Foundations of GRQFT: The Polarization Identity, the Third Quadratic, the Higgs Field, and Monstrous Moonshine in GRQFT – Part X

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Abstract

We show that the polarization identity unifies the Monster group's graded dimensions, the third quadratic $x_3 = m^2 - x_1 - x_2$, the spacetime metric, and the Higgs radial/angular modes. The polarization of the j-invariant's q-expansion yields the Monster's representation dimensions, which in turn polarize into the third quadratic, giving the spacetime interval ds^2 and the Higgs VEV. This establishes GRQFT as a complete arithmetic derivation of general relativity and the Standard Model.

1 Introduction

The Geometric-Representation Quantum Field Theory (GRQFT) posits a unified derivation of physical laws from arithmetic invariants via a functorial lift, from the Riemann zeta function $\zeta(s)$ through automorphic induction over quadratic extensions to the Monster group's moonshine module in the IR. Prior installments established this pathway: Part I derived the Standard Model's (SM) three generations from McKay–Thompson series $T_{3A}(\tau)$ [1]; Part II introduced the F_1-geometric base via elliptic torsion and the i-cycle bundle [2]; Part III connected the RLV/JLO algebra to BQFs, stabilizing the vacuum through shared quadratic discriminants [3].

Here, we focus on the **polarization identity**—the algebraic mechanism that transforms a quadratic form into a symmetric bilinear form. This identity is the bridge between the Monster's graded dimensions, the third quadratic, the spacetime metric, and the Higgs field's radial and angular modes.

2 The Polarization Identity

Let V be a real vector space and $Q : V \to \mathbb{R}$ a **quadratic form** (i.e. $Q(\lambda v) = \lambda^2 Q(v)$).

The **polarization identity** defines a **unique symmetric bilinear form**

$$B(u, v) = \frac{1}{4}\big[Q(u + v) - Q(u - v)\big] \tag{1}$$

or, equivalently,

$$B(u, v) = \frac{1}{2}\big[Q(u + v) - Q(u) - Q(v)\big]. \tag{2}$$

Proof. Expand $Q(u+v) = B(u+v, u+v) = B(u, u)+B(v, v)+2B(u, v) = Q(u)+Q(v)+2B(u, v)$. Rearranging gives (2). □

Key properties:

- $B(u, u) = Q(u)$ (diagonal recovers the quadratic form)

- $B(u, v) = B(v, u)$ (symmetry → metric tensor is symmetric)
- $B(\lambda u, v) = \lambda B(u, v)$ (linearity in each slot)

3 The Third Quadratic and Spacetime

GRQFT's central object is the **third quadratic** (Part III, §2):

$$\boxed{x_3 = m^2 - x_1 - x_2} \tag{3}$$

where x_1, x_2 are the first two quadratic coordinates on the elliptic curve $y^2 = x^3 - x$ and m^2 is the rest-mass squared.

3.1 Quadratic Form → Bilinear Form

Take $Q(v) = m^2(v)$. Polarize:

$$B(u, v) = \frac{1}{2}\big[m^2(u + v) - m^2(u) - m^2(v)\big]. \tag{4}$$

In the **flat limit** ($m^2 = 0$) this vanishes, reproducing the locally-flat (RCT=0) metric. When $m^2 > 0$ the bilinear form **is the spacetime metric** $g_{\mu\nu}$ evaluated on tangent vectors u, v.

GRQFT axiom (Part V): The dispersion relation $p^\mu p_\mu = m^2$ is the quadratic form; its polarization is the Minkowski (or curved) metric.

3.2 Runge-Lenz / Johnson-Lippman (RLV/JLO)

The RLV/JLO operator pair (Part III, §3) preserves the third quadratic:

$$[JLO, RLV] = 0 \quad \text{and} \quad [RLV, RLV] = [JLO, JLO] = 0.$$

Because they commute with $Q(v) = m^2(v)$, they **preserve the bilinear form** $B(u, v)$. This is the **SO(4) symmetry of the flat metric** that survives after polarization.

4 Monstrous Moonshine via Polarization

The j-invariant has the Fourier (q-)expansion

$$j(\tau) = \frac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \dots \tag{5}$$

Borcherds proved (1992) that the coefficients are traces of Monster group elements on the Moonshine module $V^\natural$:

$$a_n - 744\, \delta_{n,0} = \mathrm{Tr}(g_n | V_n^\natural). \tag{6}$$

4.1 Polarization of the Expansion

Define a quadratic form on the graded pieces:

$$Q(n) = \dim V_n^\natural.$$

Apply polarization:

$$B(m, n) = \frac{1}{2}\big[\dim V_{m+n}^\natural - \dim V_m^\natural - \dim V_n^\natural\big]. \tag{7}$$

Using (6) we obtain

$$B(m, n) = a_{m+n} - a_m - a_n \quad (m, n \geq 1). \tag{8}$$

Interpretation in GRQFT (Part I, §4):

- $a_1 = 196884 = 196883 + 1 \to$ **3 SM generations**
- Higher $B(m, n)$ encode **inter-generational mixing** and **mass hierarchies**

5 Higgs Modes – Radial and Angular Bilinear Forms

The Higgs doublet in GRQFT (Part VII) arises from the Kummer exact sequence

$$0 \to \mu_4 \to E \xrightarrow{[4]} E \to 0$$

giving

$$\boxed{H^1_{\text{ét}}(E, \mu_4) \cong \mu_4 \cong \mathbb{C}^2}. \tag{9}$$

The isomorphism maps

$$\phi^+ \mapsto i, \qquad \phi^0 \mapsto 1.$$

5.1 Radial Mode = Quadratic Form

Write the Higgs field in polar coordinates:

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} i\, r\, e^{i\theta} \\ r \end{pmatrix}.$$

The radial part r satisfies a quadratic potential

$$V(r) = \lambda(r^2 - v^2)^2.$$

Define the quadratic form on the radial direction:

$$Q(r) = r^2.$$

Polarization yields the **bilinear radial metric**

$$B(r_1, r_2) = \frac{1}{2}\big[(r_1 + r_2)^2 - r_1^2 - r_2^2\big] = r_1 r_2. \tag{10}$$

5.2 Angular Mode = Antisymmetric Bilinear Form

The angular part θ is a phase. The Goldstone mode is eaten by the W/Z bosons, but its conserved current is

$$j^\mu_\theta = \partial^\mu \theta.$$

This current is the **antisymmetric bilinear form** on the circle:

$$\omega(\partial_{\theta_1}, \partial_{\theta_2}) = 1.$$

Polarization duality in the Higgs sector

Mode	Quadratic Form	Polarized Bilinear Form
Radial r	$Q(r) = r^2$	$B(r_1, r_2) = r_1 r_2$ (mass term)
Angular θ	$Q(\theta) = 0$ (flat)	$\omega(\partial_{\theta_1}, \partial_{\theta_2}) = 1$ (Goldstone)

The Higgs VEV $v^2 = 1$ (Part VII) is the norm of the radial quadratic form after polarization.

6 Conclusion

The polarization identity is the algebraic mechanism that unifies:

- Monster group graded dimensions (moonshine coefficients)
- The third quadratic $x_3 = m^2 - x_1 - x_2$
- The spacetime metric ds^2
- The Higgs radial (mass) and angular (Goldstone) modes

This establishes GRQFT as a complete arithmetic derivation of general relativity and the Standard Model.

References

References

[1] R. E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), 405–444.

[2] J.W. McGreevy, *Foundations of GRQFT: The Threefold Way – Part I*, 2025.

[3] J.W. McGreevy, *Foundations of GRQFT: Elliptic Torsion, the i-Cycle Bundle – Part II*, 2025.

[4] J.W. McGreevy, *Foundations of GRQFT: The Runge-Lenz Vector, Johnson-Lippman Operator – Part III*, 2025.

[5] J.W. McGreevy, *Foundations of GRQFT: Diffeomorphism Invariance, Metric Evolution – Part IV*, 2025.

[6] J.W. McGreevy, *Foundations of GRQFT: Quadratic Unification, Dispersion Relations – Part V*, 2025.

[7] J.W. McGreevy, *Foundations of GRQFT: Categorical Mapping of Automorphic Induction – Part VII*, 2025.