Rapidity Curvature: A New Lorentz-Invariant Scalar in Special Relativity

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Abstract

We introduce a new Lorentz-invariant scalar in special relativity, the *rapidity curvature*, which quantifies the rate of change of rapidity along a timelike worldline. This scalar is defined via the hyperbolic angle between neighboring 4-velocities and provides a coordinate-independent measure of non-inertial motion. We present its exact definition, prove Lorentz invariance, and derive its series expansion in terms of proper acceleration and higher derivatives. Its geometric interpretation as the curvature of the worldline in velocity hyperboloid space is discussed, and examples including uniform acceleration and circular motion are provided. Potential extensions to curved spacetime and variational formulations are outlined, highlighting the theoretical relevance of rapidity curvature in relativistic physics.

1 Introduction

Lorentz-invariant scalars play a central role in understanding the kinematics and dynamics of particles in special relativity [1, 2]. Commonly studied invariants include the proper-time interval along a worldline and the magnitude of the 4-acceleration. These quantities are fundamental for describing motion in Minkowski spacetime and for constructing physically meaningful observables.

Rapidity, defined as the hyperbolic angle corresponding to relative velocity, provides a natural measure for composing velocities and analyzing motion in Minkowski spacetime [2]. While the rapidity between two inertial frames is a well-known concept, there has been no Lorentz-invariant scalar that quantifies the local *rate of change of rapidity* along a non-inertial worldline.

In this paper, we introduce the *rapidity curvature*, a new Lorentz-invariant scalar that measures precisely this quantity. It is defined via the hyperbolic angle between neighboring 4-velocities along a timelike worldline and can be interpreted as the curvature of the trajectory in the velocity hyperboloid.

We systematically study its properties: after defining it rigorously and proving its invariance under Lorentz transformations, we derive a series expansion in terms of proper acceleration and higher derivatives, provide a geometric interpretation, and illustrate it with examples such as uniform acceleration and circular motion. We also discuss potential extensions to curved spacetime and variational principles. The rapidity curvature complements existing invariants and provides a new tool for analyzing non-inertial relativistic motion.

2 Kinematic Preliminaries

Consider a timelike worldline $x^\mu(\tau)$ parametrized by proper time τ in Minkowski spacetime. The *4-velocity* is defined as

$$u^\mu = \frac{dx^\mu}{d\tau}, \tag{1}$$

satisfying the normalization condition

$$u^\mu u_\mu = -1. \tag{2}$$

The *4-acceleration* is the proper-time derivative of the 4-velocity:

$$a^\mu = \frac{du^\mu}{d\tau}, \tag{3}$$

which is orthogonal to the 4-velocity,

$$u^\mu a_\mu = 0. \tag{4}$$

The magnitude of the 4-acceleration $\sqrt{a^\mu a_\mu}$ corresponds to the usual proper acceleration experienced by the particle.

Higher derivatives of the worldline can also be introduced. The *4-jerk* is defined as

$$j^\mu = \frac{da^\mu}{d\tau} = \frac{d^2 u^\mu}{d\tau^2}, \tag{5}$$

and can be used to compute higher-order corrections to quantities derived from the 4-acceleration.

These objects — u^μ, a^μ, j^μ — form the basis for analyzing the local kinematics of the worldline and will be used to define the rapidity curvature and its series expansion.

3 Definition of Rapidity Curvature

3.1 Finite Proper-Time Increment

For two points on a timelike worldline separated by a proper-time interval $\Delta\tau$, we define the *rapidity increment* $\Delta\eta$ between the corresponding 4-velocities as

$$\cosh(\Delta\eta) = -u^\mu(\tau)\, u_\mu(\tau + \Delta\tau). \tag{6}$$

The *finite rapidity curvature* is then defined by

$$\rho_{\text{finite}}(\tau, \Delta\tau) = \frac{\Delta\eta}{\Delta\tau}. \tag{7}$$

This quantity provides a coordinate-independent measure of the rate at which the particle's 4-velocity changes along its worldline.

3.2 Differential Limit

The instantaneous *rapidity curvature* is obtained in the limit of infinitesimal proper-time separation:

$$\boxed{\rho(\tau) = \lim_{\Delta\tau \to 0} \frac{\Delta\eta}{\Delta\tau} = \sqrt{a^\mu a_\mu}.} \tag{8}$$

Here, $a^\mu = du^\mu/d\tau$ is the 4-acceleration. This shows that locally, the rapidity curvature coincides with the magnitude of the proper acceleration, providing a direct geometric and physical interpretation.

4 Lorentz Invariance

The Minkowski inner product of two 4-velocities is invariant under Lorentz transformations:

$$u'^{\mu} = \Lambda^{\mu}{}_{\nu} u^{\nu} \quad \Rightarrow \quad -u'(\tau) \cdot u'(\tau + \Delta\tau) = -u(\tau) \cdot u(\tau + \Delta\tau), \tag{9}$$

where $\Lambda^{\mu}{}_{\nu}$ is any Lorentz transformation and the dot denotes the Minkowski inner product $u \cdot v = \eta_{\mu\nu} u^{\mu} v^{\nu}$.

Since the finite rapidity increment is defined via

$$\cosh(\Delta\eta) = -u(\tau) \cdot u(\tau + \Delta\tau), \tag{10}$$

it follows that $\Delta\eta$ is Lorentz-invariant. Consequently, both the finite rapidity curvature

$$\rho_{\text{finite}} = \frac{\Delta\eta}{\Delta\tau} \tag{11}$$

and the instantaneous rapidity curvature

$$\rho(\tau) = \lim_{\Delta\tau \to 0} \frac{\Delta\eta}{\Delta\tau} = \sqrt{a^{\mu} a_{\mu}} \tag{12}$$

are manifestly Lorentz scalars.

This confirms that the rapidity curvature is a physically meaningful invariant, independent of the inertial frame used to describe the motion [2].

5 Series Expansion of Rapidity Curvature

To analyze the local behavior of the rapidity curvature, we expand the 4-velocity in a Taylor series about proper time τ:

$$u^{\mu}(\tau + \Delta\tau) = u^{\mu}(\tau) + a^{\mu}(\tau)\,\Delta\tau + \frac{1}{2} j^{\mu}(\tau)\,\Delta\tau^2 + \frac{1}{6} s^{\mu}(\tau)\,\Delta\tau^3 + \ldots, \tag{13}$$

where $a^{\mu} = du^{\mu}/d\tau$, $j^{\mu} = da^{\mu}/d\tau$, and $s^{\mu} = dj^{\mu}/d\tau$ is the 4-snap.

Using the definition of the rapidity increment,

$$-u(\tau) \cdot u(\tau + \Delta\tau) = 1 + \frac{1}{2}(a \cdot a)\Delta\tau^2 + \frac{1}{6}(a \cdot j)\Delta\tau^3 + O(\Delta\tau^4), \tag{14}$$

where the dot denotes the Minkowski inner product.

The instantaneous rapidity curvature is

$$\rho(\tau) = \lim_{\Delta\tau \to 0} \frac{\operatorname{arcosh}[-u(\tau) \cdot u(\tau + \Delta\tau)]}{\Delta\tau}. \tag{15}$$

Using the expansion

$$\operatorname{arcosh}(1 + x) = \sqrt{2x} - \frac{\sqrt{2}}{12} x^{3/2} + O(x^{5/2}), \tag{16}$$

we find

$$\boxed{\rho(\tau) = \sqrt{a \cdot a}\left[1 + \frac{1}{6}\frac{a \cdot j}{(a \cdot a)^{3/2}} + O\left(\frac{j^2}{a^4}\right)\right].} \tag{17}$$

This expression shows that, to leading order, the rapidity curvature coincides with the proper acceleration, while higher-order terms involve derivatives of the acceleration (jerk, snap, etc.). This expansion provides a systematic way to compute corrections for non-uniformly accelerated motion.

6 Geometric Interpretation in Velocity Space

The 4-velocity of a particle, $u^\mu(\tau)$, lies on the unit hyperboloid

$$H^3 = \{u^\mu \in \mathbb{R}^{1,3} \,|\, u^\mu u_\mu = -1\} \tag{18}$$

embedded in Minkowski spacetime. The Minkowski metric induces a natural metric on the hyperboloid:

$$ds_H^2 = du^\mu du_\mu. \tag{19}$$

The rapidity curvature $\rho(\tau)$ can be interpreted as the *curvature of the curve* $u^\mu(\tau) \subset H^3$ with respect to proper time. The infinitesimal hyperbolic distance along the 4-velocity curve is

$$ds_H = \rho(\tau)\, d\tau. \tag{20}$$

This is equivalent to the first Frenet–Serret curvature in Minkowski space:

$$\kappa_1(\tau) = \sqrt{a^\mu a_\mu} = \rho(\tau), \tag{21}$$

showing that the rapidity curvature coincides with the geometric curvature of the worldline in velocity space.

Thus, rapidity curvature provides a direct geometric measure of how sharply the 4-velocity is changing direction in Minkowski spacetime. It generalizes the notion of proper acceleration to a hyperbolic "velocity-space" picture, emphasizing the natural hyperbolic geometry of special-relativistic kinematics [2].

7 Global Rapidity Distance

The rapidity curvature can be used to define a global measure of the change in 4-velocity along a timelike worldline. For two proper-time points τ_1 and τ_2, we define the total rapidity distance as

$$\Delta\eta_{12} = \int_{\tau_1}^{\tau_2} \rho(\tau)\, d\tau. \tag{22}$$

Equivalently, using the Minkowski inner product of the 4-velocities at the endpoints, we have

$$\cosh(\Delta\eta_{12}) = -u^\mu(\tau_1)\, u_\mu(\tau_2). \tag{23}$$

This establishes a direct connection between the local curvature of the 4-velocity (measured by $\rho(\tau)$) and the global hyperbolic distance in velocity space. The total rapidity distance represents the accumulated "hyperbolic rotation" of the 4-velocity along the worldline and is Lorentz-invariant.

In this sense, $\rho(\tau)$ acts as a local density for the global hyperbolic displacement in velocity space, providing a natural bridge between local kinematics and integrated motion in Minkowski spacetime.

8 Applications and Examples

8.1 Uniform Proper Acceleration

Consider a particle undergoing constant proper acceleration a along one spatial direction. Its worldline in Minkowski spacetime is

$$x^\mu(\tau) = \frac{1}{a}\big(\sinh(a\tau), \cosh(a\tau), 0, 0\big). \tag{24}$$

The 4-velocity is

$$u^\mu = \frac{dx^\mu}{d\tau} = (\cosh(a\tau), \sinh(a\tau), 0, 0), \tag{25}$$

and the 4-acceleration is

$$a^\mu = \frac{du^\mu}{d\tau} = a(\sinh(a\tau), \cosh(a\tau), 0, 0). \tag{26}$$

The rapidity curvature is

$$\rho = \sqrt{a^\mu a_\mu} = a, \tag{27}$$

as expected for constant proper acceleration.

8.2 Uniform Circular Motion

For uniform circular motion of radius R in the x-y plane with constant speed v, the 4-velocity is

$$u^\mu = \gamma\big(1, v\cos(\omega t), v\sin(\omega t), 0\big), \quad \gamma = \frac{1}{\sqrt{1-v^2}}, \quad \omega = \frac{v}{R}. \tag{28}$$

The 4-acceleration magnitude is

$$\rho = \sqrt{a^\mu a_\mu} = \gamma^2 \frac{v^2}{R}, \tag{29}$$

which reduces to the familiar centripetal acceleration in the non-relativistic limit.

8.3 Arbitrary 3D Motion

For a general 3-dimensional velocity $\mathbf{v}(t)$, the 4-velocity is

$$u^\mu = \gamma(1, \mathbf{v}), \qquad \gamma = \frac{1}{\sqrt{1-|\mathbf{v}|^2}}. \tag{30}$$

The rapidity curvature can be expressed in terms of the 3-acceleration $\dot{\mathbf{v}}$ and the component perpendicular to velocity:

$$\rho = \gamma^3 \sqrt{|\dot{\mathbf{v}}|^2 - |\mathbf{v} \times \dot{\mathbf{v}}|^2}. \tag{31}$$

These examples illustrate that rapidity curvature reduces to familiar accelerations in simple cases while providing a general, Lorentz-invariant measure for arbitrary motion.

9 Covariant Generalization to Curved Spacetime

The concept of rapidity curvature can be extended to a curved spacetime $(\mathcal{M}, g_{\mu\nu})$ by using parallel transport to compare 4-velocities at different points along a timelike worldline.

Let $x^\mu(\tau)$ be a worldline with 4-velocity $u^\mu(\tau) = dx^\mu/d\tau$. To define the rapidity increment between τ and $\tau+\Delta\tau$, we parallel-transport the 4-velocity $u^\mu(\tau+\Delta\tau)$ back to the point τ along the worldline:

$$\tilde{u}^\mu(\tau+\Delta\tau) = P^\mu{}_\nu(\tau+\Delta\tau \to \tau)\, u^\nu(\tau+\Delta\tau), \tag{32}$$

where $P^\mu{}_\nu$ is the parallel transport operator satisfying

$$\frac{D\tilde{u}^\mu}{d\tau} = 0, \quad \tilde{u}^\mu(\tau+\Delta\tau) = u^\mu(\tau+\Delta\tau) \text{ at } \tau+\Delta\tau. \tag{33}$$

The covariant rapidity curvature is then defined as

$$\boxed{\rho(\tau) = \lim_{\Delta\tau\to 0} \frac{1}{\Delta\tau} \operatorname{arcosh}\Big[-g_{\mu\nu}(x(\tau))\, u^\mu(\tau)\, \tilde{u}^\nu(\tau+\Delta\tau)\Big]}. \tag{34}$$

In the local inertial frame, this reduces to the special relativistic definition. This formulation ensures that $\rho(\tau)$ remains a scalar under general coordinate transformations, providing a fully covariant generalization of rapidity curvature. It can be used to study accelerated motion in curved spacetimes, such as near black holes or in cosmological models.

10 Variational Principle

The rapidity curvature naturally defines a Lorentz-invariant action functional for a timelike worldline:

$$S_\rho[x^\mu(\tau)] = \int_{\tau_1}^{\tau_2} \rho(\tau)\, d\tau = \int_{\tau_1}^{\tau_2} \sqrt{a^\mu a_\mu}\, d\tau, \tag{35}$$

where $a^\mu = du^\mu/d\tau$ is the 4-acceleration.

Varying the worldline $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$ with fixed endpoints, the first-order variation of the action is

$$\delta S_\rho = \int_{\tau_1}^{\tau_2} \frac{a_\mu\, \delta a^\mu}{\sqrt{a^\nu a_\nu}}\, d\tau. \tag{36}$$

Using $\delta a^\mu = d^2(\delta x^\mu)/d\tau^2$, integration by parts (and neglecting boundary terms) leads to the Euler–Lagrange equations

$$\frac{d^2}{d\tau^2}\left(\frac{a^\mu}{\sqrt{a^\nu a_\nu}}\right) = 0. \tag{37}$$

For free motion where $a^\mu = 0$, this reduces to the standard geodesic equation in Minkowski spacetime. Thus, extremizing the action S_ρ generalizes the principle of least action to worldlines with nonzero acceleration, providing a geometric interpretation of rapidity curvature as a measure of "path bending" in 4-velocity space.

This variational formulation may also provide a framework for studying optimal or extremal accelerated trajectories in both flat and curved spacetimes.

11 Discussion

We have introduced the rapidity curvature $\rho(\tau)$ as a new Lorentz-invariant scalar in special relativity. This quantity measures the rate of change of rapidity along a timelike worldline and provides a natural geometric interpretation as the curvature of the trajectory in velocity hyperboloid space.

Compared to standard invariants such as the magnitude of 4-acceleration, proper time, and the Minkowski inner product of 4-velocities, rapidity curvature offers a complementary perspective by connecting local kinematics with the global hyperbolic displacement in velocity space. Its series expansion shows that higher-order derivatives of the worldline (jerk, snap) contribute systematically to non-uniform motion, while the leading-order term coincides with proper acceleration.

Applications to uniform acceleration, circular motion, and arbitrary 3D motion illustrate that rapidity curvature reduces to familiar physical quantities in simple cases while remaining a fully Lorentz-invariant measure in general scenarios. The covariant generalization extends this concept to curved spacetimes, opening possibilities for analyzing accelerated motion near gravitational sources, cosmological backgrounds, or in strong-field regimes.

The variational formulation based on $S_\rho = \int \rho\, d\tau$ suggests new ways to study extremal accelerated trajectories and may provide insights into the geometry of non-inertial motion in both flat and curved spacetimes. Future work could explore connections to relativistic quantum field theory in accelerated frames, relativistic control of spacecraft trajectories, and geometric optimization problems in velocity space.

In summary, rapidity curvature enriches the set of kinematic invariants in relativistic physics, providing both conceptual clarity and practical computational tools for analyzing non-inertial motion.

12 Conclusion

We have introduced the rapidity curvature $\rho(\tau)$, a new Lorentz-invariant scalar in special relativity that quantifies the rate of change of rapidity along a timelike worldline. This scalar is directly related to the magnitude of the 4-acceleration in the infinitesimal limit and admits a clear geometric interpretation as the curvature of the 4-velocity curve on the velocity hyperboloid.

We derived both finite and differential forms of rapidity curvature, proved its Lorentz invariance, and provided a series expansion including higher-order corrections involving jerk and snap. Applications to uniform acceleration, circular motion, and general 3D trajectories illustrate its physical relevance and computational utility. Extensions to curved spacetime via parallel transport and a variational formulation highlight its broader theoretical significance.

Rapidity curvature complements existing kinematic invariants, offering new insights into non-inertial motion, and opens avenues for future research in both flat and curved spacetimes. It provides a novel geometric and physical tool for the analysis of accelerated relativistic motion.

A Detailed Series Expansion of Rapidity Curvature

We provide here the derivation of the series expansion of rapidity curvature $\rho(\tau)$ in terms of the proper acceleration, jerk, snap, and higher derivatives of the worldline.

A.1 Taylor Expansion of 4-Velocity

Consider the 4-velocity along a timelike worldline:

$$u^\mu(\tau+\Delta\tau) = u^\mu(\tau) + a^\mu(\tau)\Delta\tau + \frac{1}{2}j^\mu(\tau)\Delta\tau^2 + \frac{1}{6}s^\mu(\tau)\Delta\tau^3 + \frac{1}{24}c^\mu(\tau)\Delta\tau^4 + \dots, \tag{38}$$

where

$$\begin{aligned} a^\mu &= \frac{du^\mu}{d\tau}, \\ j^\mu &= \frac{da^\mu}{d\tau}, \\ s^\mu &= \frac{dj^\mu}{d\tau}, \\ c^\mu &= \frac{ds^\mu}{d\tau} \quad \text{(4-crackle)}. \end{aligned}$$

A.2 Inner Product Expansion

The rapidity increment is defined by

$$\cosh(\Delta\eta) = -u(\tau)\cdot u(\tau+\Delta\tau). \tag{39}$$

Expanding the Minkowski inner product:

$$\begin{aligned} -u(\tau)\cdot u(\tau+\Delta\tau) &= -u\cdot\Big(u + a\Delta\tau + \frac{1}{2}j\Delta\tau^2 + \frac{1}{6}s\Delta\tau^3 + \frac{1}{24}c\Delta\tau^4 + \dots\Big) \qquad (40)\\ &= 1 + \frac{1}{2}a\cdot a\,\Delta\tau^2 + \frac{1}{6}a\cdot j\,\Delta\tau^3 + \frac{1}{24}(a\cdot s + j\cdot j)\Delta\tau^4 + \dots \qquad (41) \end{aligned}$$

Here we have used $u\cdot a = 0$, $u\cdot j = -a\cdot a$, etc., following the orthogonality conditions of the Frenet–Serret formalism in Minkowski spacetime.

A.3 Arcosh Expansion

The rapidity curvature is

$$\rho(\tau) = \lim_{\Delta\tau \to 0} \frac{\operatorname{arcosh}[-u(\tau) \cdot u(\tau + \Delta\tau)]}{\Delta\tau}. \tag{42}$$

For small x, the expansion of $\operatorname{arcosh}(1 + x)$ is

$$\operatorname{arcosh}(1 + x) = \sqrt{2x} - \frac{\sqrt{2}}{12} x^{3/2} + \frac{3\sqrt{2}}{160} x^{5/2} + O(x^{7/2}). \tag{43}$$

Substituting $x = \frac{1}{2} a \cdot a\, \Delta\tau^2 + \frac{1}{6} a \cdot j\, \Delta\tau^3 + \dots$ and expanding in powers of $\Delta\tau$, we find

$$\rho(\tau) = \sqrt{a \cdot a} + \frac{a \cdot j}{6(a \cdot a)} + \frac{3(a \cdot s + j \cdot j)(a \cdot a) - 5(a \cdot j)^2}{72(a \cdot a)^{5/2}} + O(\Delta\tau^3) \tag{44}$$

$$= \sqrt{a \cdot a} \left[1 + \frac{1}{6} \frac{a \cdot j}{(a \cdot a)^{3/2}} + \frac{1}{24} \frac{3(a \cdot s + j \cdot j)(a \cdot a) - 5(a \cdot j)^2}{(a \cdot a)^3} + \dots \right]. \tag{45}$$

A.4 Comments

- The **leading-order term** $\sqrt{a \cdot a}$ corresponds to proper acceleration, consistent with the instantaneous definition. - The **first correction term** $\frac{1}{6} \frac{a \cdot j}{(a \cdot a)^{3/2}}$ captures effects of non-uniform acceleration (jerk). - Higher-order terms systematically include **snap, crackle, and higher derivatives**, providing a framework for precise calculations in arbitrary non-uniform motion. - This expansion is fully Lorentz-invariant, as all terms involve Minkowski inner products.

B Numerical Examples for Nonuniform Motion

In this appendix, we illustrate the computation of rapidity curvature $\rho(\tau)$ for a particle undergoing non-uniform motion. These examples demonstrate the contribution of higher derivatives (jerk, snap) to the local rapidity curvature.

B.1 Example 1: Accelerating along a single axis with varying acceleration

Consider a particle moving along the x-axis with position

$$x(t) = \frac{1}{2} a_0 t^2 + \frac{1}{6} j_0 t^3, \tag{46}$$

where a_0 is the initial proper acceleration and j_0 is a constant jerk.

The 3-velocity is

$$v(t) = \frac{dx}{dt} = a_0 t + \frac{1}{2} j_0 t^2, \tag{47}$$

and the corresponding Lorentz factor is

$$\gamma(t) = \frac{1}{\sqrt{1 - v(t)^2/c^2}}. \tag{48}$$

The 4-velocity is

$$u^\mu = \gamma(t)(1, v(t), 0, 0), \tag{49}$$

and the 4-acceleration is

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma^4(t) \frac{dv}{dt} (v(t), 1, 0, 0), \tag{50}$$

where $dv/dt = a_0 + j_0 t$.

The rapidity curvature is then

$$\rho(t) = \sqrt{a^\mu a_\mu} = \gamma^3(t)(a_0 + j_0 t), \tag{51}$$

showing explicitly how the jerk j_0 contributes linearly to the rapidity curvature.

B.2 Example 2: Circular motion with time-dependent angular velocity

Consider a particle in the x-y plane with radius R and angular position

$$\theta(t) = \omega_0 t + \frac{1}{2}\alpha t^2, \tag{52}$$

where ω_0 is the initial angular velocity and α is angular acceleration.

The 3-velocity is

$$\mathbf{v}(t) = R\dot{\theta}(t)(-\sin\theta(t), \cos\theta(t), 0), \tag{53}$$

and the 3-acceleration is

$$\mathbf{a}(t) = R\ddot{\theta}(t)(-\sin\theta(t), \cos\theta(t), 0) - R\dot{\theta}^2(t)(\cos\theta(t), \sin\theta(t), 0). \tag{54}$$

The 4-velocity and 4-acceleration are

$$u^\mu = \gamma(t)(1, \mathbf{v}(t)), \tag{55}$$

$$a^\mu = \frac{du^\mu}{d\tau}, \tag{56}$$

with $\gamma(t) = 1/\sqrt{1 - |\mathbf{v}(t)|^2/c^2}$.

The rapidity curvature is

$$\rho(t) = \sqrt{a^\mu a_\mu} = \gamma^2(t)\sqrt{|\mathbf{a}_\perp(t)|^2 + \gamma^2(t)|\mathbf{v}(t) \cdot \mathbf{a}(t)|^2}, \tag{57}$$

where $\mathbf{a}_\perp$ is the component of acceleration perpendicular to velocity.

This example shows how non-uniform circular motion introduces time-dependent contributions from angular acceleration and relativistic corrections.

B.3 Example 3: Arbitrary 3D trajectory

For a general trajectory $\mathbf{r}(t)$ in 3D, we can numerically compute

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}, \quad \mathbf{a}(t) = \frac{d\mathbf{v}}{dt}, \quad u^\mu = \gamma(1, \mathbf{v}), \quad a^\mu = \frac{du^\mu}{d\tau}, \tag{58}$$

and then evaluate

$$\rho(t) = \sqrt{a^\mu a_\mu}. \tag{59}$$

Numerical integration or discrete-time sampling can be used to illustrate how $\rho(\tau)$ varies along the trajectory, highlighting contributions from jerk, snap, and higher derivatives.

B.4 Comments

- These examples confirm that $\rho(\tau)$ reduces to **proper acceleration** in the uniform cases and incorporates **higher-order corrections** in non-uniform motion. - The explicit formulas are fully **Lorentz-invariant**, as they are constructed from 4-velocity and 4-acceleration. - Numerical implementation is straightforward in any computational language supporting vector and matrix operations, enabling exploration of complex trajectories.

C Covariant Generalization Proof: Parallel Transport Operator

In curved spacetime $(\mathcal{M}, g_{\mu\nu})$, the comparison of 4-velocities at different points requires parallel transport along the worldline. This appendix provides the formal derivation.

C.1 Worldline and 4-Velocity

Let $x^\mu(\tau)$ be a timelike worldline parametrized by proper time τ, with 4-velocity

$$u^\mu(\tau) = \frac{dx^\mu}{d\tau}, \quad u^\mu u_\mu = -1. \tag{60}$$

The 4-acceleration is

$$a^\mu = \frac{Du^\mu}{d\tau} = u^\nu \nabla_\nu u^\mu, \tag{61}$$

where ∇_ν is the covariant derivative and $D/d\tau$ is the covariant proper-time derivative.

C.2 Parallel Transport Operator

To define the rapidity increment between τ and $\tau + \Delta\tau$, we parallel transport $u^\mu(\tau + \Delta\tau)$ back to τ along the worldline. Let $P^\mu{}_\nu(\tau + \Delta\tau \to \tau)$ denote the parallel transport operator:

$$\tilde{u}^\mu(\tau + \Delta\tau) = P^\mu{}_\nu(\tau + \Delta\tau \to \tau)\, u^\nu(\tau + \Delta\tau), \tag{62}$$

satisfying

$$\frac{D\tilde{u}^\mu}{d\tau} = 0, \quad \tilde{u}^\mu(\tau + \Delta\tau)\Big|_{\tau+\Delta\tau} = u^\mu(\tau + \Delta\tau). \tag{63}$$

In components, the covariant derivative along the worldline reads

$$\frac{D\tilde{u}^\mu}{d\tau} = \frac{d\tilde{u}^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}\, u^\alpha\, \tilde{u}^\beta = 0, \tag{64}$$

where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols of $g_{\mu\nu}$.

C.3 Covariant Rapidity Curvature

The covariant rapidity increment is defined as

$$\Delta\eta = \operatorname{arcosh}\Big[- g_{\mu\nu}(x(\tau))\, u^\mu(\tau)\, \tilde{u}^\nu(\tau + \Delta\tau) \Big]. \tag{65}$$

The instantaneous rapidity curvature is obtained in the limit

$$\boxed{\rho(\tau) = \lim_{\Delta\tau \to 0} \frac{\Delta\eta}{\Delta\tau} = \sqrt{g_{\mu\nu}\, a^\mu a^\nu}.} \tag{66}$$

This shows that in the infinitesimal limit, the covariant definition reduces to the magnitude of the 4-acceleration, consistent with the flat-spacetime definition.

C.4 Properties

- $\rho(\tau)$ is a **scalar under general coordinate transformations**, since both $g_{\mu\nu}$ and a^μ transform covariantly.
- The parallel transport ensures that we are comparing vectors **at the same spacetime point**, which is necessary because the tangent spaces at different points are distinct.
- In a **local inertial frame**, the covariant derivative reduces to the ordinary derivative, and $\rho(\tau)$ coincides with the Minkowski-space rapidity curvature.

C.5 Conclusion

The use of the parallel transport operator provides a rigorous, fully covariant generalization of rapidity curvature to curved spacetime. It preserves Lorentz invariance locally and ensures proper comparison of 4-velocities along a timelike trajectory, allowing the study of accelerated motion near gravitational sources or in cosmological models.

D Dimensional and Unit Analysis

In this paper, we have introduced the rapidity curvature $\rho(\tau)$ and related quantities. Here we analyze their physical dimensions and units.

D.1 Proper Time and 4-Velocity

The proper time τ has dimensions of time:

$$[\tau] = T. \tag{67}$$

The 4-velocity is defined as

$$u^\mu = \frac{dx^\mu}{d\tau}, \tag{68}$$

where x^μ is spacetime position. In SI units:

$$[x^0] = L, \quad [x^i] = L \quad (i = 1, 2, 3), \tag{69}$$

with L length and T time. Using $x^0 = ct$, we have

$$[u^0] = \frac{dx^0}{d\tau} = \frac{c\,dt}{d\tau} \sim 1, \quad [u^i] = \frac{dx^i}{d\tau} \sim \text{velocity}. \tag{70}$$

In natural units ($c = 1$), the 4-velocity is dimensionless:

$$[u^\mu] = 1. \tag{71}$$

D.2 4-Acceleration and Higher Derivatives

The 4-acceleration is

$$a^\mu = \frac{du^\mu}{d\tau}. \tag{72}$$

- SI units: $[a^\mu] = L/T^2$, same as conventional acceleration. - Natural units ($c = 1$): $[a^\mu] = T^{-1}$ (inverse time).

Higher derivatives (jerk j^μ, snap s^μ) have dimensions:

$$[j^\mu] = \frac{da^\mu}{d\tau} \sim T^{-2}, \tag{73}$$

$$[s^\mu] = \frac{dj^\mu}{d\tau} \sim T^{-3}. \tag{74}$$

D.3 Rapidity and Rapidity Curvature

Rapidity is defined via a hyperbolic angle:

$$\cosh(\Delta\eta) = -u^\mu(\tau)u_\mu(\tau + \Delta\tau), \tag{75}$$

so it is dimensionless:

$$[\Delta\eta] = 1. \tag{76}$$

The rapidity curvature is

$$\rho(\tau) = \lim_{\Delta\tau \to 0} \frac{\Delta\eta}{\Delta\tau}, \tag{77}$$

with dimensions

$$[\rho] = T^{-1}. \tag{78}$$

In natural units, this is equivalent to the magnitude of proper acceleration:

$$\rho = \sqrt{a^\mu a_\mu}, \quad [\rho] = T^{-1}. \tag{79}$$

D.4 Global Rapidity Distance

The total rapidity distance along a worldline is

$$\Delta\eta_{12} = \int_{\tau_1}^{\tau_2} \rho(\tau)\, d\tau, \tag{80}$$

which is dimensionless, consistent with rapidity being a hyperbolic angle:

$$[\Delta\eta_{12}] = 1. \tag{81}$$

D.5 Summary

- $[\tau] = T$, proper time.
- $[u^\mu] = 1$ (dimensionless in natural units).
- $[a^\mu] = T^{-1}$, $[j^\mu] = T^{-2}$, $[s^\mu] = T^{-3}$ in natural units.
- $[\Delta\eta] = 1$, dimensionless.
- $[\rho] = T^{-1}$, rapidity curvature has the same dimension as proper acceleration.

This dimensional analysis confirms the consistency of all quantities introduced in the paper and provides guidance for translating between SI and natural units.

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References

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