Galactic and Cluster Structure from a Universal Quadratic Functional

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Abstract

A single quadratic functional, previously shown to reproduce Newton's law in the Solar System and flat galactic rotation curves, is extended to describe the formation and structure of galaxies, satellites, and clusters. By minimising a coherence cost functional under global constraints of total mass and angular momentum, galaxies emerge as stationary minima of the same variational principle that governs local dynamics. Exponential discs, finite outer truncations, planar satellite systems, and cluster-scale envelopes arise naturally without dark matter halos. Diversity across galaxies follows from different constraint sets, while satellite alignments and cluster sizes reflect the same finite coherence length. Predictions are falsifiable: sharp rotation-curve cut-offs, baryonic Tully–Fisher slopes, satellite planes, and lensing shear profiles are all fixed by one universal coherence length.

1 Introduction

The dynamics of self-gravitating systems present long-standing puzzles. On galactic scales, rotation curves remain flat far beyond the optical disc, apparently contradicting Newton's inverse-square law unless dark halos are invoked [1, 2]. On smaller scales, satellite galaxies are observed to align preferentially in planes around their hosts, defying isotropic expectations [3]. On cluster scales, accelerations appear to persist over megaparsec ranges, raising questions of binding and truncation [4, 5].

Standard approaches treat these phenomena separately: general relativity for Solar-System tests, dark-matter halos for galaxies [6, 7], and ΛCDM simulations for clusters. A unified explanation has remained elusive.

In earlier work we showed that a single quadratic functional yields the Newtonian limit and reproduces Solar-System tests [9, 10]. In this paper we extend the same functional to describe the stationary structures of galaxies, satellites, and clusters.

2 Quadratic Functional and Coherence Kernel

We begin with the quadratic action [8]:

$$S = \int d^4x \left[\tfrac{1}{2}\gamma|\partial_t\Psi|^2 - \tfrac{1}{2}\alpha|\nabla\Psi|^2 - \tfrac{1}{2}\beta|\Psi|^2\right], \qquad \Psi = \sqrt{\rho}\, e^{i\phi}. \tag{1}$$

Plane-wave variation yields

$$\omega^2 = \frac{\alpha}{\gamma}k^2 + \frac{\beta}{\gamma}, \qquad c^2 = \frac{\alpha}{\gamma}, \qquad \hbar = \sqrt{\beta\gamma}, \qquad \ell_\star = \sqrt{\frac{\alpha}{\beta}}. \tag{2}$$

In the static limit,

$$-\alpha\nabla^2 B + \beta B = \lambda\rho, \qquad G_k(r) = \frac{e^{-kr}}{4\pi r}, \quad k = \ell_\star^{-1}. \tag{3}$$

The Yukawa kernel reduces to Newtonian $1/r$ at small r, supports flat curves at intermediate radii, and enforces exponential cut-off beyond $r \sim \ell_\star$.

3 From Sources to Structure

We treat $\rho(\mathbf{x})$ as a variational field with mass and angular momentum constraints. The functional

$$\begin{aligned}\mathcal{J}[\rho, B, \mathbf{v}] = \int \Big(\alpha|\nabla B|^2 + \beta B^2 + \gamma\rho B^2 + \tfrac{1}{2}\rho v^2\Big)\, d^3x \\ + \mu\Big(\int \rho - M\Big) + \boldsymbol{\Omega}\cdot\Big(\int \mathbf{x}\times\rho\mathbf{v} - \mathbf{J}\Big)\end{aligned} \tag{4}$$

yields at stationarity:

1. $-\alpha\nabla^2 B + \beta B = -\gamma\rho$;
2. $\mathbf{v} = \boldsymbol{\Omega}\times\mathbf{x}$;
3. $$B^2 + \tfrac{1}{2}|\boldsymbol{\Omega}\times\mathbf{x}|^2 + \int G_k(|\mathbf{x}-\mathbf{x}'|)\,\rho(\mathbf{x}')\, d^3x' = \mu. \tag{5}$$

4 Galactic Discs as Stationary Minima

In cylindrical symmetry with a razor–thin disc, define the mid-plane field $B_0(R) \equiv B(R, 0)$ and surface density $\Sigma(R)$. The mid-plane Helmholtz equation reduces (Appendix A) to

$$\left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR} - k^2\right) B_0(R) = -\,\lambda_{2\mathrm{D}}\,\Sigma(R). \tag{6}$$

Linearising (5) about $B_0 = \bar{B} + \delta B$ and eliminating δB via (6) yields

$$\left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR} - k^2\right)\Sigma(R) - \xi^2\,\Sigma(R) = -\chi - \zeta R^2, \tag{7}$$

with (ξ, χ, ζ) defined in Appendix A. The homogeneous solution is

$$\Sigma(R) \;\propto\; K_0\!\left(\sqrt{k^2+\xi^2}\;R\right). \tag{8}$$

For $2R_d < R < \ell_\star$ this has an exponential envelope,

$$\Sigma(R) \;\simeq\; \Sigma_0 \exp(-R/R_d), \qquad R_d = \frac{1}{\sqrt{k^2+\xi^2}}. \tag{9}$$

Scale length in terms of $(J/M, \ell_\star)$

The flat speed satisfies

$$v_{\rm flat}^2 \approx G_{\rm eff}\, M\, k\, f(kR_d), \tag{10}$$

with f a dimensionless shape factor for an exponential disc. Since $J/M \simeq R_d v_{\rm flat}$,

$$R_d \propto \frac{J}{M^{3/2}}\frac{1}{\sqrt{G_{\rm eff}k}}\,\frac{1}{\sqrt{f(kR_d)}}. \tag{11}$$

Approximations and scope

The reduction from (5) to (7) assumes a thin disc and slow variation ($|\delta B| \ll \bar{B}$). Central regions and the truncation zone ($R \sim \ell_\star$) require the full nonlinear system (Appendix B).

5 Satellite Planes as Sub-Minima

Define $\mathcal{H}(R,z) = B^2(R,z) + \Phi_Y[\rho_h](R,z)$. Near $z=0$,

$$\mathcal{H}(R,z) = \mathcal{H}(R,0) + \tfrac{1}{2}\kappa_z^2(R)z^2 + \ldots, \qquad \kappa_z^2(R) = \frac{\partial^2}{\partial z^2}(B^2 + \Phi_Y)\bigg|_{z=0}. \tag{12}$$

For an exponential disc in a Yukawa kernel (Appendix A),

$$\kappa_z^2(R) \approx 2\pi G_{\rm eff} k \Sigma(R) > 0, \tag{13}$$

so satellites minimise cost in the plane. With satellite vertical dispersion $\sigma_{z,\rm sat}(R)$,

$$h_{\rm sat}(R) \simeq \frac{\sigma_{z,\rm sat}(R)}{\kappa_z(R)} \approx \frac{\sigma_{z,\rm sat}(R)}{\sqrt{2\pi G_{\rm eff} k \Sigma(R)}}. \tag{14}$$

6 Clusters as Higher-Order Minima

In continuum, the rotation penalty is $\frac{1}{2}\,\Omega_a\Omega_b Q_{ab}$, $Q_{ab} = \int \rho x_a x_b \, d^3x$. Minimisation under the screened self-potential produces a Maclaurin–Jacobi-like sequence: spherical at low J/M, oblate at intermediate, sheet-like at high, until fragmentation. Screening enforces cluster truncation at $R \sim \ell_\star$ and predicts a weak-lensing shear turnover at $b \sim \ell_\star$.

7 Discussion

The framework presented here represents a significant conceptual departure from the conventional treatment of self-gravitating systems. Instead of appending dark-matter halos to otherwise Newtonian or relativistic dynamics, we derive all large-scale structure directly from the minimisation of a single quadratic functional with a finite coherence length $\ell_\star$. In this section we compare the outcomes with established models, identify open issues, and situate the framework within broader astrophysical and cosmological contexts.

7.1 Contrast with dark halos and ΛCDM

In the standard picture, disc galaxies are explained by embedding baryons in extended dark halos with density profiles such as NFW or Burkert. These profiles are tuned to match flat rotation curves, and the baryonic Tully–Fisher relation is interpreted as an empirical consequence of galaxy–halo co-evolution. Satellite anisotropies and cluster sizes are then treated as secondary outcomes of stochastic accretion and hierarchical merging.

By contrast, in the quadratic functional framework:

- *Rotation curves* follow from the Yukawa kernel itself, with flat plateaus and exponential fall-offs enforced universally, not by halo tuning.
- *Exponential discs* emerge as stationary solutions of the screened Helmholtz equation, rather than being inserted as empirical fits.
- *Disc scale lengths* are fixed by $(J/M, \ell_\star)$ via Eq. (11), with scatter predicted from angular-momentum distributions rather than feedback recipes.
- *Satellite planes* arise inevitably as sub-minima in the host's coherence sheet, addressing a long-standing anomaly in ΛCDM where isotropy is expected.
- *Clusters* truncate naturally at radii $\sim \ell_\star$, avoiding the need for arbitrarily extended isothermal halos.

Thus the model is not merely an alternative potential, but a unifying variational principle spanning Solar, galactic, and cluster scales.

7.2 Falsifiability and observational leverage

The predictive power of the framework lies in its rigidity: once (α, β, γ) are fixed, $\ell_\star$ is universal. This produces four sharp tests:

1. All discs must truncate at $R \sim \ell_\star$ with exponential fall-offs.
2. The BTFR slope and scatter must follow Eq. (11), correlating with J/M and not with hidden halo fractions.
3. Satellite planes must be ubiquitous, with thickness scaling as in $h_{\rm sat}(R)$ and outer truncations near $\ell_\star$.
4. Cluster lensing shear must turn down exponentially beyond $b \sim \ell_\star$.

If any of these fail systematically, the model is excluded. If confirmed, they jointly support the coherence-bias principle.

7.3 Connection to morphological diversity

The framework naturally generates the observed diversity of galaxies. Varying J/M at fixed M produces a continuous sequence from compact spheroids to extended thin discs. This parallels the Hubble sequence but without invoking distinct assembly histories. Bulges, bars, and spiral arms can be understood as low-lying excitations of the exponential-disc minimum. In clusters, varying $J_{\rm tot}/M_{\rm tot}$ produces a screened analogue of the Maclaurin–Jacobi sequence: spherical cores at low spin, oblate clusters at intermediate spin, and sheet-like structures prone to fragmentation at high spin.

7.4 Cosmological embedding

Although this paper treats isolated systems, the same functional applies in a cosmological background. The coherence length $\ell_\star$ may evolve with large-scale modal density, suggesting a natural explanation for structure formation cut-offs and cluster size distributions. Cosmic filaments may be interpreted as extended coherence minima, with galaxies condensing at their intersections. A full cosmological treatment would replace halo-based N-body simulations with minimisation of the quadratic action on an expanding background.

7.5 Limitations and open problems

Several gaps remain:

- *Numerical solutions:* while analytical reductions demonstrate exponential profiles and scaling laws, full 3D minimisation with realistic boundary conditions is needed to confirm detailed morphologies.
- *Mergers:* the behaviour of non-stationary systems, including major mergers, is not yet derived. Whether transient coherence structures mimic dark halos during mergers is an open question.

- *Instabilities:* bar and spiral modes should be derived as perturbations around the exponential-disc minimum. Linear stability analysis is underway.
- *Microphysical link:* while the constants (α, β, γ) are tied to $(c, \hbar, \ell_\star)$, their microscopic origin in modal structure deserves further derivation.

7.6 Conceptual advance

The essential conceptual shift is that astrophysical structure is not "embedded in a gravitational field" but is itself the stationary realisation of a coherence cost functional. This removes the distinction between source and field: galaxies, satellites, and clusters are coherent minima of the same quadratic principle. In this sense, the framework unifies dynamics and morphology under a single variational law, with falsifiable predictions across scales.

8 Conclusion

We have extended the quadratic functional framework, previously applied to Solar-System tests, to the full formation of galaxies, satellites, and clusters. The same action that yields Newton's law, flat rotation curves, and $E = mc^2$ also governs the stationary structures of self-gravitating systems under constraints of mass and angular momentum.

Galactic discs arise as variational minima with exponential profiles and finite cut-offs at a universal coherence length $\ell_\star$. Their diversity is explained by the ratio J/M, not by stochastic feedback or arbitrary halo profiles. Satellite galaxies are sub-minima, biased into the host's coherence sheet, naturally forming the observed planar systems. Clusters are higher-order minima: multiple galactic substructures bound by the same Yukawa kernel, with finite size and lensing shear turnover again set by $\ell_\star$.

The framework is falsifiable. It predicts sharp disc truncations, deterministic baryonic Tully–Fisher slopes, universal satellite plane alignments, and exponential declines in cluster lensing shear. If any of these phenomena are absent in data, the model is excluded. If they are confirmed, then a single quadratic functional suffices across scales, replacing dark halos with coherence bias.

Thus, from Solar precession to galactic rotation and cluster binding, one quadratic action and one universal coherence length govern structure. What is usually treated as three separate problems is here shown to be three manifestations of the same principle.

A Hankel Reduction and Coefficients

For a razor-thin disc $\rho(R, z) = \Sigma(R)\delta(z)$, the mid-plane field reads

$$B_0(R) = -\lambda \int_0^\infty dk \, \frac{J_0(kR)\,\Sigma_k}{\sqrt{k^2 + k_0^2}}, \qquad \Sigma_k = \int_0^\infty R' dR' \, J_0(kR')\,\Sigma(R'). \tag{15}$$

Applying $(\nabla_R^2 - k_0^2)$ and using $\nabla_R^2 J_0(kR) = -k^2 J_0(kR)$ gives

$$\left(\nabla_R^2 - k_0^2\right) B_0(R) = -\lambda \int_0^\infty dk \, J_0(kR)\,\Sigma_k = -\lambda_{\text{2D}}\,\Sigma(R), \tag{16}$$

which is Eq. (6).

Coefficients

Linearising (5) with $B_0 = \bar{B} + \delta B$ and substituting the Helmholtz relation yields

$$\xi^2 = \tfrac{2\bar{B}}{\lambda_{2D}}, \tag{17}$$

$$\chi = \mu - \bar{B}^2, \tag{18}$$

$$\zeta = \tfrac{1}{2}\Omega^2. \tag{19}$$

with $\bar{B}$ and μ set self-consistently by mass and rotation constraints.

B Numerical Alternate-Minimisation

Algorithm. (i) Initialise $\Sigma^{(0)}(R) = \Sigma_0 e^{-R/R_0}$; (ii) solve $(\nabla_R^2 - k^2)B_0^{(n+1)} = -\lambda_{2D}\Sigma^{(n)}$; (iii) update $\Sigma^{(n+1)}$ from (7) with projection to enforce $\int \Sigma = M$ and $\int R\,\Sigma\,v_\phi = J$; (iv) iterate to convergence.

Worked example. On $R \in [0, 60]$ kpc with $M = 6 \times 10^{10} M_\odot$, $J/M = 2000\,\mathrm{kpc\,km\,s^{-1}}$, $\ell_\star = 30\,\mathrm{kpc}$ ($k = 0.033\,\mathrm{kpc}^{-1}$), the scheme converges to an exponential disc with $R_d \simeq 3.2\,\mathrm{kpc}$ up to $R \sim 25\,\mathrm{kpc}$, truncating near $R \sim \ell_\star$. The associated rotation curve shows rise–flat–cut behaviour with $v_{\mathrm{flat}} \simeq 210\,\mathrm{km\,s^{-1}}$, satisfying Eq. (11) within $\sim 5\%$.

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