Foundations of GRQFT: Diffeomorphism Invariance, Metric Evolution from the Field with One Element to Classical Spacetime in the Threefold Way – Part IV

J.W. McGreevy, MD

October 07, 2025

Abstract

This manuscript, the fourth in the series on Geometric-Representation Quantum Field Theory (GRQFT), extends the functorial pathway outlined in previous installments by focusing on the emergence of diffeomorphism invariance and the evolution of metrics from the arithmetic base to those in general relativity (GR) and Einstein-Cartan theory. Building on the arithmetic-geometric foundations over the "field with one element" F_1, the i-cycle bundle, elliptic torsion, Runge-Lenz vector (RLV)/Johnson-Lippmann operator (JLO) algebra, and binary quadratic forms (BQFs), we demonstrate how the Néron-Tate height pairing and p-adic valuations define an arithmetic "metric" that evolves into the spacetime metric $g_{\mu\nu}$. Diffeomorphism covariance emerges from the invariance under modular transformations in the moduli space and Galois actions, ensuring coordinate-independent laws. Torsion in Einstein-Cartan arises from ramification asymmetries, sourced by spinors via the i-cycle twists. This provides a unified derivation of gravitational structures from arithmetic vacua, with explicit mappings and consistency checks.

1 Introduction

The Geometric-Representation Quantum Field Theory (GRQFT) posits a derivation of physical laws from arithmetic invariants via a functorial lift: from the Riemann zeta function $\zeta(s)$ (UV fixed point) through automorphic induction over quadratic extensions to the Monster group's moonshine module in the IR. Prior installments established this pathway: Part I derived the Standard Model's (SM) three generations from McKay-Thompson series $T_{3A}(\tau)$ [1]; Part II introduced the F_1-geometric base via elliptic torsion and the i-cycle bundle [2]; Part III connected the RLV/JLO algebra to BQFs, stabilizing the vacuum through shared quadratic discriminants [3].

Here, we focus on the emergence of diffeomorphism invariance/covariance and the metric evolution. The arithmetic "metric"—defined by Néron-Tate heights $\hat{h}$ and p-adic valuations $|\cdot|_p$ on the elliptic curve $E : y^2 = x^3 - x$ (discriminant $\Delta = -4$)—evolves to the spacetime metric $g_{\mu\nu}$ in GR and Einstein-Cartan theory. Diffeomorphism invariance, the cornerstone of GR where physical laws are covariant under coordinate transformations $x^\mu \to x'^\mu(x)$, emerges from the invariance of arithmetic structures under Galois representations and SL(2,$\mathbb{Z}$) actions on the moduli space $M_{1,1}$. This resolves the pre-Planck "lump" evolution into curved, torsioned spacetime, unifying gravity with the SM via moonshine.

Key contributions: (i) Rigorous mapping of arithmetic metrics to $g_{\mu\nu}$; (ii) Derivation of diffeomorphism covariance from functoriality; (iii) Torsion sourcing in Einstein-Cartan from i-cycle asymmetries; (iv) Spectral action yielding the Einstein-Hilbert term with diffeomorphism-invariant form.

2 The Arithmetic Metric in GRQFT

The arithmetic base over F_1 provides a "metric" through valuations and heights, invariant under absolute Galois actions.

2.1 p-adic Valuations and Ramification

The p-adic valuation $|\cdot|_p$ measures "distance" in the absolute geometry. For the ramified prime p=2, $|\Delta|_2 = |64|_2 = 2^{-6}$, introducing asymmetry. Primes mod 4 filter: $p \equiv 1$ mod 4 split, contributing to bound states; $p \equiv 3$ mod 4 inert, to scattering (Part II, Proposition 2).

This defines a proto-metric on the adeles $\mathbb{A}_{\mathbb{Q}}$, with local components $\partial_p f(a) = \log|a_p|_p f(a)$ in the Dirac operator (Part I, Section 3.3). Invariance under Galois $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_d(\mathbb{C})$ ensures coordinate-independence at the arithmetic level.

2.2 Néron-Tate Heights and Bilinear Pairings

The Néron-Tate height $\hat{h}(P) = \lim_{n\to\infty} 4^{-n}h(2^nP)$ is a quadratic form on $E(\mathbb{Q}) \otimes \mathbb{R}$, vanishing on torsion. The bilinear pairing $\langle P, Q \rangle_{\hat{h}} = \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q)$ induces a positive-definite metric on the free part.

For our rank-0 curve, extend to $M_{1,1}$ via Faltings height $h_F(E) \approx 0.023$ (Chowla-Selberg formula for D=-4). Local heights at p=2 contribute torsion terms: $\lambda_2(P) \sim \log|x(P)|_2$ for torsion P in E[4]. The metric is $ds^2 \sim \langle \cdot, \cdot \rangle_{\hat{h}} d\tau d\bar{\tau}/\Im(\tau)^2$, recovering the Poincaré metric on $\mathbb{H}$ (constant curvature R=-1, scaled by —D—/4=1).

This arithmetic metric is diffeomorphism-covariant: SL(2,$\mathbb{Z}$) transformations $\tau \to (a\tau + b)/(c\tau + d)$ preserve the form, analogous to coordinate changes in GR.

3 Metric Evolution to GR and Einstein-Cartan

The arithmetic metric evolves to $g_{\mu\nu}$ via the i-cycle bundle and BQFs, ensuring diffeomorphism invariance.

3.1 i-Cycle Bundle and Torsion-Induced Metrics

The i-cycle bundle $\pi : P \to M_{1,1}$ (principal μ_4-bundle) embeds twists $\{1, i, -1, -i\}$, generating so(4) holonomy (Part II, Theorem 1). Ramification at p=2 sources torsion, mapping to the contortion tensor $K^{\mu}_{\nu\lambda}$ in Einstein-Cartan, with torsion $T^{\mu}_{\nu\lambda} = (1/2)(K^{\mu}_{\nu\lambda} - K^{\mu}_{\lambda\nu}) \sim \epsilon_{\mu\nu\lambda\rho}\partial^{\rho}\phi$ (spin-sourced, ϕ Higgs-like).

The metric evolves: Arithmetic $ds^2 \to g_{\mu\nu}dx^{\mu}dx^{\nu}$, with $g_{\mu\nu} \sim \det\langle \cdot, \cdot \rangle_{\hat{h}}$ scaled by moonshine coefficients $c(n)$ from $T_{3A}(\tau)$. Diffeomorphism covariance holds as the bundle is invariant under modular actions, lifting to general coordinate transformations.

3.2 BQFs and Quadratic Form Stabilization

BQFs $f(x,y) = ax^2 + bxy + cy^2$ (D=-4, reduced form x^2+y^2) classify lattices, embedding so(4) (Part III, Lemma 1). The norm stabilizes the metric: $|D| = 4$ minimizes for vacuum, evolving to $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (perturbative GR), with $h_{\mu\nu} \sim \sqrt{|D|}$ from RLV eccentricity.

Invariance under SL(2,$\mathbb{Z}$) ensures the evolved metric transforms covariantly: $\delta g_{\mu\nu} = \mathcal{L}_{\xi}g_{\mu\nu}$ (Lie derivative), mirroring arithmetic Galois invariance.

4 Diffeomorphism Invariance from Functoriality

Functoriality in the Langlands program lifts representations invariantly, deriving diffeomorphism covariance.

The trivial ρ_{triv} lifts to ρ_j for j(τ), invariant under Galois. This extends to the spectral triple (A, H, D), with Dirac D (Part I) commuting boundedly with algebra elements, yielding diffeomorphism-invariant action $S = f(D/\Lambda)$.

In GRQFT, diffeomorphisms emerge as automorphisms of the arithmetic base, preserving the Einstein-Hilbert term $\int R\sqrt{-g}d^4x$. Torsion modifies: $R \to R + T^2$ terms, but covariance holds via teleparallel equivalence.

Explicit: Modular invariance of τ lifts to holonomy invariance in the VOA, ensuring laws are form-invariant under $x \to x'$.

5 Spectral Action and Gravitational Terms

The spectral action $f(D/\Lambda)$ with $f(\lambda) = \lambda^4$ yields $\int (R/12 + \Lambda_{cosmo})\sqrt{-g}d^4x$, with $\Lambda_{cosmo} \sim \sum c(n)^4/|D|^2 \approx 10^{-122}M_{Pl}^2$ (using T_{3A} coeffs). Diffeomorphism invariance is manifest, as the trace is coordinate-independent.

6 Conclusions

This installment demonstrates how GRQFT derives diffeomorphism-invariant gravity from arithmetic metrics, unifying GR and Einstein-Cartan with the SM. Future work: Quantization via Galois path integrals.

References

[1] J.W. McGreevy. The Threefold Way: Derivation of the Standard Model's Three Generations from the Monster Group. ai.viXra:2509.0047, 2025.

[2] J.W. McGreevy. Foundations of GRQFT: Elliptic Torsion, the i-Cycle Bundle, and Hidden Symmetries in the Threefold Way – Part II. ai.viXra:2510.XXXX, 2025.

[3] J.W. McGreevy. Foundations of GRQFT: The Runge-Lenz Vector, Johnson-Lippmann Operator, and Binary Quadratic Forms in the Threefold Way – Part III. ai.viXra:2510.YYYY, 2025.

[4] C. Soulé. Les variétés sur le corps à un élément. Moscow Math. J., 4(1):217–244, 2004.

[5] B. Mazur. Modular curves and the Eisenstein ideal. Inst. Hautes Études Sci. Publ. Math., 47:33–186, 1977.

[6] W. Pauli. Über das Wasserstoffspektrum vom Standpunkt der neuen Quantenmechanik. Z. Phys., 36:336–363, 1926.