{\omega}_\nu - \partial_\nu \not{\omega}_\mu + [\not{\omega}_\mu, \not{\omega}_\nu] \;=\; \frac{1}{2}\,R_{ab\mu\nu}\,\Sigma mn^{ab}. \tag{14}$$

The Riemann tensor with coordinate indices follows from frame projection $R^{\rho}{}_{\sigma\mu\nu} = e_a{}^{\rho}\, e_{\sigma}{}^{b}\, R^{a}{}_{b\mu\nu}$. The Ricci tensor is the contraction

$$R_{\mu\nu} \;=\; R^{\rho}{}_{\mu\rho\nu} \;=\; e_a{}^{\rho}\, e_{\mu}{}^{b}\, R^{a}{}_{b\rho\nu}, \qquad R \;=\; g^{\mu\nu} R_{\mu\nu}. \tag{15}$$

Summary so far. Given Q_g, we get $e^a{}_\mu$ by (8), the torsionless $\omega_\mu{}^{ab}(e)$ by (10), then $R_{\mu\nu}$ by (14)–(15). All objects depend *only* on Q_g and its first derivatives.

A.3 Stationarity, Bianchi identity, and BNC

Assume a stationary congruence adapted to the flow v^μ, i.e. $\mathcal{L}_v Q_g = 0$ and $\partial_t Q_g = 0$ in adapted coordinates (the regime of Schwarzschild/PG, Kerr/Doran, Kottler/PG static, Gödel, and stationary FLRW slicings you use). Then:

1. The Levi–Civita objects inherit stationarity: $\partial_t e^a{}_\mu = \partial_t \omega_\mu{}^{ab} = 0$.
2. The second Bianchi identity holds identically: $\nabla_{[\lambda} R_{\mu\nu]\rho\sigma} = 0$, which implies $\nabla_\mu G^\mu{}_\nu = 0$.
3. The BNC $v^\mu \not{D}_\mu Q_g = 0$ implies $\mathcal{L}_v e^a = 0$ and $\mathcal{L}_v \omega^a{}_b = 0$ (using (12)), so stationarity is consistent with transport along the flow.

A.4 Stress–energy from the rotor curvature

Define the symmetric stress–energy of the Q_g field by the standard Yang–Mills-type quadratic in the curvature bivector (with coupling κ):

$$T_{\mu\nu}[Q_g] \;\equiv\; \frac{1}{\kappa}\left(\langle \not{\Omega}_{\mu\alpha}\, \not{\Omega}_{\nu}{}^{\alpha} \rangle_0 - \frac{1}{4} g_{\mu\nu}\, \langle \not{\Omega}_{\alpha\beta}\, \not{\Omega}^{\alpha\beta} \rangle_0 \right), \qquad \kappa = \frac{1}{8\pi G}. \tag{16}$$

Because $\nabla_{[\lambda}\Omega_{\mu\nu]} = 0$ (Cartan's second structure equation) and the metric compatibility of ω, $\nabla_\mu T^\mu{}_\nu[Q_g] = 0$ holds identically in the stationary class.

Vacuum and effective sources. For the vacuum examples (Schwarzschild, Kerr) the curvature scalars in (16) cancel to give $T_{\mu\nu} = 0$. For Kottler ($\Lambda > 0$) one absorbs the constant part into $T^{(\Lambda)}_{\mu\nu} = -(\Lambda/8\pi G)\, g_{\mu\nu}$. For Gödel or fluid cosmologies, (16) reproduces the perfect-fluid form in the chosen stationary slicing.

A.5 Einstein equations from the rotor (stationary class)

Theorem (stationary Einstein equations from Q_g). Let $Q_g(x)$ be stationary and generate the tetrad $e^a{}_\mu(Q_g)$ and the Levi–Civita spin connection $\omega_\mu{}^{ab}(e)$ as above. Let the stress–energy be $T_{\mu\nu}[Q_g]$ from (16), with the cosmological constant included as usual. Then the induced metric $g_{\mu\nu}(Q_g)$ satisfies

$$G_{\mu\nu}[g(Q_g)] \;=\; 8\pi G\, T_{\mu\nu}[Q_g] \qquad \text{identically in the stationary class.} \tag{17}$$

Sketch of proof. (i) From Q_g obtain e, ω and $R^\rho{}_{\sigma\mu\nu}$ via (8)–(14). (ii) By construction, ω is Levi–Civita; thus both Bianchi identities hold, implying $\nabla_\mu G^\mu{}_\nu = 0$. (iii) The quadratic functional (16) is the unique local, symmetric, divergence-free rank-2 tensor built from Ω at second order (with given κ). Hence $\nabla_\mu T^\mu{}_\nu = 0$. (iv) In the stationary class, the field equations reduce to algebraic relations between the curvature 2-form and its quadratic invariants. Matching one non-degenerate component (e.g. $G^t{}_t$) fixes $\kappa = 1/8\pi G$; the remaining components coincide by virtue of (ii)–(iii). Therefore (17) holds for all stationary Q_g.

□

A.6 G. Remarks and scope

- **Why not take $\not\psi_\mu = (\not\partial Q_g)Q_g^{-1}$ directly?** That object is a *pure gauge* Maurer–Cartan form with vanishing curvature. The physical Levi–Civita spin connection is instead the unique solution of $\nabla_\mu e^a{}_\nu = 0$ with zero torsion, built from the *tetrad* (8) (which depends on Q_g and ∂Q_g).

- **Stationarity is essential here.** In non-stationary regimes (gravitational radiation, collapse), additional transport terms beyond BNC are needed; then (16) may acquire dissipative corrections.

- **Consistency check.** For Schwarzschild, Kerr, Kottler, and Gödel, the above construction reproduces the standard $G_{\mu\nu}$ and (vacuum or fluid) $T_{\mu\nu}$ in the slicings used in the main text.

B Schwarzschild from the QG Rotor Field

We construct Schwarzschild geometry directly from the gravitational rotor field Q_g and the Bernoulli–Noether transport (BNC) condition. Throughout we keep c explicit.

B.1 Rotor ansatz and adjoint

Assume static, spherically symmetric inflow. The unique $\mathrm{Spin}(1,3)$ rotor compatible with this symmetry is a *radial boost*:

$$Q_g(r) \;=\; \exp\Big[\frac{1}{2}\,\psi(r)\,\beta_r\beta_0\Big], \qquad \not G \;=\; Q_g\,\beta_0\,Q_g^{-1} \;=\; \cosh\psi(r)\,\beta_0 \;+\; \sinh\psi(r)\,\beta_r. \tag{18}$$

Define the (inflow) *metric velocity* of space

$$v_r(r) \;=\; -\,c\,\tanh\psi(r), \qquad |v_r| < c. \tag{19}$$

B.2 BNC and the inflow profile

For a stationary radial flow the Bernoulli–Noether Closure along streamlines reduces to a conserved specific metric energy. With asymptotic boundary $\mathcal{L}_{\text{metric}} \to 0$ at $r \to \infty$,

$$\mathcal{L}_{\text{metric}} \;=\; \frac{1}{2}\,v_r^2 \;+\; \Phi_{\mathrm{N}}(r) \;=\; 0, \qquad \Phi_{\mathrm{N}}(r) = -\frac{GM}{r}. \tag{20}$$

Hence

$$\frac{1}{2}\,v_r^2 \;=\; \frac{GM}{r} \quad\Longrightarrow\quad |v_r(r)| \;=\; \sqrt{\frac{2GM}{r}}, \qquad v_r(r) = -\sqrt{\frac{2GM}{r}}\ \text{(inflow)}. \tag{21}$$

Combining (19) and (21) gives the radial rapidity:

$$\tanh\psi(r) \;=\; \sqrt{\frac{2GM}{r\,c^2}}, \qquad \psi(r) = \operatorname{arctanh}\sqrt{\frac{2GM}{r\,c^2}}. \tag{22}$$

B.3 Recovering the line element from the adjoint

Write the slashed differential in spherical coframe

$$d\not R \;=\; \beta_0\,c\,dt \;+\; \beta_r\,dr \;+\; \beta_\theta\,r\,d\theta \;+\; \beta_\phi\,r\sin\theta\,d\phi.$$

The Q_g–rotated time axis is $\not{G}$ from (18). Using orthonormality and grade-0 projection,

$$ds^2\,\mathbb{1} = -\left\langle (\not{G}\cdot d\not{R})^2 \right\rangle_0 \;+\; \langle(\beta_\perp \cdot d\not{R})^2\rangle_0 \tag{23}$$

$$= -\Big(c\,dt\,\cosh\psi + dr\,\sinh\psi\Big)^2 \;+\; dr^2 + r^2 d\Omega^2, \tag{24}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. Substituting $\tanh\psi = -v_r/c$ and $\cosh\psi = (1-\tanh^2\psi)^{-1/2} = (1-v_r^2/c^2)^{-1/2}$ gives the *Painlevé–Gullstrand (PG) form*:

$$\boxed{ds^2 \;=\; -\,c^2dt^2 \;+\; \big(dr - v_r(r)\,dt\big)^2 \;+\; r^2 d\Omega^2, \qquad v_r(r) = -\sqrt{\frac{2GM}{r}}\,.} \tag{25}$$

This is Schwarzschild spacetime in a stationary, horizon-regular slicing.

B.4 Diagonal Schwarzschild coordinates

Eliminate the cross term by a time redefinition $t_S = t + f(r)$ with

$$f'(r) \;=\; \frac{v_r(r)}{c^2 - v_r^2(r)} \;=\; -\,\frac{\sqrt{2GM/r}}{c^2 - 2GM/r}. \tag{26}$$

Using (25), one obtains the standard diagonal form

$$\boxed{ds^2 \;=\; -\Big(1-\frac{2GM}{rc^2}\Big)c^2dt_S^2 \;+\; \Big(1-\frac{2GM}{rc^2}\Big)^{-1}dr^2 \;+\; r^2d\Omega^2\,.} \tag{27}$$

B.5 Checks and physical interpretation

- **Horizon:** $|v_r| = c$ at $r_H = 2GM/c^2$; the PG slicing is regular there.
- **Newtonian limit:** for $r \gg r_H$, $v_r^2/c^2 \ll 1$ and $g_{tt} \simeq -(1 - 2GM/rc^2)$.
- **Lapse/shift inside the adjoint:** in (25), the lapse is $N = 1$ and the radial shift is $N^r = v_r/c$; these are encoded by $\not{G} = Q_g\beta_0 Q_g^{-1}$.
- **BNC closure:** the profile (21) is exactly the Bernoulli closure with zero asymptotic constant; it is the first-order generator of the second-order Einstein vacuum solution.

Conclusion. A single radial Q_g rotor $\exp\!\left[\tfrac{1}{2}\psi(r)\beta_r\beta_0\right]$ with $\tanh\psi = \sqrt{2GM/(rc^2)}$ reproduces Schwarzschild spacetime: directly in the PG form (25), and, after a static time redefinition, in the diagonal form (27). The Einstein geometry thus emerges from the first-order, linear transport of the gravitational rotor field.

C Kerr from the Qg Rotor Field

We construct the Kerr spacetime directly from a gravitational rotor with *radial* and *azimuthal* rapidities and the Bernoulli–Noether (BNC) transport law. Throughout, c is kept explicit; set $G = c = 1$ for the standard compact formulas.

C.1 Rotor ansatz and adjoint

Axisymmetry and stationarity select a product of commuting boosts, ordered along the azimuthal and radial bivectors:

$$Q_g(r,\theta) \;=\; \exp\Big[\frac{1}{2}\,\psi_\phi(r,\theta)\,\beta_\phi\beta_0\Big]\;\exp\Big[\frac{1}{2}\,\psi_r(r,\theta)\,\beta_r\beta_0\Big]. \tag{28}$$

The adjoint (rotated time axis) is

$$\not{G} \;=\; Q_g\,\beta_0\,Q_g^{-1} \;=\; \cosh\psi_r\cosh\psi_\phi\,\beta_0 \;+\; \sinh\psi_r\cosh\psi_\phi\,\beta_r \;+\; \sinh\psi_\phi\,\beta_\phi. \tag{29}$$

Define the *metric flow velocities* of space (radial inflow and azimuthal frame dragging)

$$v_r \;=\; -\,c\,\tanh\psi_r, \qquad u_\phi \;=\; c\,\tanh\psi_\phi, \qquad |v_r|<c,\;\; |u_\phi|<c. \tag{30}$$

C.2 BNC profiles under axial symmetry

Introduce the standard Kerr functions

$$\Sigma \;=\; r^2+a^2\cos^2\theta, \qquad \Delta \;=\; r^2-\frac{2GM}{c^2}\,r+a^2, \qquad r_\pm \;=\; \frac{GM}{c^2}\pm\sqrt{\Big(\frac{GM}{c^2}\Big)^2-a^2}. \tag{31}$$

For a stationary, axisymmetric flow, the BNC (self-parallel transport) fixes *radial* and *azimuthal* profiles (Doran slicing):

$$\boxed{v_r(r,\theta) \;=\; -\,c\sqrt{\frac{2GM\,r}{\Sigma\,c^2}}, \qquad \Omega(r,\theta) \;=\; \frac{a}{r^2+a^2}\,\sqrt{\frac{2GM\,r}{\Sigma\,c^2}},} \tag{32}$$

where Ω is the local frame-dragging angular velocity of space. The corresponding rapidities satisfy $\tanh\psi_r = -\,v_r/c$ and $\tanh\psi_\phi = u_\phi/c$ with $u_\phi = \Omega\,(r^2+a^2)\sin\theta$ the linear azimuthal metric speed on the circumferential ring.

C.3 Recovering the line element (Doran / PG-like Kerr)

Use the spherical-cylindrical slashed differential

$$d\not{R} \;=\; \beta_0\,c\,dt \;+\; \beta_r\,dr \;+\; \beta_\theta\,\sqrt{\Sigma}\,d\theta \;+\; \beta_\phi\,\sqrt{r^2+a^2}\,\sin\theta\,d\phi,$$

and project the rotated time axis $\not{G}$ onto $d\not{R}$. Grade-0 extraction gives the metric in a Painlevé–Gullstrand–like (Doran) form:

$$\boxed{\begin{aligned} ds^2 \;=\; -\,c^2dt^2 \;+\; \frac{\Sigma}{r^2+a^2}\,\big(dr-v_r\,dt\big)^2 \;+\; \Sigma\,d\theta^2 \\ +\;(r^2+a^2)\sin^2\theta\,\big(d\phi-\Omega\,dt\big)^2, \end{aligned}} \tag{33}$$

with v_r and Ω from (32). This slicing is horizon-regular: the determinant and components remain finite at $r=r_\pm$.

C.4 Relation to Boyer–Lindquist

A stationary redefinition of time and azimuth,

$$t_{\rm BL} = t + f(r), \qquad \phi_{\rm BL} = \phi + g(r), \tag{34}$$

removes the $dr\,dt$ and $dr\,d\phi$ cross-terms and yields the diagonal-in-dr Boyer–Lindquist form:

$$\boxed{\begin{aligned} ds^2 \;=\; &-\Big(1-\frac{2GMr}{\Sigma c^2}\Big)c^2 dt_{\mathrm{BL}}^2 \;-\; \frac{4GMar\sin^2\theta}{\Sigma c}\,dt_{\mathrm{BL}}\,d\phi_{\mathrm{BL}} \\ &+\frac{\Sigma}{\Delta}\,dr^2 \;+\;\Sigma\,d\theta^2\;+\;\Big(r^2+a^2+\frac{2GMa^2r\sin^2\theta}{\Sigma c^2}\Big)\sin^2\theta\,d\phi_{\mathrm{BL}}^2. \end{aligned}} \tag{35}$$

By construction, (35) and (33) are the same geometry written in different (regular vs. orthogonally-separated) slicings, both generated by the rotor (28).

C.5 Horizon, ergosurface, and limits

- **Horizons:** located at $\Delta = 0$, i.e. $r = r_\pm$. In the flow picture, $|v_r| = c$ on the *equator* at $r = r_+$, while $|v_r| < c$ off-equator due to Σ.
- **Ergosurface:** given by $g_{tt} = 0 \;\Leftrightarrow\; 1 - 2GMr/(\Sigma c^2) = 0$; inside, all timelike worldlines co-rotate with the metric medium.
- **Schwarzschild limit:** $a \to 0$ gives $v_r \to -c\sqrt{2GM/(rc^2)}$, $\Omega \to 0$ and reduces (33) to the Schwarzschild PG metric.
- **Asymptotics:** for $r \gg r_+$, $v_r/c \sim \sqrt{2GM/(rc^2)} \ll 1$ and $\Omega \sim a\,\sqrt{2GM/(r^5c^2)}$, recovering flat spacetime with weak Lense–Thirring drag.

C.6 Physical interpretation in the Q_g language

Equation (33) shows the lapse/shift encoded *inside the adjoint*:

$$N = 1, \qquad N^r = \frac{v_r}{c}, \qquad N^\phi = \Omega.$$

The BNC fixes the *first-order* flow profiles (32); the *second-order* Einstein geometry then follows. The azimuthal rapidity ψ_ϕ stores rotational (frame-dragging) energy; the radial rapidity ψ_r stores inflow energy. On the outer horizon, the radial rapidity diverges ($|v_r|/c \to 1$) while ψ_ϕ remains finite, capturing the null character of the horizon with finite spin.

Conclusion. A two-rapidity rotor $Q_g = \exp\!\big[\tfrac{1}{2}\psi_\phi\,\beta_\phi\beta_0\big]\,\exp\!\big[\tfrac{1}{2}\psi_r\,\beta_r\beta_0\big]$ with BNC-determined profiles $v_r = -c\sqrt{2GMr/(\Sigma c^2)}$ and $\Omega = \frac{a}{r^2+a^2}\sqrt{2GMr/(\Sigma c^2)}$ reproduces the Kerr spacetime directly in the horizon-regular Doran (PG-like) form (33), and, after a stationary redefinition, in the Boyer–Lindquist form (35). The Einstein nonlinearity thus emerges from the quadratic projection of a first-order linear rotor transport law.

D Kottler–de Sitter (Schwarzschild–de Sitter) from the Qg Rotor

We derive the Schwarzschild–de Sitter (Kottler) metric directly from the gravitational rotor with a *single radial rapidity* and the Bernoulli–Noether (BNC) transport law. A positive cosmological constant $\Lambda > 0$ corresponds to a constant Hubble scale $H \equiv \sqrt{\Lambda c^2/3}$.

D.1 Rotor ansatz and adjoint (spherical symmetry)

Stationarity and spherical symmetry select a radial boost rotor:

$$Q_g(r) \;=\; \exp\Big[\tfrac{1}{2}\,\psi(r)\,\beta_r\beta_0\Big], \qquad \not{G} \;=\; Q_g\,\beta_0\,Q_g^{-1} = \cosh\psi\,\beta_0 + \sinh\psi\,\beta_r. \tag{36}$$

Define the *metric flow speed* (radial drift of the spacetime medium)

$$v_r(r) \;=\; -\,c\,\tanh\psi(r), \qquad |v_r| < c. \tag{37}$$

D.2 BNC with cosmological term: inflow profile

For a static spherical field with mass M and Λ, the effective Newtonian potential is $\Phi_{\text{eff}}(r) = -GM/r - \frac{1}{6}\,\Lambda c^2 r^2$. The BNC (conservation of specific metric energy along streamlines) with vanishing constant at the reference boundary fixes

$$\frac{1}{2}\,v_r^2 \;=\; \frac{GM}{r} \;+\; \frac{1}{6}\,\Lambda c^2 r^2 \;\Longrightarrow\; \boxed{v_r(r) \;=\; -\sqrt{\frac{2GM}{r} \;+\; H^2 r^2}\,, \qquad H^2 = \frac{\Lambda c^2}{3}\,.} \tag{38}$$

Combining (37) and (38) gives the radial rapidity $\tanh\psi = -v_r/c$.

D.3 PG / Lemaître form from the adjoint

With $d\not{R} = \beta_0\,c\,dt + \beta_r\,dr + \beta_\theta\,r\,d\theta + \beta_\phi\,r\sin\theta\,d\phi$, grade–0 projection of $(\not{G}\cdot d\not{R})^2$ yields the Painlevé–Gullstrand/Lemaître form

$$\boxed{ds^2 \;=\; -\,c^2dt^2 \;+\; \big(dr - v_r(r)\,dt\big)^2 \;+\; r^2 d\Omega^2, \qquad v_r^2 = \frac{2GM}{r} + H^2 r^2\,.} \tag{39}$$

The lapse is $N = 1$ and the radial shift is $N^r = v_r/c$, both encoded inside the adjoint $\not{G} = Q_g\beta_0 Q_g^{-1}$.

D.4 Static Kottler (diagonal) coordinates

Eliminate the cross term by a radius–dependent time shift $t_S = t + f(r)$ with

$$f'(r) \;=\; \frac{v_r(r)}{c^2 - v_r^2(r)} \;=\; -\frac{\sqrt{2GM/r + H^2 r^2}}{c^2 - (2GM/r + H^2 r^2)}. \tag{40}$$

Using (39) one obtains the *static* Kottler form:

$$\boxed{ds^2 \;=\; -\Big(1 - \frac{2GM}{rc^2} - \frac{\Lambda r^2}{3}\Big)c^2 dt_S^2 \;+\; \Big(1 - \frac{2GM}{rc^2} - \frac{\Lambda r^2}{3}\Big)^{-1} dr^2 \;+\; r^2 d\Omega^2\,.} \tag{41}$$

D.5 Checks, horizons, and limits

- **Horizons:** roots of $1 - \frac{2GM}{rc^2} - \frac{\Lambda r^2}{3} = 0$ give the black–hole and cosmological horizons $r_{H,\pm}$. In the PG slicing (39) these are the radii where $|v_r| = c$.
- **Schwarzschild limit:** $\Lambda \to 0 \Rightarrow v_r \to -\sqrt{2GM/r}$, recovering the Schwarzschild PG metric and its diagonal form.
- **de Sitter limit:** $M \to 0 \Rightarrow v_r \to -Hr$, and (41) reduces to the static de Sitter patch.
- **Weak field:** for $r \gg r_{H,+}$, $g_{tt} \simeq -(1 - 2GM/(rc^2) - \Lambda r^2/3)$.

Conclusion. A single radial Q_g rotor $\exp\left[\frac{1}{2}\psi(r)\beta_r\beta_0\right]$ with Bernoulli–Noether profile $v_r^2 = 2GM/r + H^2r^2$ ($H^2 = \Lambda c^2/3$) reproduces the Kottler (Schwarzschild–de Sitter) spacetime. The PG/Lemaître form (39) follows directly from the adjoint $\not{G}$, and a stationary time redefinition yields the static diagonal metric (41). The second–order Einstein geometry thus emerges from the first–order, linear rotor transport of the Q_g field.

E Constructing the Gödel Metric from a Pure Azimuthal Gravitational Rotor

The Gödel spacetime is a stationary, rigidly rotating universe. In canonical cylindrical coordinates (t, r, ϕ, z) it can be written as

$$ds^2 \;=\; -\big(dt + H(r)\,d\phi\big)^2 \;+\; dr^2 \;+\; D^2(r)\,d\phi^2 \;+\; dz^2, \tag{42}$$

with

$$H(r) \;=\; 4\Omega\,\sinh^2\!\Big(\frac{r}{2R}\Big), \qquad D(r) \;=\; R\,\sinh\!\Big(\frac{r}{R}\Big), \qquad R^2 = \frac{1}{2\Omega^2} \;\;(\text{in } c=1). \tag{43}$$

E.1 Rotor ansatz (pure azimuthal rapidity).

In the Q_g framework, Gödel geometry is generated by a *purely azimuthal* gravitational rapidity field acting on the time basis:

$$Q_g(r) \;=\; \exp\Big[\frac{1}{2}\,\psi_\phi(r)\,\beta_\phi\beta_0\Big], \qquad \not{G} \;=\; Q_g\,\beta_0\,Q_g^{-1} \;=\; \cosh\psi_\phi\,\beta_0 + \sinh\psi_\phi\,\beta_\phi. \tag{44}$$

This rotor produces a stationary time-axis tilt into the ϕ-direction (frame dragging) with no radial inflow: $v_r = 0$.

E.2 Recovering the line element from the adjoint.

Using $d\not{R} = \beta_0\,dt + \beta_r\,dr + \beta_\phi\,D(r)\,d\phi + \beta_z\,dz$, the Q_g-induced line element is

$$ds^2 \;=\; -\Big\langle \big(\not{G}\cdot d\not{R}\big)^2 \Big\rangle_0 \;=\; -\big(dt + \tanh\psi_\phi(r)\,D(r)\,d\phi\big)^2 \;+\; dr^2 \;+\; D^2(r)\,d\phi^2 \;+\; dz^2. \tag{45}$$

Matching (42) gives the identification

$$\tanh\psi_\phi(r) \;=\; \frac{H(r)}{D(r)} \;=\; \frac{4\Omega\,\sinh^2(\frac{r}{2R})}{R\,\sinh(\frac{r}{R})}. \tag{46}$$

For small r, $\psi_\phi(r) \approx 2\Omega r$, i.e. a uniform local vorticity.

E.3 Bernoulli–Noether transport.

The flow field is rigid rotation with streamlines $r = \text{const}$, so along those lines

$$v^\mu \not{D}_\mu Q_g \;=\; 0, \qquad v^\mu\partial_\mu \propto \partial_\phi, \tag{47}$$

is satisfied provided $\psi_\phi = \psi_\phi(r)$ is stationary (no ϕ-dependence). Thus Gödel spacetime is a stationary BNC solution with $v_r = 0$, $v_\phi = \text{const}$ on each cylinder.

E.4 Physical content in Q_g language.

Gödel spacetime corresponds to a *vorticity condensate* of the gravitational rotor: uniform azimuthal rapidity, no expansion ($\dot{\psi}_r = 0$), no inflow ($\psi_r = 0$). It is the rotationally homogeneous limit of the same rapidity field that yields Kerr (when $\psi_r \neq 0$) and galactic disks (radial balance with $\psi_r \approx 0$, ψ_ϕ varying).

E.5 Relation to known metrics.

Pure ψ_ϕ reproduces Gödel; mixed (ψ_r, ψ_ϕ) yields Kerr-like rotating inflow; pure $\psi_r(t)$ yields de Sitter expansion. All arise from the same rotor mechanism, with (46) fixing the Gödel parameters.

F FLRW Cosmology from the Qg Rotor Field

We derive the homogeneous–isotropic (FLRW) cosmological metric directly from the gravitational rotor Q_g and the Bernoulli–Noether transport (BNC) condition. We keep c explicit and first present the spatially flat case $k = 0$, then indicate the extension to $k = \pm 1$.

F.1 Symmetry, rotor ansatz, and adjoint

Cosmic homogeneity and isotropy forbid any preferred spatial direction. At each point, the local Hubble flow is purely *radial* with speed proportional to the areal radius. The unique rotor compatible with these symmetries is a time–dependent, direction–averaged radial boost,

$$Q_g(t,r) \;=\; \exp\Big[\frac{1}{2}\,\psi(t,r)\,\beta_r\beta_0\Big], \qquad \not{G} \;=\; Q_g\,\beta_0\,Q_g^{-1} \;=\; \cosh\psi\,\beta_0 + \sinh\psi\,\beta_r. \tag{48}$$

Define the *metric outflow velocity* of space (Hubble flow)

$$v_r(t,r) \;=\; c\,\tanh\psi(t,r), \qquad |v_r| < c. \tag{49}$$

Isotropy then enforces the Hubble law

$$v_r(t,r) \;=\; H(t)\,r, \qquad H(t) = \frac{\dot a(t)}{a(t)}, \tag{50}$$

with $a(t)$ the scale factor (to be recovered below).

F.2 BNC in a homogeneous expanding medium

For a stationary-in-space, time–dependent flow, the BNC reads $v^\mu \not{D}_\mu Q_g = 0$ with $v^\mu\partial_\mu = \partial_t + v_r\partial_r$. With $\psi = \psi(t,r)$ constrained by (50), the transport equation integrates to a rotor whose rapidity is linear in r at fixed t:

$$\tanh\psi(t,r) \;=\; \frac{H(t)\,r}{c}, \qquad \psi(t,r) \;=\; \operatorname{arctanh}\Big(\frac{H(t)\,r}{c}\Big). \tag{51}$$

This encodes the uniform Hubble outflow inside the adjoint $\not{G}$.

F.3 Recovering the line element: PG-like cosmology ($k = 0$)

Use the slashed differential in spherical coframe

$$d\not{R} \;=\; \beta_0\,c\,dt + \beta_r\,dr + \beta_\theta\,r\,d\theta + \beta_\phi\,r\sin\theta\,d\phi.$$

Project the rotated time axis $\not{G}$ of (48) onto $d\not{R}$ and extract the scalar part:

$$ds^2\,\not{1} \;=\; -\Big\langle(\not{G}\cdot d\not{R})^2\Big\rangle_0 \;+\; \Big\langle(\beta_\perp\cdot d\not{R})^2\Big\rangle_0 \;=\; \Big(-\,c^2dt^2 \;+\; (dr - v_r\,dt)^2 \;+\; r^2 d\Omega^2\Big)\,\not{1}, \tag{52}$$

with $v_r = H(t)r$ from (50) and $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$. Equation (52) is a Painlevé–Gullstrand–like form for a spatially flat expanding universe: the lapse is $N = 1$ and the radial shift is $N^r = v_r/c = H(t)r/c$, both encoded inside $\not{G}$.

F.4 Diagonal FLRW form via comoving radius

Introduce the comoving radius χ by

$$r(t,\chi) \;=\; a(t)\,\chi, \qquad dr \;=\; a\,d\chi + \dot a\,\chi\,dt \;=\; a\,d\chi + Hr\,dt. \tag{53}$$

Insert into (52). The cross term cancels:

$$dr - v_r dt \;=\; (a\,d\chi + Hr\,dt) - Hr\,dt \;=\; a\,d\chi,$$

and the metric becomes

$$\boxed{ds^2 \;=\; -\,c^2dt^2 \;+\; a^2(t)\left(d\chi^2 + \chi^2 d\Omega^2\right),} \tag{54}$$

which is the standard spatially flat FLRW line element.

F.5 Including spatial curvature $k = \pm 1$

For constant-curvature spatial slices, replace the areal radius by $r = a(t)\,S_k(\chi)$ with

$$S_k(\chi) = \begin{cases} \sin\chi, & k = +1, \\ \chi, & k = 0, \\ \sinh\chi, & k = -1, \end{cases} \qquad \frac{dr - v_r dt}{a(t)} \;=\; d\chi.$$

The PG-like form generalises to

$$ds^2 \;=\; -\,c^2dt^2 \;+\; \left(dr - Hr\,dt\right)^2 \;+\; r^2 d\Omega^2 \;+\; \mathcal{O}(k)\,, \tag{55}$$

and the diagonal FLRW form follows as

$$\boxed{ds^2 \;=\; -\,c^2dt^2 \;+\; a^2(t)\left[\frac{d\chi^2}{1 - k\,\chi^2} \;+\; \chi^2 d\Omega^2\right],} \tag{56}$$

after the usual constant-curvature identification $d\chi^2/(1 - k\chi^2) \leftrightarrow dS_k^2$.

F.6 Dynamics from the rotor: Friedmann content

The kinematics above are fixed by the rotor $\psi(t,r)$ via $v_r = H(t)r = c\,\tanh\psi$. The *dynamics* (Friedmann equations) arise by projecting the quadratic Q_g-stress onto a symmetric tensor $T_{\mu\nu}[Q_g]$ and using the integrability (Bianchi) identity implied by BNC:

$$\nabla_\mu G^\mu{}_\nu[g(Q_g)] \;=\; 0 \;=\; \nabla_\mu T^\mu{}_\nu[Q_g].$$

For a perfect-fluid projection, this yields the standard relations

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \qquad \dot H = -4\pi G\Big(\rho + \frac{p}{c^2}\Big) + \frac{kc^2}{a^2},$$

with (ρ, p) the effective energy density and pressure extracted from $T_{\mu\nu}[Q_g]$.

F.7 Physical interpretation in the Q_g language

The cosmological expansion is a *time-dependent rotor tilt* of the local time axis encoded in $\not{G}$. The shift $N^r = Hr/c$ represents Hubble outflow, while the lapse $N = 1$ reflects the cosmological proper-time slicing. Spatial curvature k appears as the constant-curvature choice of the spatial coframe, not as an external ingredient: the geometry is recovered from the adjoint action of the same rotor that encodes the flow.

Conclusion. A single time–dependent Q_g rotor with rapidity $\tanh \psi(t, r) = H(t)r/c$ generates the PG-like cosmological metric (52). The comoving transformation $r = a(t)\chi$ removes the shift and yields the standard FLRW line element (54) ($k = 0$) or (56) ($k = \pm 1$). Thus, FLRW cosmology emerges directly from first-order rotor transport, with Friedmann dynamics obtained from the quadratic Q_g-stress and the Bianchi identity implied by BNC.

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