Oscillatory Coulomb Potential: Static 3D Point-Charge Interactions with Alternating Attraction and Repulsion

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Abstract

We propose a novel reinterpretation of the classical Coulomb interaction by introducing an oscillatory component in three-dimensional space. The resulting potential for a point charge is given by

$$\varphi(r) = \frac{q}{4\pi\varepsilon_0} \frac{\cos(\kappa r)}{r},$$

where κ is a tunable parameter controlling the wavelength of oscillations. This form naturally arises from a Helmholtz-type generalization of Poisson's equation,

$$(\nabla^2 + \kappa^2)\varphi(\mathbf{r}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{r}).$$

The corresponding force vector,

$$\mathbf{F}(\mathbf{r}) = \frac{q_1 q_2}{4\pi\varepsilon_0 r^3}\left[\cos(\kappa r) + \kappa r \sin(\kappa r)\right]\mathbf{r},$$

introduces alternating attractive and repulsive zones with a characteristic length scale $\lambda = 2\pi/\kappa$. This potential recovers the classical Coulomb law in the limit $\kappa \to 0$ while providing new equilibrium points in many-body systems. Possible applications include engineered materials, optical lattices, and theoretical studies of static interactions with modulated forces. The framework presented here provides a novel perspective on Coulomb-like interactions and opens avenues for both analytical and numerical investigations of oscillatory force landscapes.

1 Introduction

The classical Coulomb law describes the electrostatic interaction between point charges in three-dimensional space, given by a potential proportional to $1/r$ and a force that is purely attractive or repulsive depending on the charge signs. While this law has been remarkably successful in describing electromagnetic interactions at macroscopic and microscopic scales, it does not allow for alternating zones of attraction and repulsion in static systems.

Modifications of the Coulomb potential have been studied in various contexts, such as screened Coulomb (Yukawa) potentials in plasma physics and condensed matter, and quantum-modulated potentials in atomic and optical systems. These modifications typically introduce exponential decay or boundary-induced oscillations, but a fundamental, static three-dimensional Coulomb-like potential with intrinsic oscillatory behavior has not been widely explored.

In this work, we introduce an oscillatory Coulomb potential derived from a Helmholtz-type generalization of Poisson's equation. The resulting potential exhibits alternating attractive and repulsive regions while reducing to the classical Coulomb potential in the limit of vanishing oscillation parameter. This formulation provides a simple analytical framework to explore novel equilibrium points and force landscapes in many-body systems. Possible applications include engineered metamaterials, optical lattices, and theoretical investigations into static interactions with modulated forces.

The paper is organized as follows: In Section 2, we present the theoretical framework and derivation of the oscillatory Coulomb potential. Section 3 discusses possible extensions and generalizations. Section 4 provides illustrative results and numerical examples. Section 5 offers a discussion of physical implications and potential applications, and Section 6 concludes with a summary and outlook.

2 Theoretical Framework

We begin with the classical Poisson equation for the electrostatic potential $\varphi(\mathbf{r})$ generated by a point charge q at the origin:

$$\nabla^2\varphi(\mathbf{r}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{r}), \tag{1}$$

where ε_0 is the vacuum permittivity. The well-known solution of this equation yields the standard Coulomb potential:

$$\varphi_{\text{Coulomb}}(r) = \frac{q}{4\pi\varepsilon_0 r}. \tag{2}$$

To introduce oscillatory behavior into the potential, we generalize Poisson's equation by adding a Helmholtz term:

$$\left(\nabla^2 + \kappa^2\right)\varphi(\mathbf{r}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{r}), \tag{3}$$

where $\kappa > 0$ is a tunable parameter controlling the wavelength of the oscillations.

The Green's function for the operator $(\nabla^2 + \kappa^2)$ in three dimensions is given by:

$$G(\mathbf{r}) = \frac{e^{i\kappa r}}{4\pi r}. \tag{4}$$

Taking the real part to obtain a physically interpretable potential yields the **oscillatory Coulomb potential**:

$$\varphi(r) = \frac{q}{4\pi\varepsilon_0}\frac{\cos(\kappa r)}{r}. \tag{5}$$

The corresponding force on a second point charge q_2 located at $\mathbf{r}$ is obtained via the gradient:

$$\mathbf{F}(\mathbf{r}) = -q_2\nabla\varphi(r) = \frac{q_1 q_2}{4\pi\varepsilon_0 r^3}\left[\cos(\kappa r) + \kappa r\sin(\kappa r)\right]\mathbf{r}, \tag{6}$$

where $\mathbf{r}$ is the vector from the source charge to the test charge.

This formulation preserves the classical Coulomb potential in the limit $\kappa \to 0$:

$$\lim_{\kappa \to 0} \varphi(r) = \frac{q}{4\pi\varepsilon_0 r}, \quad \lim_{\kappa \to 0} \mathbf{F}(\mathbf{r}) = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}. \tag{7}$$

The introduction of the oscillatory term $\cos(\kappa r)$ produces alternating attractive and repulsive zones with a characteristic length scale:

$$\lambda = \frac{2\pi}{\kappa}, \tag{8}$$

allowing for the possibility of static equilibrium points in multi-particle systems.

3 Extensions and Generalizations

The oscillatory Coulomb potential introduced in Section 2 can be generalized in several ways to model more complex systems or incorporate additional physical effects.

3.1 Screened Oscillatory Coulomb Potential

A natural extension is to introduce exponential screening, similar to the Yukawa potential. This gives:

$$\varphi_{\text{screened}}(r) = \frac{q}{4\pi\varepsilon_0} \frac{e^{-\lambda r}\cos(\kappa r)}{r}, \tag{9}$$

where $\lambda > 0$ controls the screening length. The corresponding force becomes:

$$\mathbf{F}_{\text{screened}}(\mathbf{r}) = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} e^{-\lambda r} \left[(1 + \lambda r)\cos(\kappa r) + \kappa r \sin(\kappa r)\right] \hat{\mathbf{r}}. \tag{10}$$

This potential can model interactions in plasmas or materials with screening effects while preserving the oscillatory character.

3.2 Anisotropic Oscillatory Potential

To account for directional dependence in engineered or lattice systems, one can introduce anisotropy in the Laplacian operator:

$$\left(\partial_x^2 + \gamma\partial_y^2 + \delta\partial_z^2 + \kappa^2\right)\varphi(\mathbf{r}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{r}), \tag{11}$$

where γ, δ are dimensionless anisotropy parameters. This yields potentials that vary differently along each spatial axis, suitable for modeling layered materials or directional interactions.

3.3 Fractional Helmholtz Operator

Fractional derivatives allow modeling of nonlocal or fractal-like interactions. Replacing the Laplacian by a fractional operator $(-\nabla^2)^{\alpha/2}$ leads to:

$$\left((-\nabla^2)^{\alpha/2} + \kappa^2\right)\varphi(\mathbf{r}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{r}), \tag{12}$$

with $0 < \alpha \leq 2$. The resulting potential exhibits a long-range power-law decay modulated by oscillations:

$$\varphi_\alpha(r) \sim r^{-\alpha}\cos(\kappa r). \tag{13}$$

This generalization can be useful in systems with nonlocal interactions, complex media, or effective fractal geometries.

3.4 Fourier-Space Representation

The oscillatory potential is also conveniently represented in Fourier space. Taking the Fourier transform of Eq. (3) gives:

$$\varphi(\mathbf{k}) = \frac{q}{\varepsilon_0} \frac{1}{k^2 - \kappa^2}, \tag{14}$$

with the inverse transform yielding the real-space potential. This form is particularly useful for numerical simulations of many-body systems, as well as for calculating lattice sums or interactions under periodic boundary conditions.

4 Results

To illustrate the properties of the oscillatory Coulomb potential, we consider two representative cases for the oscillation parameter κ.

4.1 Case 1: Oscillatory behavior ($\kappa = 10^{10}$)

For a large value of $\kappa = 10^{10}\,\mathrm{m}^{-1}$, the potential exhibits rapid spatial oscillations, leading to alternating attractive and repulsive zones in the corresponding force. Figures 1 and 2 show the potential $\varphi(r)$ and radial force $F(r)$ as functions of distance r. The characteristic wavelength of the oscillations is $\lambda = 2\pi/\kappa$.

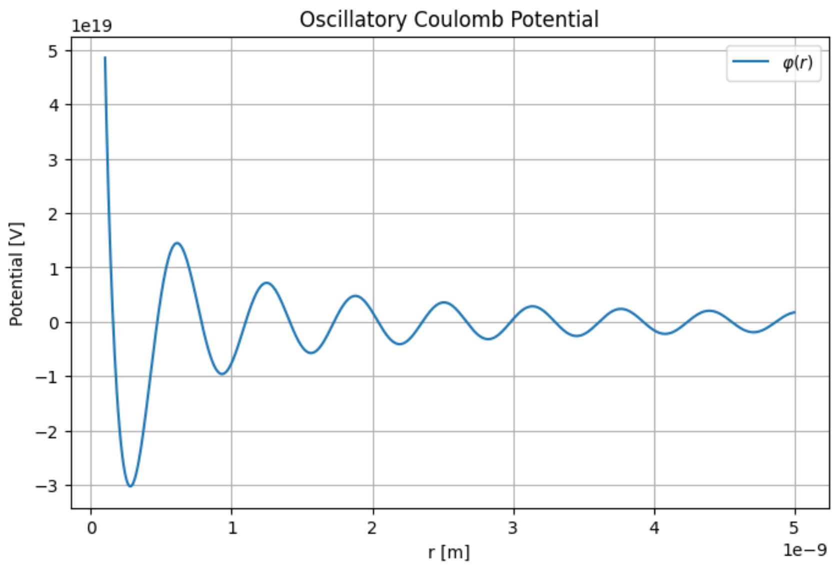

Figure 1: Oscillatory Coulomb potential $\varphi(r)$ for $\kappa = 10^{10}\,\mathrm{m}^{-1}$.

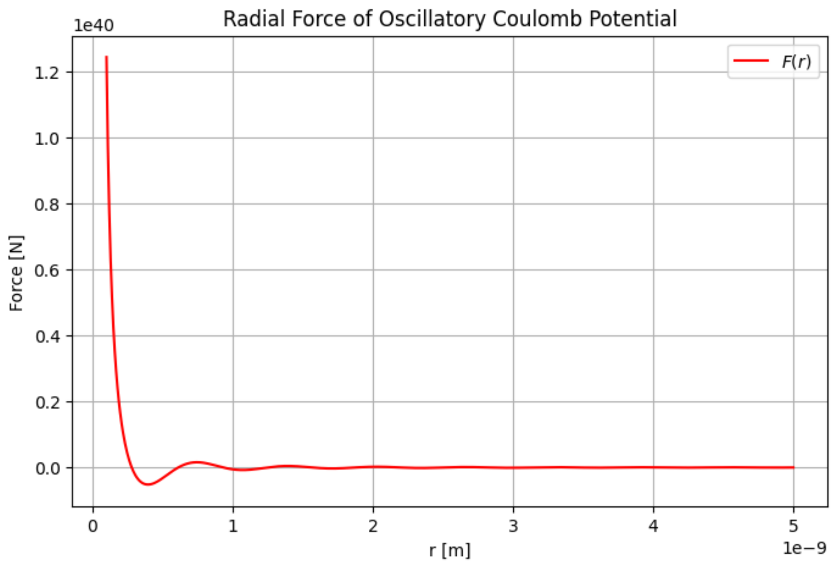

Figure 2: Radial force $F(r)$ corresponding to the oscillatory Coulomb potential for $\kappa = 10^{10}\,\mathrm{m}^{-1}$.

4.2 Case 2: Classical Coulomb limit ($\kappa = 1$)

For a small value of $\kappa = 1\,\mathrm{m}^{-1}$, the oscillations are extremely slow and the potential and force closely approach the classical Coulomb behavior. Figures 3 and 4 illustrate the potential and force, showing that the standard $1/r$ potential and $1/r^2$ force are effectively recovered.

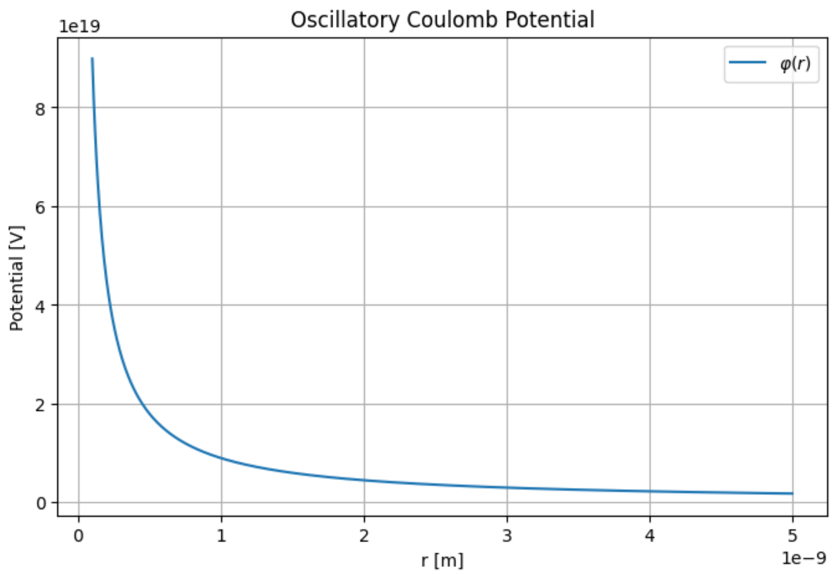

Figure 3: Oscillatory Coulomb potential $\varphi(r)$ for $\kappa = 1\,\mathrm{m}^{-1}$, approaching classical Coulomb.

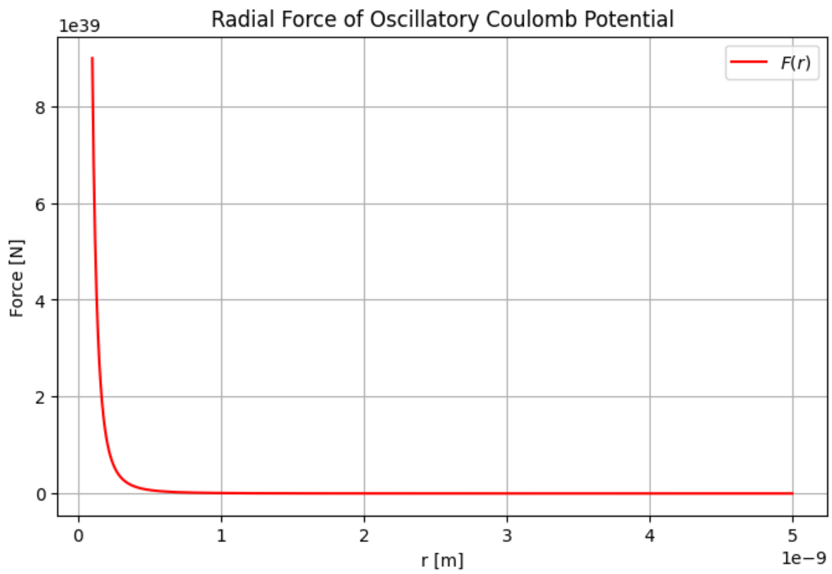

Figure 4: Radial force $F(r)$ corresponding to the potential for $\kappa = 1\,\mathrm{m}^{-1}$, approaching classical Coulomb.

5 Discussion

The results presented in Section 4 illustrate the distinctive features of the oscillatory Coulomb potential. In the case of a large oscillation parameter ($\kappa = 10^{10}\,\mathrm{m}^{-1}$), the potential exhibits rapid spatial oscillations, producing alternating regions of attraction and repulsion in the radial force. These alternating zones can generate stable equilibrium points in multi-particle systems, a behavior absent in the classical Coulomb interaction. Such features may be useful in engineered lattices or self-assembling structures where controlled force modulation is desired.

In contrast, for a small value of $\kappa = 1\,\mathrm{m}^{-1}$, the potential approaches the classical Coulomb form, confirming that the oscillatory term reduces to the standard $1/r$ potential in the appropriate limit. This illustrates the **continuous interpolation** between classical and oscillatory behavior, allowing the framework to model both conventional and modified electrostatic interactions.

The oscillatory Coulomb potential can be extended to include screening effects, anisotropy, or fractional operators, broadening its applicability to plasmas, metamaterials, and systems with nonlocal interactions. Additionally, the Fourier-space representation provides a convenient tool for numerical simulations of many-body systems, enabling efficient calculation of forces in lattices or periodic boundary conditions.

Overall, the oscillatory potential introduces **novel static interaction landscapes** that combine familiar Coulomb behavior with modulated attraction and repulsion. This opens opportunities for exploring equilibrium structures, energy landscapes, and force-driven assembly in both theoretical and experimental contexts.

6 Conclusion

In this work, we introduced an oscillatory generalization of the classical Coulomb potential by incorporating a Helmholtz term into Poisson's equation. The resulting potential,

$$\varphi(r) = \frac{q}{4\pi\varepsilon_0}\frac{\cos(\kappa r)}{r},$$

exhibits alternating attractive and repulsive regions with a characteristic wavelength $\lambda = 2\pi/\kappa$, while reducing to the standard Coulomb potential in the limit $\kappa \to 0$.

We demonstrated the behavior of the potential and radial force for two representative cases: a large $\kappa = 10^{10}\,\mathrm{m}^{-1}$, showing pronounced oscillations, and a small $\kappa = 1\,\mathrm{m}^{-1}$, recovering classical Coulomb interactions. The oscillatory nature introduces novel equilibrium points, which may have applications in engineered materials, optical lattices, and theoretical studies of static interaction landscapes.

Extensions to screened, anisotropic, or fractional forms, as well as Fourier-space representations, provide additional flexibility for modeling complex systems and many-body interactions. Overall, the oscillatory Coulomb potential offers a **simple yet versatile framework** for exploring modulated electrostatic forces and their consequences in physical and mathematical systems.

Future work could include detailed numerical simulations of multi-particle equilibrium structures, experimental realization in nanostructures or cold atom systems, and exploration of stability and energy landscapes arising from the oscillatory interactions.

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