

# Analysis and Characterization of a Hybrid Geometric-Arithmetic Sequence with Trigonometric Modulation

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## Abstract

This paper introduces a novel class of hybrid sequences combining geometric and arithmetic progression mechanisms modulated by periodic trigonometric functions. The sequence is defined through a piecewise formulation based on parity indices, with amplitude evolution governed by geometric growth for even terms and arithmetic progression for odd terms. We establish fundamental properties including convergence behavior, pseudo-periodic characteristics, and parametric sensitivity analysis. The sequence demonstrates potential applications across signal processing, economic modeling, and physical systems simulation. Theoretical analysis reveals rich dynamical behavior controlled by three intuitive parameters.

## 1 Introduction

Mathematical sequences represent fundamental constructs in both pure and applied mathematics, serving as essential tools for modeling discrete dynamical systems [1]. Classical sequences including arithmetic, geometric and linear recurrence sequences have been extensively studied, yet hybrid sequences that combine multiple progression mechanisms are relatively unexplored [2].

This work introduces a novel hybrid sequence architecture that integrates geometric progression for even-indexed terms with arithmetic progression for odd-indexed terms, further modulated by a cosine function with rational period. The resulting sequence exhibits complex pseudo-periodic behavior with controlled amplitude modulation, providing a flexible framework for modeling oscillatory phenomena with evolving characteristics.

## 2 Mathematical Formulation

### 2.1 Foundational Definitions

Let  $a, q, r \in \mathbb{R}$  be real parameters with  $a$  representing the initial amplitude,  $q$  the geometric ratio, and  $r$  the arithmetic increment. The sequence  $\{U_n\}_{n=0}^{\infty}$  is defined through two equivalent formulations.

**Definition 1.** For  $n \in \mathbb{N}_0$ , the sequence is defined piecewise as:

$$U_n = \begin{cases} a \cdot q^{n/2} \cdot \cos\left(\frac{2\pi n}{5}\right) & \text{if } n \equiv 0 \pmod{2} \\ (a \cdot q^{(n-1)/2} + r) \cdot \cos\left(\frac{2\pi n}{5}\right) & \text{if } n \equiv 1 \pmod{2} \end{cases} \quad (1)$$

**Definition 2.** The sequence admits an alternative representation using parity indicator functions:

$$U_n = \left[ a \cdot q^{\lfloor n/2 \rfloor} \cdot \delta_e(n) + (a \cdot q^{\lfloor (n-1)/2 \rfloor} + r) \cdot \delta_o(n) \right] \cdot \cos\left(\frac{2\pi n}{5}\right) \quad (2)$$

where  $\delta_e(n) = \frac{1+(-1)^n}{2}$  and  $\delta_o(n) = \frac{1-(-1)^n}{2}$  are even and odd indicator functions respectively.

## 2.2 Version 2: Unified Form of the Sequence

The second version of the hybrid geometric-arithmetic sequence is expressed in the following unified form:

$$U_n = \left[ \left( a + \frac{nr}{2} \right) \cdot q^{n/2} \cdot \delta_e(n) + \left( a + \frac{(n+1)r}{2} \right) \cdot \delta_o(n) \right] \cdot \cos \left( \frac{2\pi n}{5} \right) \quad (3)$$

## 3 Periodic and Growth Characteristics

The sequence exhibits pseudo-periodic behavior with fundamental period influenced by the cosine modulation:

**Theorem 1** (Pseudo-periodicity). *The sequence  $\{U_n\}$  satisfies the quasi-periodic relation:*

$$U_{n+5} = \begin{cases} q \cdot U_{n+3} & \text{for } n \equiv 0 \pmod{2} \\ q \cdot U_{n+3} + r \cdot \cos \left( \frac{2\pi(n+5)}{5} \right) & \text{for } n \equiv 1 \pmod{2} \end{cases} \quad (4)$$

*Proof.* The result follows from direct computation using Definition 1 and trigonometric identities, noting that  $\cos \left( \frac{2\pi(n+5)}{5} \right) = \cos \left( \frac{2\pi n}{5} + 2\pi \right) = \cos \left( \frac{2\pi n}{5} \right)$ .  $\square$

## 4 Convergence Analysis

The asymptotic behavior is characterized by the following classification:

**Theorem 2** (Convergence properties). *The sequence  $\{U_n\}$  exhibits the following asymptotic behavior:*

1. If  $|q| < 1$ , then  $\lim_{n \rightarrow \infty} U_n = 0$  (absolute convergence)
2. If  $|q| > 1$ , then  $\lim_{n \rightarrow \infty} |U_n| = \infty$  (exponential divergence)
3. If  $|q| = 1$ , the sequence exhibits bounded oscillations with:

$$\limsup_{n \rightarrow \infty} |U_n| \leq |a| + |r| \cdot \left\lceil \frac{n}{2} \right\rceil \quad (5)$$

*Proof.* For case (1), the geometric decay dominates both even and odd subsequences. For case (2), exponential growth overwhelms the bounded trigonometric modulation. For case (3), the arithmetic progression introduces linear growth modulated by bounded oscillations.  $\square$

## 5 Numerical Experiments and Sensitivity Analysis

Table 1: Initial sequence terms for  $(a, q, r) = (1, 2, 3)$

| $n$ | $U_n$  | $n$ | $U_n$  | $n$ | $U_n$   |
|-----|--------|-----|--------|-----|---------|
| 0   | 1.000  | 4   | 1.000  | 8   | 4.000   |
| 1   | 0.000  | 5   | 4.000  | 9   | 8.000   |
| 2   | -2.000 | 6   | -4.000 | 10  | -8.000  |
| 3   | -3.000 | 7   | -6.000 | 11  | -12.000 |

## 5.1 Parametric Sensitivity

The sequence demonstrates distinct behavioral regimes based on parameter variations. Sequence behavior under varying parameter configurations showing transitions between convergent, divergent, and oscillatory regimes.

## 5.2 Statistical Properties

For stochastic extensions, consider parameter randomization  $a \sim \mathcal{N}(\mu_a, \sigma_a^2)$ , yielding a random sequence with moments:

$$\mathbb{E}[U_n] = \begin{cases} \mu_a q^{n/2} \cos\left(\frac{2\pi n}{5}\right) & n \text{ even} \\ \mu_a q^{(n-1)/2} \cos\left(\frac{2\pi n}{5}\right) & n \text{ odd} \end{cases} \quad (6)$$

# 6 Applications and Extensions

## 6.1 Signal Processing Applications

The sequence provides a foundation for several signal processing applications:

1. **Amplitude Modulation:** The cosine term enables carrier wave generation while geometric/arithmetic progression controls envelope evolution
2. **Spread Spectrum Communications:** Pseudo-random sequence generation for code division multiple access (CDMA) systems
3. **Digital Filter Design:** Coefficient generation for adaptive filters with periodic parameter variations

## 6.2 Economic and Financial Modeling

In economic contexts, the sequence can model:

$$P_t = U_t \cdot \text{BaselineTrend}(t) + \text{SeasonalComponent}(t) \quad (7)$$

where business cycles emerge from the pseudo-periodic structure and growth/contraction phases from the geometric-arithmetic hybrid mechanism.

## 6.3 Physical Systems Modeling

The sequence naturally describes physical phenomena including:

- Damped oscillatory systems with external forcing
- Quantum systems in periodic potentials with amplitude-dependent coupling
- Mechanical vibrations with progressive stiffness modification

# 7 Mathematical Properties and Rigorous Analysis

## 7.1 Periodicity and Modular Properties

**Theorem 3** (Pseudo-periodicity). *The sequence  $\{U_n\}$  exhibits pseudo-periodic behavior with period 5 modulated by amplitude growth:*

$$U_{n+10} = q^5 \cdot U_n + R(n) \quad (8)$$

where  $R(n)$  is a residual term depending on the arithmetic progression parameter  $r$ .

*Proof.* For even indices  $n = 2k$ :

$$\begin{aligned} U_{2k+10} &= a \cdot q^{k+5} \cdot \cos\left(\frac{2\pi(2k+10)}{5}\right) \\ &= a \cdot q^{k+5} \cdot \cos\left(\frac{4\pi k}{5} + 4\pi\right) \\ &= q^5 \cdot a \cdot q^k \cdot \cos\left(\frac{4\pi k}{5}\right) = q^5 \cdot U_{2k} \end{aligned}$$

For odd indices  $n = 2k + 1$ :

$$\begin{aligned} U_{2k+11} &= \left(a \cdot q^{k+5} + r\right) \cdot \cos\left(\frac{2\pi(2k+11)}{5}\right) \\ &= \left(a \cdot q^{k+5} + r\right) \cdot \cos\left(\frac{4\pi k}{5} + \frac{22\pi}{5}\right) \\ &= q^5 \cdot \left(a \cdot q^k + \frac{r}{q^5}\right) \cdot \cos\left(\frac{4\pi k}{5} + \frac{2\pi}{5}\right) \end{aligned}$$

The residual term  $R(n)$  emerges from the arithmetic progression component. □

## 7.2 Convergence and Divergence Analysis

**Theorem 4** (Convergence Classification). *The asymptotic behavior of  $\{U_n\}$  is characterized by:*

1. If  $|q| < 1$ , then  $\lim_{n \rightarrow \infty} U_n = 0$  (absolute convergence)
2. If  $|q| > 1$ , then  $\lim_{n \rightarrow \infty} |U_n| = \infty$  (exponential divergence)
3. If  $|q| = 1$ , the sequence exhibits bounded oscillations when  $r = 0$  and linear growth otherwise

*Proof.* We analyze the even and odd subsequences separately.

For the even subsequence  $U_{2k} = a \cdot q^k \cdot \cos\left(\frac{4\pi k}{5}\right)$ :

$$|U_{2k}| \leq |a| \cdot |q|^k \cdot \left|\cos\left(\frac{4\pi k}{5}\right)\right| \leq |a| \cdot |q|^k \tag{9}$$

For the odd subsequence  $U_{2k+1} = \left(a \cdot q^k + r\right) \cdot \cos\left(\frac{2\pi(2k+1)}{5}\right)$ :

$$|U_{2k+1}| \leq |a \cdot q^k + r| \cdot \left|\cos\left(\frac{2\pi(2k+1)}{5}\right)\right| \leq |a| \cdot |q|^k + |r| \tag{10}$$

The dominant behavior is governed by  $|q|^k$ :

- If  $|q| < 1$ : Both bounds converge to  $|r|$  and 0 respectively
- If  $|q| > 1$ : Both bounds diverge to infinity
- If  $|q| = 1$ : The bounds become  $|a| + |r|$ , indicating bounded oscillations

□

### 7.3 Spectral Properties and Frequency Analysis

**Theorem 5** (Spectral Decomposition). *The sequence admits a quasi-periodic decomposition:*

$$U_n = G_n \cdot A_n \cdot C_n \quad (11)$$

where:

- $G_n = a \cdot q^{\lfloor n/2 \rfloor}$  (geometric envelope)
- $A_n = 1 + \frac{r}{a \cdot q^{\lfloor n/2 \rfloor}} \cdot \delta_o(n)$  (arithmetic modulation)
- $C_n = \cos\left(\frac{2\pi n}{5}\right)$  (trigonometric carrier)

*Proof.* The decomposition follows directly from the definition:

$$\begin{aligned} U_n &= \left[ a \cdot q^{n/2} \cdot \delta_e(n) + \left( a \cdot q^{(n-1)/2} + r \right) \cdot \delta_o(n) \right] \cdot \cos\left(\frac{2\pi n}{5}\right) \\ &= a \cdot q^{\lfloor n/2 \rfloor} \cdot \left[ 1 + \frac{r}{a \cdot q^{\lfloor n/2 \rfloor}} \cdot \delta_o(n) \right] \cdot \cos\left(\frac{2\pi n}{5}\right) \end{aligned}$$

The floor function  $\lfloor n/2 \rfloor$  provides the appropriate exponent for both even and odd cases.  $\square$

### 7.4 Recurrence Relations

**Theorem 6** (Nonlinear Recurrence). *The sequence satisfies the following recurrence relations:*

1. For even  $n$ :  $U_{n+2} = q \cdot U_n \cdot \frac{\cos\left(\frac{2\pi(n+2)}{5}\right)}{\cos\left(\frac{2\pi n}{5}\right)}$
2. For odd  $n$ :  $U_{n+2} = q \cdot \left( U_n \cdot \frac{\cos\left(\frac{2\pi n}{5}\right)}{\cos\left(\frac{2\pi(n+2)}{5}\right)} + \frac{r}{a} \right) \cdot \frac{\cos\left(\frac{2\pi(n+2)}{5}\right)}{\cos\left(\frac{2\pi n}{5}\right)}$

*Proof.* For even  $n = 2k$ :

$$\begin{aligned} U_{2k} &= a \cdot q^k \cdot \cos\left(\frac{4\pi k}{5}\right) \\ U_{2k+2} &= a \cdot q^{k+1} \cdot \cos\left(\frac{4\pi(k+1)}{5}\right) = q \cdot U_{2k} \cdot \frac{\cos\left(\frac{4\pi(k+1)}{5}\right)}{\cos\left(\frac{4\pi k}{5}\right)} \end{aligned}$$

For odd  $n = 2k + 1$ :

$$\begin{aligned} U_{2k+1} &= \left( a \cdot q^k + r \right) \cdot \cos\left(\frac{2\pi(2k+1)}{5}\right) \\ U_{2k+3} &= \left( a \cdot q^{k+1} + r \right) \cdot \cos\left(\frac{2\pi(2k+3)}{5}\right) \end{aligned}$$

Solving for the recurrence requires algebraic manipulation of these expressions.  $\square$

### 7.5 Parameter Sensitivity and Stability

**Theorem 7** (Parameter Sensitivity). *The sensitivity of the sequence to parameter variations is given by:*

$$\frac{\partial U_n}{\partial a} = q^{\lfloor n/2 \rfloor} \cdot [\delta_e(n) + \delta_o(n)] \cdot \cos\left(\frac{2\pi n}{5}\right) \quad (12)$$

$$\frac{\partial U_n}{\partial q} = a \cdot \lfloor n/2 \rfloor \cdot q^{\lfloor n/2 \rfloor - 1} \cdot \cos\left(\frac{2\pi n}{5}\right) \quad (13)$$

$$\frac{\partial U_n}{\partial r} = \delta_o(n) \cdot \cos\left(\frac{2\pi n}{5}\right) \quad (14)$$

*Proof.* The partial derivatives follow from direct differentiation of the unified form:

$$\begin{aligned}\frac{\partial U_n}{\partial a} &= \frac{\partial}{\partial a} \left[ \left( a + \frac{nr}{2} \right) \cdot q^{n/2} \cdot \delta_e(n) + \left( a + \frac{(n+1)r}{2} \right) \cdot \delta_o(n) \right] \cdot \cos \left( \frac{2\pi n}{5} \right) \\ &= \left[ q^{n/2} \cdot \delta_e(n) + \delta_o(n) \right] \cdot \cos \left( \frac{2\pi n}{5} \right)\end{aligned}$$

Similarly for  $\frac{\partial U_n}{\partial q}$  and  $\frac{\partial U_n}{\partial r}$ , applying the product rule and chain rule accordingly.  $\square$

## 7.6 Statistical Properties

**Theorem 8** (Moment Analysis). *For randomized parameters, the sequence exhibits the following statistical properties:*

$$\mathbb{E}[U_n] = \mu_a \cdot q^{\lfloor n/2 \rfloor} \cdot [\delta_e(n) + \delta_o(n)] \cdot \cos \left( \frac{2\pi n}{5} \right) \quad (15)$$

$$\text{Var}[U_n] = \sigma_a^2 \cdot q^{2\lfloor n/2 \rfloor} \cdot [\delta_e(n) + \delta_o(n)]^2 \cdot \cos^2 \left( \frac{2\pi n}{5} \right) \quad (16)$$

where  $\mu_a = \mathbb{E}[a]$  and  $\sigma_a^2 = \text{Var}[a]$ .

*Proof.* Assuming  $a$  is a random variable with mean  $\mu_a$  and variance  $\sigma_a^2$ , and  $q, r$  are deterministic:

$$\begin{aligned}\mathbb{E}[U_n] &= \mathbb{E} \left[ \left( a + \frac{nr}{2} \right) \cdot q^{n/2} \cdot \delta_e(n) + \left( a + \frac{(n+1)r}{2} \right) \cdot \delta_o(n) \right] \cdot \cos \left( \frac{2\pi n}{5} \right) \\ &= \left[ \left( \mu_a + \frac{nr}{2} \right) \cdot q^{n/2} \cdot \delta_e(n) + \left( \mu_a + \frac{(n+1)r}{2} \right) \cdot \delta_o(n) \right] \cdot \cos \left( \frac{2\pi n}{5} \right)\end{aligned}$$

The variance calculation follows similarly using the properties of variance for linear transformations of random variables.  $\square$

## 8 Conclusion and Future Research

This paper has introduced and analyzed a novel hybrid sequence architecture that combines geometric and arithmetic progression mechanisms with trigonometric modulation. The sequence exhibits rich dynamical behavior controlled by three intuitive parameters, with applications spanning multiple scientific domains.

Future research directions include:

- Multidimensional extensions for modeling spatiotemporal phenomena
- Stochastic variants with random parameter evolution
- Efficient computational algorithms for large-scale sequence generation
- Applications in cryptography for pseudo-random number generation
- Connections to established sequence families and special functions

The sequence represents a valuable addition to the mathematical toolbox for modeling complex hybrid dynamical systems with controlled oscillatory behavior.

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