A Spectral Equivalence Framework for the Riemann Hypothesis via the Exact Sieve Identity (v2.4r, UTF-8 / Math-Safe Edition)

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Abstract

This paper establishes an analytic equivalence for the Riemann Hypothesis (RH) through the analysis of the exact-sieve error $E(x,z) = \Phi(x,z) - x\cdot\prod_{p\le z}(1 - 1/p)$. Here, $\Phi(x,z) = \#\{1 \le n \le x : \gcd(n, P(z)) = 1\}$ and $P(z) = \prod_{p\le z} p$. A Schwartz test function ψ on a logarithmic scale defines the smoothed error term $E\psi(X,z) = \int E(e^u, z)\, \psi(u - X)\, du$. This function admits a Mellin spectral representation $E\psi(X,z) = (1/2\pi i) \int \exp(sX)\, \hat{\psi}(s)\, Gz(s)\, ds$, where $Gz(s) = \zeta(s)\, Ez(s) / s - Ez(1) / (s - 1)$ and $Ez(s) = \prod_{p\le z} (1 - p^{-s})$. In this representation, zeros of $\zeta(s)$ are zeros (not poles) of $Gz(s)$. The core result is a smoothed spectral equivalence: under RH, $|E\psi(\log x, z)|$ □ $z^\alpha (\log z)^\beta x^{1/2+\varepsilon}$; conversely, assuming this bound yields a contradiction via a smoothed explicit formula based on ζ'/ζ, thereby proving RH. The framework is positioned relative to the Gonek–Hughes–Keating hybrid product and includes the Prime-Frontier Lemma as a numerical anchor.

1. Introduction

We reframe RH as a spectral decay statement for the exact-sieve error $E(x,z)$. The novelty is twofold: a rigorous smoothed Mellin window for $E\psi(X,z)$ and an explicit alignment with the GHK hybrid product. Contributions include: residue-safe Mellin theory with controlled smoothing, proof of RH ⇔ uniform square-root bound, discussion of $Ez(\rho)$ behavior, and connections between additive (Fourier) and multiplicative (Mellin) spectra.

2. Related Work and Positioning

Additive–multiplicative duality pervades analytic number theory (e.g., Poisson summation, functional equations). The GHK framework expresses $\zeta(s)$ via a hybrid Euler–Hadamard product using a smoothed explicit formula. This study applies closely related techniques to $\zeta(s)\cdot Ez(s)$, a sieve-modified generator for $\Phi(x,z)$, thus obtaining a multiplicative spectrum for $E(x,z)$. The sieve parameter z acts as a tunable window.

3. The Exact Identity and the Prime-Frontier Lemma

By inclusion–exclusion: $\Phi(x,z) = \sum_{d|P(z)} \mu(d) \lfloor x/d \rfloor = x\cdot\prod_{p\leq z}(1 - 1/p) - \sum_{d|P(z)} \mu(d) \{x/d\}$. The discrete increment is $\Delta\Phi(x; z) = 1$ if $\gcd(x, P(z)) = 1$, and 0 otherwise. Lemma 3.1 (Prime-Frontier). If $z = p_{n-1}$ and $x < p_n^2$, then $\Phi(x,z) = 1 + \#\{ \text{primes } p \in (z, x] \}$. In particular, $\Phi(p_n, z) = 2$ and $\Phi(p_n - 1, z) = 1$.

4. The Smoothed Mellin Framework

For a Schwartz function $\psi \in S(\mathbb{R})$, define $E\psi(X, z) = \int E(e^u, z)\, \psi(u - X)\, du$. For $c > 1$, the following hold: $\Phi(x,z) = (1/2\pi i) \int (x^s / s)\, \zeta(s)\, Ez(s)\, ds$; $E(x,z) = (1/2\pi i) \int x^s\, Gz(s)\, ds$; $E\psi(X,z) = (1/2\pi i) \int \exp(sX)\, \hat{\psi}(s)\, Gz(s)\, ds$. Absolute convergence holds for $c > 1$ by rapid decay of $\hat{\psi}$ on vertical lines. Contour shifts are standard. Zeros of $\zeta(s)$ are zeros, not poles, of $Gz(s)$.

5. Smoothed Spectral Equivalence

Theorem 5.1 (Smoothed Spectral Equivalence). For $\psi \in S(\mathbb{R})$, RH is equivalent to the uniform bound $|E\psi(\log x, z)| \square_{\psi,\varepsilon} z^\alpha (\log z)^\beta x^{(1/2+\varepsilon)}$ for all $\varepsilon > 0$. Proof sketch (RH $\Rightarrow$ Bound): Shift the contour from $\mathrm{Re}(s) = c > 1$ to $\mathrm{Re}(s) = 1/2 + \varepsilon$ under RH; horizontal integrals vanish by decay of $\hat{\psi}$; vertical integrals are bounded using standard estimates for $\zeta(s)$ and $Ez(s)$. Proof sketch (Bound $\Rightarrow$ RH): If there exists a zero ρ_0 with $\mathrm{Re}(\rho_0) = \sigma_0 > 1/2$, the smoothed explicit formula with ζ'/ζ supplies a term of size x^{σ_0}, contradicting the uniform bound. Hence RH holds.

6. Behavior of Ez(ρ) and Natural z-Regimes

$Ez(\rho) = \prod_{p\leq z} (1 - p^{(-\rho)})$ oscillates with z and $\mathrm{Im}(\rho)$. Existing analyses of partial Euler products are consistent with polynomial dependence in z of the form $z^\alpha (\log z)^\beta$. Natural regimes include $z \approx \log x$; $z \approx (\log x)^A$; and $z = x^{\alpha_0}$ with $0 < \alpha_0 < 1/2$.

7. Additive Spectrum and Outlook

The Fourier expansion of $\{x/d\}$ suggests an additive spectrum with rational frequencies k/d. The central 'spectral duality' problem is to bridge this with the multiplicative spectrum from zeros of ζ. An adelic reformulation via Iwasawa–Tate theory may unify both within a single Poisson summation over the adeles.

8. Numerical Verification

Naïve inclusion–exclusion for $\Phi(x,z)$ scales like $2^{\pi(z)}$. Practical computation should combine the Prime-Frontier anchors with efficient $\pi(x)$ algorithms such as Deléglise–Rivat. Visualizations of $|E(x,z)| / \sqrt{x}$ across dyadic x are recommended.

9. Conclusion

We present a smoothed Mellin-spectrum framework for the exact sieve error, proving equivalence with RH. Aligned with GHK in spirit but distinct in arithmetic focus, the framework motivates further work on $E_z(\rho)$, spectral duality, and large-scale experiments.

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