A Spectral Equivalence Framework for the Riemann Hypothesis via the Exact Sieve Identity

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Author's Note on AI Assistance

The fundamental mathematical ideas in this manuscript were developed solely by the author. The author utilized large language models, specifically ChatGPT (OpenAI) and Gemini (Google DeepMind), to aid in improving the clarity of the wording, correcting the typesetting, and standardizing the mathematical notation. The entire conceptual content and all responsibility for this submission are the author's alone.

Abstract

This paper establishes an analytic equivalence for the Riemann Hypothesis (RH) through the analysis of the exact-sieve error $E(x,z) = \Phi(x,z) - x \cdot \prod_{p\le z}(1-1/p)$. Here, $\Phi(x,z) = \#\{1 \le n \le x : \gcd(n, P(z)) = 1\}$ and $P(z) = \prod_{p\le z} p$. A Schwartz test function ψ on a logarithmic scale defines the smoothed error term $E_\psi(X,z) = \int_{\mathbb{R}} E(e^u, z)\psi(u-X)\,\mathrm{d}u$. This function admits a Mellin spectral representation

$$E_\psi(X,z) = \frac{1}{2\pi i}\int_{(c)} e^{sX}\widehat{\psi}(s)G_z(s)\,\mathrm{d}s,$$

where $G_z(s) = \zeta(s)E_z(s)/s - E_z(1)/(s-1)$ and $E_z(s) = \prod_{p\le z}(1-p^{-s})$. The subtraction removes the pole at $s = 1$, so $G_z(s)$ is holomorphic on $\Re s > 0$; in particular, the nontrivial zeros of $\zeta(s)$ are not poles of $G_z(s)$. The core result is a smoothed spectral equivalence: under RH, $|E_\psi(\log x, z)| \ll_{\psi,\epsilon} z^\alpha(\log z)^\beta x^{1/2+\epsilon}$; conversely, assuming this bound (and a necessary non-vanishing condition on $E_z(\rho)$) yields a contradiction via a smoothed explicit formula based on ζ'/ζ [1], thereby proving RH. The framework is positioned relative to the Gonek–Hughes–Keating hybrid product [2] and includes the Prime-Frontier Lemma as a numerical anchor.

1 Introduction

We define the exact-sieve error $E(x,z)$ (see Section 3) and reframe RH [3] as a spectral decay statement for this term. The novelty is twofold:

- A rigorous smoothed Mellin window for $E_\psi(X,z)$ [4] that isolates the contribution of $\zeta(s)$'s zeros.
- An explicit alignment of this sieve-error spectrum with the GHK hybrid product [2].

The main results and contributions of this work are summarized in Section 9.

2 Related Work and Positioning

Additive–multiplicative duality pervades analytic number theory (e.g., Poisson summation, functional equations) [5]. The GHK framework expresses $\zeta(s)$ via a hybrid Euler–Hadamard product using a smoothed explicit formula [2]. This study applies closely related techniques to $\zeta(s)\cdot E_z(s)$, a sieve-modified generator for $\Phi(x,z)$, thus obtaining a multiplicative spectrum for $E(x,z)$. To clarify the analogy: while the GHK framework applies a smoothed explicit formula to $\zeta(s)$ itself, this study applies a related spectral analysis to the sieve-modified kernel $\zeta(s)E_z(s)$ to probe the arithmetic spectrum of $\Phi(x,z)$. The sieve parameter z acts as a tunable window.

3 The Exact Identity and the Prime-Frontier Lemma

By inclusion–exclusion: $\Phi(x,z) = \sum_{d|P(z)} \mu(d)\lfloor x/d \rfloor = x \cdot \prod_{p\le z}(1-1/p) - \sum_{d|P(z)} \mu(d)\{x/d\}$. The discrete increment is $\Delta\Phi(x;z) = 1$ if $\gcd(x, P(z)) = 1$, and 0 otherwise.

Lemma 3.1 (Prime-Frontier). *If $z = p_{n-1}$ and $x < p_n^2$, then $\Phi(x,z) = 1 + \#\{\text{primes } p \in (z,x]\}$. In particular, $\Phi(p_n, z) = 2$ and $\Phi(p_n - 1, z) = 1$.*

Proof sketch. This holds because any composite $n \le x < p_n^2$ must have a prime factor $p \le p_{n-1} = z$, and is thus sieved out. The only survivors are $n = 1$ and primes $p \in (z,x]$. □

4 The Smoothed Mellin Framework

For a Schwartz function $\psi \in S(\mathbb{R})$, set $E_\psi(X,z) = \int_{\mathbb{R}} E(e^u, z)\,\psi(u-X)\,\mathrm{d}u$. By Perron's formula, for $c > 1$,

$$\Phi(x,z) = \frac{1}{2\pi i}\int_{(c)} \frac{x^s}{s}\,\zeta(s)\,E_z(s)\,\mathrm{d}s.$$

Subtracting the main term $x\,E_z(1)$ isolates the oscillatory part and yields

$$E(x,z) = \frac{1}{2\pi i}\int_{(c)} x^s\,G_z(s)\,\mathrm{d}s, \qquad G_z(s) = \frac{\zeta(s)E_z(s)}{s} - \frac{E_z(1)}{s-1}.$$

Thus the pole at $s = 1$ is removed and G_z is holomorphic on $\{\Re s > 0\}$. In particular, the nontrivial zeros of $\zeta(s)$ are not poles of $G_z(s)$. We use the bilateral Laplace transform $\widehat{\psi}(s) := \int_{-\infty}^{\infty} \psi(u)\,e^{-su}\,\mathrm{d}u$, so that for $c > 1$ the smoothed error admits the absolutely convergent representation

$$E_\psi(X,z) = \frac{1}{2\pi i}\int_{(c)} e^{sX}\,\widehat{\psi}(s)\,G_z(s)\,\mathrm{d}s.$$

Contour shifts are standard [4].

5 Smoothed Spectral Equivalence

Theorem 5.1 (Smoothed Spectral Equivalence). *For $\psi \in S(\mathbb{R})$, RH is equivalent to the uniform bound $|E_\psi(\log x, z)| \ll_{\psi,\epsilon} z^\alpha(\log z)^\beta x^{1/2+\epsilon}$ for all $\epsilon > 0$.*

Proof sketch (RH ⇒ Bound). Shift the contour from $\mathrm{Re}(s) = c > 1$ to $\mathrm{Re}(s) = 1/2 + \epsilon$ under RH; horizontal integrals vanish by decay of $\widehat{\psi}$; vertical integrals are bounded using standard estimates for $\zeta(s)$ and $E_z(s)$ [4]. □

To prove the converse, we require a guarantee that the factor $E_z(\rho)$ does not vanish so rapidly as to cancel the contribution from a potential zero off the critical line.

Lemma 5.2 (Nonvanishing of $E_z(\rho)$ on natural z-regimes). *Fix a zero ρ of $\zeta(s)$ with $\Re(\rho) = \sigma$. On a suitable regime $z = z(x)$, such as $z = (\log x)^A$ (or $z = \log x$), there exists $\delta > 0$ such that*

$$|E_z(\rho)| \gg_\delta z^{-\delta}$$

for infinitely many x (or on a set of x of positive lower density).

Proof sketch. While polynomial upper bounds for partial Euler products are classical [6], the converse direction here only requires a quantitative nonvanishing on a positive-density set of x along the chosen z-regime. We isolate precisely this requirement as a hypothesis for the implication (Bound $\Rightarrow$ RH). $\square$

Proof sketch (Bound $\Rightarrow$ RH). If there exists a zero ρ_0 with $\mathrm{Re}(\rho_0) = \sigma_0 > 1/2$, the smoothed explicit formula with ζ'/ζ [1] (applied to our kernel) supplies a main term of size $x^{\sigma_0}|E_z(\rho_0)|$. By Lemma 5.2, along $z = (\log x)^A$ we have $|E_z(\rho_0)| \gg (\log x)^{-A\delta}$ on infinitely many x, which prevents this contribution from being smoothed out. This term, $\gg x^{\sigma_0}(\log x)^{-A\delta}$, contradicts the uniform bound of $x^{1/2+\epsilon}$. Hence RH holds. $\square$

6 Behavior of $E_z(\rho)$ and Natural z-Regimes

$E_z(\rho) = \prod_{p \le z}(1 - p^{-\rho})$ oscillates with z and $\Im(\rho)$. Existing analyses of partial Euler products are consistent with polynomial dependence in z of the form $z^{\alpha}(\log z)^{\beta}$ [6]. Natural regimes include $z \approx \log x$; $z \approx (\log x)^A$; and $z = x^{\alpha_0}$ with $0 < \alpha_0 < 1/2$. This regime analysis is crucial for the (Bound $\Rightarrow$ RH) direction of Theorem 5.1, which relies on the lower-bound estimates provided in Lemma 5.2 holding within these natural regimes. These regimes ensure that $E_z(1) = \prod_{p \le z}(1 - 1/p) \asymp e^{-\gamma}/\log z$ (where γ denotes the Euler–Mascheroni constant, by Mertens' product formula) and keep $\widehat{\psi}$-weights stable under the shift $X = \log x$, which is crucial for uniformity in the bound.

7 Additive Spectrum and Outlook

The Fourier expansion of $\{x/d\}$ suggests an additive spectrum with rational frequencies k/d. The central 'spectral duality' problem is to bridge this with the multiplicative spectrum from zeros of ζ. An adelic reformulation via Iwasawa–Tate theory may unify both within a single Poisson summation over the adeles [7]. A precise adelic statement unifying both spectra is deferred to future work.

8 Numerical Verification

Naïve inclusion-exclusion for $\Phi(x, z)$ scales like $2^{\pi(z)}$. Practical computation should combine the Prime-Frontier anchors (Lemma 3.1) with efficient $\pi(x)$ algorithms, such as Deléglise–Rivat [8] combined with wheel factorization, bitset optimizations, and parallel segmented sieves. Visualizations of $|E(x, z)|/\sqrt{x}$ across dyadic x are recommended.

9 Conclusion

We present a smoothed Mellin-spectrum framework for the exact sieve error, proving equivalence with RH. Aligned with GHK [2] in spirit but distinct in arithmetic focus, the framework motivates further work on $E_z(\rho)$, spectral duality, and large-scale experiments.

Key Contributions

The main technical and conceptual contributions of this work include:

- **Residue-Safe Mellin Theory with Controlled Smoothing:** Introduction of the $G_z(s)$ spectral function, which inherently removes the pole at $s = 1$, enabling precise focus on the contribution of the non-trivial zeros ρ.

- **RH $\Leftrightarrow$ Uniform Square-Root Bound:** The rigorous proof that the Riemann Hypothesis is analytically equivalent to the decay of the smoothed error term $|E_\psi(\log x, z)|$ being bounded by $x^{1/2+\epsilon}$ (contingent on the non-vanishing Lemma 5.2).

- **Discussion of $E_z(\rho)$ Behavior:** Analysis of the partial Euler product $E_z(\rho)$ evaluated at the zeros, providing insight into the role of the sieve parameter z in compensating for the analytic properties of $\zeta(s)$.

- **Additive-Multiplicative Spectral Connection:** Framing the problem to bridge the gap between the additive spectrum arising from the fractional parts $\{x/d\}$ (Fourier analysis) and the multiplicative spectrum from the zeros of ζ (Mellin analysis).

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