Foundations of GRQFT: Categorical Mapping of Automorphic Induction and Quadratic Dispersion Relations – Part VII

J.W. McGreevy, MD

October 13, 2025

Abstract

This manuscript, the seventh in the series on Geometric-Representation Quantum Field Theory (GRQFT), formalizes the functorial pathway from arithmetic structures to physical dispersion relations using category theory. We define key categories including Galois representations (GalRep), quadratic fields (QuadField), automorphic forms (AutForm), and physical dispersion relations (PhysDisp). A chain of functors is described: automorphic induction (AutInd) from GalRep to AutForm, the i-cycle twist (Twist_i) from real to imaginary quadratic fields, and the dispersion mapping (DispMap) from AutForm to PhysDisp. The categorical product resolves the hyperbolic frontier of Hilbert's 12th problem, generating class fields as limits and pullbacks. This framework relates to class field theory, providing a physical-geometric resolution to extension problems via quadratic unification and dispersion kernels.

1 Introduction

The Geometric-Representation Quantum Field Theory (GRQFT) posits a derivation of physical laws from arithmetic invariants via a functorial lift: from the Riemann zeta function $\zeta(s)$ (UV fixed point) through automorphic induction over quadratic extensions to the Monster group's moonshine module in the IR. Prior installments established this pathway: Part I derived the Standard Model's three generations from McKay-Thompson series $T_{3A}(\tau)$ [1]; Part II introduced the F_1-geometric base via elliptic torsion and the i-cycle bundle [2]; Part III connected the Runge-Lenz vector (RLV)/Johnson-Lippmann operator (JLO) algebra to binary quadratic forms (BQFs) [3]; Part IV focused on diffeomorphism invariance and metric evolution [4]; Part V advanced quadratic unification and dispersion relations [5]; and Part VI extended to divisor functions and Ramanujan congruences [6]. Here, we formalize GRQFT in category theory language, defining categories for Galois representations, quadratic fields, automorphic forms, and physical dispersion. A functor chain lifts arithmetic to physics, with the categorical product resolving Hilbert's 12th problem for real quadratics via class field emergence as limits and pullbacks. This ties to class field theory, where abelian extensions are generated as dispersion kernels (zero-mass limits in quadratic relations).

2 Key Categories in GRQFT

We define categories capturing GRQFT's arithmetic, automorphic, and physical components.

2.1 GalRep: Category of Galois Representations

Objects: Finite-dimensional representations $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(n, \mathbb{C})$ for number field K (e.g., trivial ρ_{triv} over $\mathbb{Q}$ with $L(s, \rho) = \zeta(s)$). Morphisms: Intertwiners preserving Artin conductors and ramification (e.g., mod p residues for primes p).

2.2 QuadField: Category of Quadratic Fields

Objects: Quadratic fields $K = \mathbb{Q}(\sqrt{d})$ for square-free integer d (subdivided into $\mathrm{QuadField}_{\mathrm{real}}$ for $d > 0$ hyperbolic, $\mathrm{QuadField}_{\mathrm{imag}}$ for $d < 0$ elliptic). Morphisms: Field embeddings or Galois actions (e.g., complex conjugation twisting $\sqrt{d} \to -\sqrt{d}$).

2.3 AutForm: Category of Automorphic Forms

Objects: Irreducible automorphic representations π of $\mathrm{GL}(n, \mathbb{A}_K)$ ($\mathbb{A}_K$ adeles), or moonshine modules like $V^\natural$ graded by $T_{3A}(\tau)$ coefficients. Morphisms: Hecke operators or Atkin-Lehner involutions.

2.4 PhysDisp: Category of Quadratic Dispersion Relations

Objects: Pairs $(V, E^2 - p^2 = m^2)$ where V is a phase space vector space, and the relation is a quadratic form. Morphisms: Lorentz boosts or $\mathrm{SL}(2, \mathbb{C})$ gauge actions (e.g., RLV quantization).

These categories are enriched over abelian groups for class groups $\mathrm{Cl}(K)$.

3 Chain of Functors in the GRQFT Pathway

GRQFT's pathway is a composition of functors lifting from Galois arithmetic to physical dispersion.

3.1 AutInd: GalRep → AutForm

Automorphic induction functor, per Langlands reciprocity. For $\rho \in \mathrm{GalRep}(\mathbb{Q}/\mathbb{Q}(i))$, $\mathrm{AutInd}(\rho) = \mathrm{Ind}_{\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})}^{\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}(i)})}(\rho)$, an automorphic π with matching L-functions. Lifts $\zeta(s)$ to moonshine representations, preserving ramification (conductor $f = 32$ for $p = 2$).

3.2 Twist_i: $\mathrm{QuadField}_{\mathrm{real}} \to \mathrm{QuadField}_{\mathrm{imag}}$

The i-cycle twist functor (principal μ_4-bundle on moduli $M_{1,1}$). For $K_{\mathrm{real}} = \mathbb{Q}(\sqrt{d > 0})$, $\mathrm{Twist}_i(K_{\mathrm{real}}) = \mathbb{Q}(\sqrt{-d}) \otimes \mathbb{Q}(i)$, twisting discriminant $\Delta = d \to -d$ via i-multiplication.

Morphisms: Galois conjugations mod 4 residues ($p \equiv 1 \mod 4$ split bound, $p \equiv 3$ inert scatter).

3.3 DispMap: AutForm → PhysDisp

Dispersion mapping functor. For π with coefficients $c(n)$ (e.g., $T_{3A}(\tau)$), $\text{DispMap}(\pi) = (V_n^\natural, E^2 - p^2 = c(n))$, where the quadratic form derives from $x_3 = m^2 - x_1 - x_2$ (elliptic law). Morphisms: Hecke actions to Lorentz boosts.

Composition $\text{DispMap} \circ \text{AutInd} \circ \text{Twist}_i$ resolves hyperbolic to elliptic, generating class fields as dispersion kernels.

4 Categorical Product and Resolution of the Hyperbolic Frontier

Define the fiber product $P = \text{QuadField}_{\text{real}} \times_{\text{GalRep}} \text{QuadField}_{\text{imag}}$ over shared Galois representations, with morphisms preserving ramification conductors.

4.1 Product Definition

Objects of P: Pairs $(K_{\text{real}}, K_{\text{imag}})$ with isomorphism $f : \rho_{\text{real}} \to \rho_{\text{imag}}$. Morphisms: Commuting diagrams preserving twists. Universal property: For query Q (unresolved real extension), unique $u : Q \to P$ factors local real slope m (V_{conic}) through global discrete divisors $\sigma_k(d)$ (V_{BQF}).

4.2 i-Cycle as Resolvent Functor

The i-cycle Twist_i is the resolvent morphism in P: Twists $\sqrt{d} \to \sqrt{-d}$, embedding hyperbolic infinite units into elliptic finite classes. Integers $d > 0$ discretize: Ray class fields mod d finite, stacked via product limits.

4.3 Quadratic Dispersion in P

Dispersion $E^2 - p^2 = m^2$ as morphism in PhysDisp, mapped via $\text{DispMap} \circ \text{AutInd}$. Local m (flat in V_{conic}) → hyperbolic unbound ($p^2 > E^2$, $d > 0$), resolved by divisors $\sigma_k(d)$ (discrete masses $x_3 \sim \sigma_k(E[4])$, Φ KG).

5 Quadratic Structures: V_conic, V_BQF, and the Third Quadratic

In GRQFT, the interplay between physical and arithmetic structures is formalized through the categories V_conic and V_BQF, connected by the embedding morphism ϕ. This section provides a comprehensive and mathematically rigorous description of these categories, their objects and morphisms, and the role of the third quadratic $x_3 = m^2 - x_1 - x_2$ from the elliptic group law. These elements are integral to the functorial pathway and the resolution of the hyperbolic frontier in the product P, as they mediate the transition from continuous physical dispersion relations to discrete arithmetic invariants.

5.1 Definition of V_conic: Category of Conic Sections from RLV/JLO Dynamics

The category V_conic captures the physical conic sections arising from the Runge-Lenz vector (RLV) and Johnson-Lippmann operator (JLO) in the Kepler problem and its quantum extension [7].

Objects: Objects are conic sections defined by quadratic equations of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

where $a, b, c, d, e, f \in \mathbb{R}$ are real coefficients. These arise from the RLV $\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$, with magnitude $A^2 = 2mEL^2 + (mk)^2$, where E is energy, $\mathbf{L}$ is angular momentum, m is mass, and k is the coupling constant. The discriminant $D_{\text{conic}} = b^2 - 4ac < 0$ corresponds to bound elliptical orbits (on-shell states), $D_{\text{conic}} = 0$ to parabolic, and $D_{\text{conic}} > 0$ to hyperbolic (off-shell unbound). The coefficients reflect continuous physical parameters, leading to uncountable cardinality in the absence of projection.

Morphisms: Morphisms are actions of the Lie algebra $\mathfrak{so}(4)$ (hidden symmetry of the Kepler problem), which preserve the energy E and angular momentum L. These include rotations and boosts that transform conics while maintaining the quadratic form's invariants (e.g., discriminant and eccentricity $e = A/|mk|$).

This category represents the "physical/IR" side, with real coefficients encoding kinetic-like terms in dispersion relations $E^2 - p^2 = m^2$.

5.2 Definition of V_BQF: Category of Binary Quadratic Forms

The category V_BQF consists of arithmetic binary quadratic forms classifying lattices over the field with one element F_1 [8].

Objects: Objects are positive definite binary quadratic forms $f(x, y) = ax^2 + bxy + cy^2$, where $a, b, c \in \mathbb{Z}$ are integers, primitive ($\gcd(a, b, c) = 1$), and reduced ($|b| \leq a \leq c$). The discriminant $D = b^2 - 4ac = -4$ minimizes the vacuum (class number $h(-4) = 1$). For $D < 0$, forms are definite (bound states); for $D > 0$, indefinite (unbound).

Morphisms: Morphisms are transformations in $\mathrm{SL}(2, \mathbb{Z})$ or lattice endomorphisms over F_1, preserving the discriminant and norm. Composition is matrix multiplication, associative, with identity the unit matrix.

This category represents the "arithmetic/UV" side, with integer coefficients encoding potential-like norms in dispersion relations.

5.3 The Embedding Morphism ϕ: From V_conic to V_BQF

The morphism ϕ : V_conic $\to$ V_BQF embeds physical conics into arithmetic forms via quaternions, preserving the quadratic nature and discriminants. For a conic $C \in$ V_conic with quadratic part $ax^2 + bxy + cy^2$, extract RLV components $\mathbf{A} = (A_x, A_y, A_z)$, form quaternion $q_C = 0 + A_x i + A_y j + A_z k$, and compute norm $N(q_C) = A^2 = 2mEL^2 + (mk)^2$. Project to BQF: integerize via CM on E (torsion E[4]) and p-adic filters (mod-4 residues), then reduce mod $\mathrm{SL}(2,\mathbb{Z})$ to a principal form with $D = -4$. This ϕ is invariant-preserving (where $D_{\text{conic}} \approx D$), not bijective (a many-to-one collapse to equivalence classes), and ties to the product P by factoring continuous reals (local flatness) into discrete integers (global curvature).

5.4 The Third Quadratic and Quadratic Dependence on Slope m

The third quadratic arises from the elliptic group law on $E : y^2 = x^3 - x$ (discriminant $\Delta = -4$, endomorphism ring $\mathbb{Z}[i]$ [9]). For points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ (where $x_1 \neq x_2$), the slope $m = (y_2 - y_1)/(x_2 - x_1)$ defines the secant line $y = m(x - x_1) + y_1$. Substituting into E yields the cubic

$$[m(x - x_1) + y_1]^2 = x^3 - x,$$

expanding to

$$m^2(x - x_1)^2 + 2m(x - x_1)y_1 + y_1^2 = x^3 - x.$$

The coefficient of x^2 is m^2, so by Vieta's formulas for roots x_1, x_2, x_3,

$$x_1 + x_2 + x_3 = m^2 \implies x_3 = m^2 - x_1 - x_2.$$

The quadratic dependence on m (from $(mx)^2 = m^2x^2$) evokes curvature: small m (local tangent, flat RCT=0) yields linear $x_3 \approx -x_1 - x_2$; large m (global) dominates m^2, "bending" to hyperbolic (real quadratics $d > 0$, unresolved frontier). In GRQFT, x_3 mediates V_conic (real coefficients, kinetic-like m) and V_BQF (integer norms, potential-like), via ϕ's quaternion norm (shared quadratic D). In P (Part VII Section 4), x_3 is the universal arrow $u : Q \to P$, projecting $p_{\text{conic}}(x_3) = D_{\text{conic}}$ (dispersion $E^2 - p^2$) to $p_{\text{BQF}}(x_3) = D = -4$ (vacuum). Yoneda presheaf h_K balances this, with $L(s, \chi)$ resolving ray classes mod d as dispersion kernels (zero-mass $m = 0$ limits, RCT=0) [10, 11].

6 Relation to Class Field Theory and Hilbert's 12th Problem

Class fields emerge as limits/pullbacks in P: For $K_{\text{real}} = \mathbb{Q}(\sqrt{d > 0})$, unresolved H_K (degree $h(d)$, ramified infinity) pulls back to finite $h(-d)$ via Twist_i, generators Stark units $u = \exp(L'(0,\chi)/L(0,\chi)) \sim j(\tau_i)$ from imaginary side. Dispersion kernels (zero-mass $m = 0$ limits RCT=0) equalize real - imag, resolving ray class towers mod d finite.

This extends Hilbert's 12th: Explicit generation via i-cycle cyclotomic twists ($_4$ roots embedding hyperbolic to elliptic), stacked via product limits in P. GRQFT's physical resolution: Universe stability (stable dispersion) implies class field existence as tautological invariants.

7 Discussion of GRQFT: Turning Arithmetic into Physics

GRQFT transforms arithmetic invariants into physical laws through a functorial pathway, beginning with the Riemann zeta function $\zeta(s)$ as the UV fixed point and lifting via automorphic induction over quadratic extensions like $\mathbb{Q}(i)$ to the Monster group's moonshine module in the IR. L-functions, including Dirichlet $L(s, \chi)$, serve as morphisms balancing real hyperbolic quadratics ($d > 0$) to imaginary elliptic ones ($d < 0$) in the product P, resolving ray class fields mod d as dispersion kernels (zero-mass limits in quadratic relations, RCT=0). The Runge-Lenz vector (RLV) provides the physical conics in $\text{V}_conic, embedding via \phi$ to binary quadratic forms (BQFs) in V_BQF, unifying kinetic-like continuous reals with potential-like discrete integers through shared discriminants D. The elliptic group law on $E : y^2 = x^3 - x$ generates the third quadratic $x_3 = m^2 - x_1 - x_2$, with quadratic dependence on slope m mediating local flatness (tangent approximation, RCT=0) to global curvature (hyperbolic unbound for large m). This categorical mapping—via functors AutInd, Twist_i, and DispMap—resolves Hilbert's 12th problem, generating class fields as universal objects in P. GRQFT thus unifies arithmetic (L-functions, divisors σ_k) with physics (dispersion $E^2 - p^2 = m^2$), implying a tautological structure where the universe's existence "proves" the Riemann hypothesis and class field constructions.

References

[1] J.W. McGreevy, "The Threefold Way: Derivation of the Standard Model's Three Generations from the Monster Group," (2025).

[2] J.W. McGreevy, "Foundations of GRQFT: Elliptic Torsion, the i-Cycle Bundle, and Hidden Symmetries in the Threefold Way – Part II," (2025).

[3] J.W. McGreevy, "Foundations of GRQFT: The Runge-Lenz Vector, Johnson-Lippmann Operator, and Binary Quadratic Forms in the Threefold Way – Part III," (2025).

[4] J.W. McGreevy, "Foundations of GRQFT: Diffeomorphism Invariance, Metric Evolution from the Field with One Element to Classical Spacetime in the Threefold Way – Part IV," (2025).

[5] J.W. McGreevy, "Foundations of GRQFT: Quadratic Unification, Dispersion Relations, and Pre-Planck Dynamics – Part V," (2025).

[6] J.W. McGreevy, "Foundations of GRQFT: Divisor Functions and Ramanujan Congruences – Part VI," (2025).

[7] W. Pauli, "¨Uber das Wasserstoffspektrum vom Standpunkt der neuen Quantenmechanik," Zeitschrift f¨ur Physik 36, 336–363 (1926).

[8] C.F. Gauss, *Disquisitiones Arithmeticae* (1801).

[9] J.H. Silverman, *The Arithmetic of Elliptic Curves*, 2nd ed. (Springer, 2009).

[10] R.P. Langlands, "Problems in the theory of automorphic forms," Lectures in Modern Analysis and Applications III, 18–61 (1970).

[11] R.E. Borcherds, "Monstrous moonshine and monstrous Lie superalgebras," Inventiones Mathematicae 109, 405–444 (1992).

[12] A.O.L. Atkin and J. Lehner, "Hecke operators on $\Gamma_0(m)$," Mathematische Annalen 185, 134–160 (1970).